From ‘clever hoax’ to ‘crown jewel’

A collection of papers relating to the accomplishments of the Mādhava/Kerala school of astronomy and mathematics

Saṅgamagrāma Mādhava declared sometime around 1400 CE:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

C. M. Whish published this Indian discovery in 1834.
Augustus De Morgan shouted in 1843: Whish’s paper was ‘a clever hoax’.
David Mumford proclaimed in 2010: The contributions of the Mādhava school
Is ‘the crown jewel’ of Indian mathematics.

Compiled by
Dr. V. N. Krishnachandran

(Strictly for private circulation only.)

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Preface

This is a collection of papers and documents like excerpts of chapters and pages from books all relating to the mathematical contributions of the Kerala school of astronomy and mathematics. The school has variously been called as the "Kerala schhol" (K. V. Sarma), "Mādhava school" (Kim Plofker) and the "Nila school" (P. P. Divakaran) by different historians of Indian mathematics.

The collection has been divided into three parts. In Part I, there is only one paper and it deals with the historiography of the Kerala school of astronomy and mathematics. It is a fairly long paper with thirty-seven pages. In Part II, we have collected together five papers, three of which are authored by K. V. Sarma, all pertaining to the life, times and work of Saṅgamagrāma Mādhava the celebrated founder of the Kerala school of astronomy and mathematics. The first document in this part is an extract from *Kerala Sāhiyta Caritram* (History of Kerala Literature in Malayalam) by Ullor S. Parameswara Aiyer published posthumously in 1953. This is likely to be the first time an account of the life of Mādhava appears in print. But there is no mention of the times in which Mādhava flourished. The first time an account of the times of Mādhava appeared in print was in an article authored by K. V. Sarma and published in the Malayalam weekly *Mathrubhumi* in November 1956. A scanned copy of this article has been included as the second paper in Part II of this collection. The remaining three papers in Part II throw more light on the life and times of Mādhava.

Part III, with forty-two papers, constitutes the core of this collection. The story of the modern unravelling of the mathematical contributions of this school begins with a brief reference to the different infinite series representations of π discovered by the members of the school in the book *Kāla Saṃkalita* authored by John Warren and published in 1825. The first paper in Part III is the extracts from this book which discuss these series. The papers in this part are all historically important and represent the different stages in the understanding of the mathematical contributions of the Kerala school. Only those documents that are publicly available and accessible in the online world have been included in this part. Consequently, a few important papers that should have been included in this collection were not included as they could not be accessed from the world of internet. A list of such papers is given at the end of this collection. Each paper in this collection is a reproduction of the text as printed by the original publisher.

The appendix in the collection contains three items: Appendix A is a reproduction of the *Yuktibhāṣā* derivation of the Leibniz series and Appendix B is the *Yuktibhāṣā* derivation of the power series expansions of the sine and cosine functions. Both these derivations were extracted from the English translation of *Yuktibhāṣā* published in 2008. Appendix C is a list of "missing papers", that is, papers that should have been in this collection, but were not.

This collection has been compiled for the benefit of students, teachers and researchers who are interested in reading the original papers and documents related to the unravelling

of the contributions of the Kerala school of astronomy and mathematics. It is intended strictly for private circulation only.

July 2024 Dr. V. N. Krishnachandran

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Part I

HISTORIOGRAPHY

Paper 1

Historiography of the Kerala School of Astronomy and Mathematics, 1820–2021

M. D. Srinivas, "Emergence of a New Era in the History of Indian Mathematics Historiography of the Kerala School of Astronomy and Mathematics, 1820–2021: Some Highlights"

Bhāvanāā, The mathematics magazine, Volume 6, Issue 1, January 2022.

M. D. Srinivas

M. D. Srinivas was born in Bengaluru in the year 1950. He had his early education in various institutions in Bengaluru and secured his Master's degree from Bangalore University. He joined the University of Rochester for a PhD programme in 1972 and secured the PhD degree in 1976. After returning in 1976, he joined the Department of Theoretical Physics at the University of Madras where he spent the next 20 years. While at the University of Madras, he along with a few of his friends established the "Patriotic and People-oriented Science and Technology (PPST) Group" and launched the *PPST Bulletin*. Around 1990, he with a few like-minded persons established a research institute called the "Centre for Policy Studies". He opted for voluntary retirement from the university in 1996. Since 1996, he has been working at the Centre for Policy Studies.

Source: Bhavana: The Mathematics Magazine

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Bhāvanā
The mathematics magazine

A publication of the Bhavana Trust
Volume 6 Issue 1 January 2022 Historical

Emergence of a New Era in the History of Indian Mathematics

Historiography of the Kerala School of Astronomy and Mathematics, 1820–2021: Some Highlights

By M.D. Srinivas

Courtesy: Royal Asiatic Society of Great Britain and Ireland

It is now well established that the school of astronomy and mathematics, founded by Mādhava in Kerala in the 14th century, came up with two of the major discoveries, namely calculus and non-geocentric planetary models, a couple of centuries before they were to become the hallmarks of the modern European scientific revolution. The seminal discovery of the Kerala school, of the infinite series for $\pi\pi$ was brought to the notice of modern scholarship for the first time in the book *Kālasan˙n˙kalita* of John Warren, published from Madras in 1825. This was soon followed by the detailed account presented in the pioneering articles of Charles Whish, which were published during 1827–1834. However, the work of the Kerala school was totally ignored by modern scholarship for over a hundred years. This was because Whish's paper and the Kerala work on infinite series were denigrated and suppressed by the British academic establishment so much so that even reputed Indologists such as Albrecht Weber, and George Thibaut, and renowned historians of mathematics such as David Smith chose not to say anything about the work of the Kerala school even when they referred to the articles of Whish in their scholarly writings.

The work of the Kerala school was resurrected by the pioneering article on the "Hindu values of $\pi\pi$" by Bibhutibhushan Datta in 1926. This was followed by the article "On the use of series in Hindu Mathematics" by Datta's colleague, Avadhesh Narayan Singh, which was published in the first issue of *Osiris* in 1936. During 1940–1952, C.T. Rajagopal and his collaborators wrote a series of important articles explaining, in modern mathematical terms, all the important demonstrations for the infinite series for $\pi\pi$, and the sine and the cosine functions, as given in the Malayalam text *Yuktibhāṣā*. Equally significant was the publication in 1948 of the mathematics part of this classic text, which was edited along with detailed notes in Malayalam,

by Rama Varma Tampuran and Akhileswaraiyer. When the contents of these important publications were taken note of in the 1950s and 1960s, by some of the well-known historians of mathematics such as J.E. Hoffmann (1953), C.N. Srinivasiengar (1958, 1967), A.P. Yushkevich (1961, 1964), D.T. Whiteside (1961, 1968), H.L.L. Busard (1962), T.A. Sarasvati Amma (1963, 1979), A.K. Bag (1966, 1979) and M.E. Baron (1969), one could clearly see the emergence of a new era in the historiography of Indian mathematics.

Starting from the publication of *Bhaṭadīpikā* commentary of Parameśvara on *Āryabhaṭīya* by H. Kern in 1874, around fifteen source-texts of the Kerala school were published in the next eighty years. Still, there remained several gaps in our understanding of the history and achievements of the Kerala school. It is the monumental work of K.V. Sarma, whose prolific writings have contributed substantially to our present understanding of the history and contributions of the Kerala school. Starting from 1953, Sarma edited and published about twenty very important works of the leading savants of the Kerala school. Sarma also brought out a comprehensive history and bibliography of Kerala works on Astronomy and Mathematics in 1972, where he listed around 400 works. It was Sarma's relentless efforts to bring out works which present yuktis, or demonstrations, which led to the publication of the Sanskrit commentaries, *Kriyākramakarī* and *Yuktidīpikā* of Śan˙n˙kara, the compilation *Gaṇitayuktayaḥ* (containing 27 small tracts of yuktis), and the critical edition of *Yuktibhāṣā* of Jyeṣṭhadeva, along with English translation and explanations. Sarma's publication of the seminal works of Nīlkaṇṭha such as *Golasāra*, *Siddhāntadarpaṇa* and *Grahasphuṭānayane Vikṣepavāsanā*, led to a clearer understanding of the revised planetary model that was proposed by Nīlkaṇṭha in his *Tantrasan˙n˙graha* and *Āryabhaṭīyabhāṣya*. One of Sarma's landmark publications has been the *Jyotirmīmāṃsā* of Nīlkaṇṭha, which discusses the crucial role of continuous parīkṣā, or examination of the theories by observation, and the consequent process of revision and preparation of newer and more accurate karaṇas or computational manuals.

Currently, there are still many important source-works of the Kerala school–-including works composed by legendary figures such as Mādhava, Parameśvara, Putumana Somayājī, Śan˙n˙kara and Acyuta–-which are yet to be edited and published. Further, among the fifty major source-works (and about thirty small tracts) which have been published, only about a dozen have been translated; and even among them, there are only perhaps half a dozen works which have been studied in depth to explicate their full technical content. It will be befitting to the legacy of K.V. Sarma, and his illustrious predecessors such as Datta and Singh, Rama Varma Tampuran, C.T. Rajagopal and others, that our young scholars imbibe their great dedication and respect for our tradition, and carry forward their mission to ensure that the entire corpus of works of the Kerala school is rigorously and comprehensively studied and brought to light.

John Warren (1769-1830) Portrait as a boy Wikimedia Commons

Denigration and Suppression of the Kerala work on Calculus, 1825–1925

The first notice of Indians' knowledge of the infinite series for π (1825)

Lt. Col. John Warren (1769–1830), a Frenchman who was in charge of the observatory at Madras during 1805–10, seems to have been the first European observer to report in the 1820s that some of the Indian astronomers that he had met seemed to know the infinite series for ππ [1], [2]. In his book *Kālasan˙n˙kalita*, which was published in Madras in 1825, Warren states that [3][pp.92–93]:

On their manner of resolving geometrically the ratio of the diameter to the circumference of a circle, I never saw any Indian demonstration: the common opinion, however, is, that they approximate it in the manner of the ancients, by exhaustion; that is, by means of inscribed and circumscribed polygons. However, a native astronomer who was a perfect stranger to European Geometry gave me the well-known series, 1−13+15−17+19−111+113+⋯1−3
1+51−71+91−111+13
1+⋯ to unit (∗∗) being the ratio of the area of the circle to the square of its diameter, or that of an arc of 45∘45∘ to Radius unit… This person reduced the first five terms of the series before me, which he called *Bagah-Anoobandha*, or *Bagah-Apovacha*; to show that he understood its use. This proves at least, that the Hindus are not ignorant of the doctrine of the series; …

In the footnote marked (∗∗) to the above passage, Warren included a note from George Hyne (1800–1826), an officer in the East India Company's medical service.[1] The note referred to a communication made to the Madras Literary Society by Charles Whish, who was in the East India Company's service since 1812 and was posted at Malabar (northern Kerala). Hyne mentions that Whish, in his communication, had listed a number of infinite series for ππ that were employed by the Indian astronomers. However, Hyne was very emphatic that [3][p.93]:

The Hindus never invented this series; it was communicated with many others, by Europeans, to some learned Natives in modern times. Mr. Whish sent a list of the various methods of demonstrating the ratio of the diameter and circumference of a Circle employed by the Hindus to the Literary Society, being impressed with the notion that they were the inventors. I requested him to make further enquiries, and his reply was that he had reasons to believe them entirely modern and derived from Europeans, observing that not one of them who used the Rules could demonstrate them. Indeed the pretensions of Hindus to such a knowledge of geometry is too ridiculous to deserve refutation.

After reproducing Hyne's note, Warren states that [3][p.93]:

I join in substance to Mr. Hyne's opinion, but do not admit that the circumstance that none of the Sastras mentioned by Mr. Whish, who used these series could demonstrate them, would alone be conclusive. It cannot be denied, that the inventors of the system of Hindu Astronomy possessed a knowledge of Geometry which their successors have not entirely preserved, and if we bring the question home to ourselves we are compelled to acknowledge, that thousands (even among the well informed) use Laplace's formulae without understanding the principles of their construction.

Later, in an Appendix to the *Kālasan˙n˙kalita*, titled "*On certain infinite series collected in different parts of India, by various Gentlemen, from various Native Astronomers…*", Warren reproduces a subsequent letter (dated August 17, 1825) from Hyne. Warren prefaces the communication of Hyne with the following comments [3][p.330]:

I have stated … that in Mr. Hyne's opinion the Hindus never invented the Series referring to the Quadrature of the Circle which were found in their possession in various parts of India; and that Mr. Whish, from whom he had obtained some of those which were communicated to the Madras Literary Society, after having first expressed a belief that they were indigenous, had subsequently reasons for thinking them entirely modern, and derived from the Europeans; observing that not one of the *Jyautish Sastras* who used these rules, were capable of demonstrating them.

Since the time that I wrote the Note referred to, Mr. Hyne has done me the favour to communicate to me an account of the Series which had come to his knowledge; and I now lay the same before the reader in that Gentleman's own language, being well persuaded that it cannot fail to interest much all the votaries of science.

The Appendix then reproduces the letter of Hyne, which presents seven infinite series for $\pi\pi$ [3][pp.330–331]. It may be seen that these series are all part of the famous paper of Whish which was read at the Royal Asiatic Society, London, in December 1832, and published in its *Transactions* in 1834.

From all these passages in the *Kālasan˙n˙kalita*, we can clearly conclude the following:

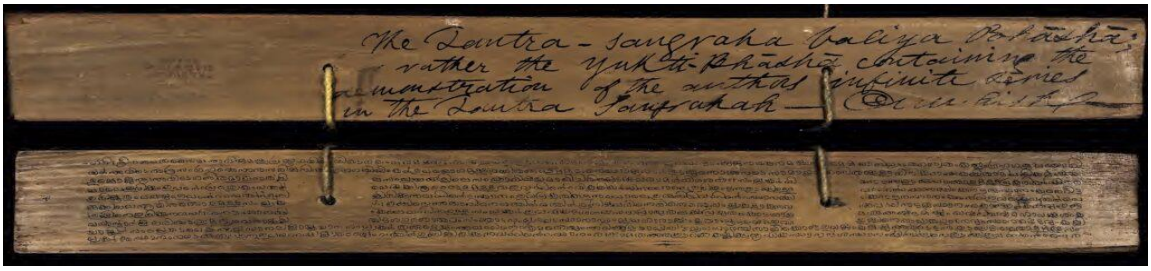
1. In the 1820s, both Warren (who was mainly in Madras and Pondicherry) and Whish (who was stationed in Malabar or northern Kerala) had met several Indian astronomers who were conversant with some of the infinite series for $\pi\pi$.
2. By 1825, the Madras Literary Society had received a communication from Whish wherein he had referred to several Indian texts (śāstras) which mentioned these series.
3. Whish was however persuaded by Hyne who was a part of the elite establishment in Madras that, since the Indian astronomers as well as their śāstras seemed to be unaware of the demonstrations of these series, they must have been communicated to the Indians by Europeans in the modern times.

The career of Charles Whish

Charles Mathew Whish (1794–1833) came to India in 1812 after studying at the East India College in London. His career in India in the service of the East India Company is summarised below [4][p.153]:

- 1813: Writer
- 1815: Register of the Zillah of South Malabar
- 1823: Register of the Zillah of Malabar
- 1827: Assistant Judge and Joint Criminal Judge of Malabar
- 1830: Assistant Judge and Joint Criminal Judge of Cuddapah
- 1831: Out of employment
- 1832: Acting Judge and Joint Criminal Judge of Cuddapah
- 1833: Died on 14th April 1833, at Cuddapah.

Whish spent over 15 years in Malabar and seems to have written a Malayalam grammar and dictionary. He also collected around 300 Malayalam and Sanskrit manuscripts of important literary, philosophical and scientific works. After his death, these manuscripts were presented to the Royal Asiatic Society in 1836 by J.C. Whish, the brother of Charles Whish.

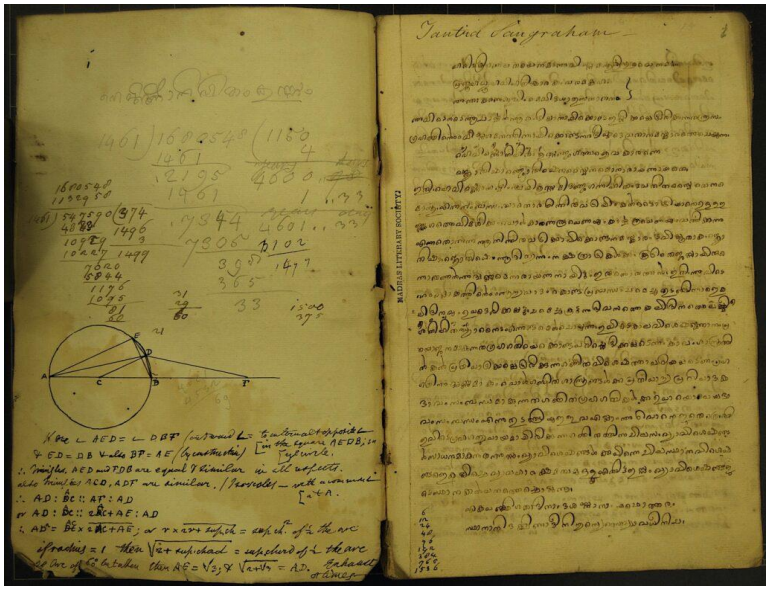

The Tantra-sangraha Valiya Bhashya,
rather the Yukti-bhasha containing the
Demonstration of the Authors infinite series
in the Tantra Sangrahah

Whish inscribed a note on the verso of the first blank palm-leaf of Manuscript No. 124 Royal Asiatic Society of Great Britain and Ireland

The notebook of Whish (c.1820)

We have already referred to the communication from Whish sent around 1825 to the Madras Literary Society regarding the Kerala work on infinite series. Some idea of the background of this communication may be had from an uncatalogued and unnumbered paper manuscript in the form of a notebook preserved in the Whish Collection at the Royal Asiatic Society, London. This is a manuscript of the Malayalam Commentary *Kriyākalāpa* by Śan˙n˙kara Vāriyar on the *Tantrasan˙n˙graha* of Nīlakaṇṭha. It also contains some notes by Whish in English, which seem to have been written from 1820 onwards. The details of this notebook have been presented in a recent study of *Kriyākalāpa* [5]:

This manuscript contains 182 folios, each folio having about 24 lines. The text is quite readable. However, the manuscript is not complete and abruptly ends in the fifth chapter of *Tantrasan˙n˙graha*, with the noting at the end: “vide other book for the remainder.” On the opposite page, there is a signature of Whish with the year marked as 1820. There is a seal of “Madras Literary Society” on the margin of the first page of the manuscript.



Pages from Whish's notebook **courtesy:** Royal Asiatic Society of Great Britain and Ireland.Photographed by K. Ramasubramanian

In *Kriyākalāpa*, while commenting on the verses 3–6 of the second chapter (*Sphuṭaprakaraṇa*) of *Tantrasaṅgraha*, Śaṅkara makes a long excursus where he cites several Sanskrit verses–-which describe the Mādhava series for $\pi 44\,\pi$ and its many transformed versions–-along with his Malayalam commentary. The Whish notebook mentioned above contains notes by Whish giving the mathematical form of all these series, and it may be seen that they appear in the same order in his famous article of 1834. In that article, Whish wrongly attributed the Sanskrit verses on infinite series, which are mentioned in the commentary *Kriyākalāpa*, to be a part of the *Tantrasaṅgraha* itself–-an error that has been repeated in several later studies even after reliable editions of the *Tantrasaṅgraha* were published by S.K. Pillai in 1958 and K.V. Sarma in 1977.

Initial publications of Whish (1827)

While his first communication on the Kerala work on infinite series was summarily rejected by Hyne, Whish did manage to get two of his papers published in 1827 in the very first issue of the *Transactions of the Madras Literary Society*. The first article on the "Alphabetic Notation of the Hindus" [6] deals with different systems of numeration employed in the Indian texts, such as the system used in the *Āryabhaṭīya* and the Kaṭapayādi system used in south India. As in illustration of the Kaṭapayādi system, Whish presents the verse from the *Sadratnamāla* of Śaṅkaravarman which gives the value of $\pi\pi$ correct to 17 decimal places. Whish also cites the verses of Mādhava giving the table of 24 sine values which are accurate to the thirds. Again, Whish attributes the latter verses to the *Tantrasaṅgraha*, though they are found only in the commentaries. In the natural course of events, just these two illustrations of the remarkable achievements of the Kerala school should have been sufficient to evoke considerable interest to locate and study the works of this school, even at this stage when the more detailed 1834 paper of Whish had not yet seen the light of the day.

The other paper of Whish's is on a very different note and is titled, "On the Origin and Antiquity of the Hindu Zodiack". At the beginning of this paper, Whish declares [7][pp.63–64]:

My intention is to prove, that the Indians have borrowed the dodecatemory [twelve-part] division of the heavens, with the figures etc., of the zodiacal constellations, immediately from the Greeks. Sir William Jones says,[2] "The Brahmins were always too proud to borrow their science from the Greeks, Arabs, Muguls, or any nation of Mlechchhas, as they call those who are ignorant of the Vedas, and those who have not studied the language of the Gods…". I shall endeavour to refute these, and the few arguments of the same nature succeeding it, in the essay from which I quote.

At this point, there is a footnote inserted in the paper, attributed to "G.H.", which states [7][p.64]:

In an essay embracing the various objects of an extensive speculation, the facts necessarily appear detached, and somewhat desultory. The chain of argument adopted by the author has, however, an obvious connection. He first endeavours to prove, by a comparison of the Greek and Sanscrit languages, that the duodenary zodiack of the Greeks was communicated, at an earlier period, to the Hindus; and he afterwards adduces testimony to show that, prior to this period, the Hindus were unacquainted with a duodenary division of heavens.

The unsigned editorial "Notice", at the beginning of this volume of the *Transactions of the Madras Literary Society* mentions that G.H. refers to George Hyne. In fact, this article of Whish has several footnotes by G.H. inserted at different places. It seems that, by this article, which purported to show that the twelve-part Zodiac was communicated to the Indians by the Greeks, Whish had managed to some extent to win the approval of George Hyne. In any case, Hyne seems to have passed away sometime in 1826, as the same editorial note at the beginning of the *Transactions* also records its tribute to George Hyne, "by whose lamented death science has been deprived of an ardent and profound investigator".

Whish's landmark paper (1832)

On the title page of Whish's landmark paper "On the Hindu Quadrature of the Circle", it is mentioned that it was communicated by the "Madras Literary Society and Auxiliary Royal Asiatic Society" and was read at the Royal Asiatic Society, London, on the 15th of December 1832 [9][p.509]. It was published in the third volume of the *Transactions of Royal Asiatic Society* in 1834.[3]

Whish begins his paper with the famous verse from *Āryabhaṭīya* which gives an approximate value for ππ, and then goes on to cite some of the very accurate approximations given in the Kerala works, *Tantrasaṅgraha*, *Karaṇapaddhati*, and *Sadratnamāla*. Whish then states [9][p.512]:

The approximations to the true values of the circumference with a given diameter, exhibited in these three works, are so wonderfully correct, that the European mathematicians, who seek for such proportions in the doctrine of fluxions, or in the more tedious continual bisection of an arc, will wonder by what means the *Hindū* has been able to extend the proportion to so great a length. Some quotations which I shall make from these three books will shew that a system of fluxions peculiar to their authors alone among the *Hindūs*, has been followed by them in establishing their quadratures of the circle; and a few more verses, which I shall hereafter treat of and explain, will prove, that by the same mode also, the sines, cosines etc. are found with the greatest accuracy.

Whish then presents the verses which describe the Mādhava series for ππ with the three successive end-correction terms, corresponding to increasing levels of accuracy. As we mentioned earlier, Whish again wrongly attributes these verses, and the next fourteen verses that he cites (which give many transformed series for ππ), to *Tantrasaṅgraha* without realising that they are being cited by Śaṅkara Vāriyar as a part of his commentary *Kriyākalāpa*.

Whish then goes on to cite a few verses from *Karaṇapaddhati* and *Sadratnamāla* which give some more series for ππ, and the series (the so-called Gregory Series) for the arc-tan. Before going on to a discussion of the authorship and the period of these works, Whish declares rather emphatically [9][p.521]:

First, then, it is a fact that which I have ascertained beyond a doubt, that the invention of infinite series of these forms has originated in Malabar, and is not, even to this day, known to the eastward of the range of *Ghāts*, which divides that country, called in the earliest times *Ceralam*, from the countries of *Madura*, *Coimbatore*, *Mysore*, and those in succession, to that northward of these provinces.

It is not clear on what basis, and under what compulsions, Whish is making this unusually emphatic claim, which has often been repeated in the later studies. For, as may be seen from the passage that was cited from the *Kālasaṅkalita* (with which Whish would have been very familiar), Warren states that he had met an Indian astronomer (either in Madras or Pondicherry) who knew some of these series. Warren also mentions that these series were collected "from various parts of India by Gentlemen". As regards a historically earlier period, it is well known that the sixteenth-century astronomer Sundararāja of Kanchipuram has mentioned, in his commentary on *Vākyakaraṇa*, that he was in correspondence with the great savant, Nīlakaṇṭha, who was kind enough to compose a tract *Sundararājapraśnottara* in answer to his queries. It may also be noted that there are Telugu and Grantha manuscripts of *Karaṇapaddhati*, and a manuscript of a Sanskrit version of the Malayalam text *Yuktibhāṣā* is deposited in GOML,[4] Madras, which has been edited and published by K.V. Sarma in 2004.

Proceeding to a discussion of the date and authorship of the texts, Whish first notes the following regarding *Sadratnamāla* and its author [9][p.521]:

The author of *Sadratnamālah* is Sancara Varma, the younger brother of the present *Rājā* of *Cadattanāda* near *Tellicherry*, a very intelligent man and acute mathematician.[5] This work, which is a complete system of *Hindū* astronomy, is comprehended in two hundred and eleven verses of different measures, and abounds with fluxional forms and series, to be found in no work of foreign or other Indian countries.

Whish then mentions 1733 as the time of Putumana Somayājī, the author of *Karaṇapaddhati*. As regards *Tantrasaṅgraha*, Whish mentions [9][p.522]:

The author of the *Tantra Sangraha*, who was educated in the college of *Brahmanas* at Trichur (which college still exists, and which is a place of reputed sanctity, and certainly of respectable learning), has, among his works on arithmetic and astronomy, laid the foundations for a complete system of fluxions, and opened a mine of wealth in mathematics … The copy of the work, which I have obtained with some difficulty, by frequent intercourse with this interesting society, bears in itself marks of antiquity, the commentary in the vulgar[6] tongue [Malayalam] being written in a language which is not now generally current in Malabar, and the forms of many letters differing materially from those of the present day.

The author of the work is Talaculattūra Nambūtiri, of *Cerala*, … he flourished in the forty-sixth century of the present age or *Caliyuga* … the date which is shown in the commencement of the work itself, namely, the year 4600 of *Caliyuga*[7] …

In the above passage, Whish refers to a college which "still exists, which is a place of reputable sanctity, and certainly of reputable learning". In this context, it may be noted that the report dated August 5, 1823, sent by J. Vaughan, the Collector of Malabar, as a part of a survey of indigenous education ordered by Thomas Munro the then Governor of Madras Presidency, reproduces a note by the Samudrin Rāja of Kozhikode on the College being maintained by him in Tirunāvāya which had 75 scholars studying "Theology & Law". This report of the Collector also mentions that, in addition, there were 808 students pursuing higher studies in "Astronomy" and 194 students studying "Medical Science" under reputed individual scholars [10][pp.199–203].

As regards *Yuktibhāṣā*, Whish mentions that it is a commentary on the *Tantrasaṅgraha* and that its author, Cellalura Nambutiri, wrote another work *Dṛkkaraṇa* in 1608 CE. Though Whish

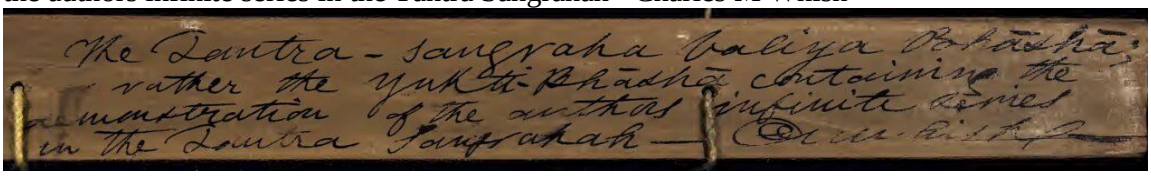
promised at the beginning of the article that he will discuss the series for sine and cosine, his paper seems to end rather abruptly with a promise that they will be taken up in a further publication, along with the demonstrations of all the series as given in the *Yuktibhāṣā* [9][p.523]:

A further account of the *Yucti-Bhāshā*, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper: I shall therefore conclude this, by submitting a simple and curious proof of the 47th proposition [Theorem of Pythagoras] of Euclid, extracted from the *Yucti-Bhāshā*.[8]

Unfortunately, Whish passed away in 1833, without fulfilling the above promise of presenting the demonstrations of the various series as given in *Yuktibhāṣā*. Around this period, there seems to have been some personal or other problem due to which Whish was "out of service" in 1831, after being transferred to Cuddapah in 1830. He rejoined as Acting Judge in Cuddapah in 1832, but passed away soon on April 14, 1833, without seeing his article come out in print.

It appears that Whish was clear from the various responses he received from around 1825, that his European peers and superiors would not accept any number of quotations from the Indian śāstras describing different infinite series as satisfactory evidence that the Indians had discovered these series, unless he could also show that some of the Indian texts also presented valid demonstrations for these series. And he was well aware of the proofs contained in the Malayalam text, *Yuktibhāṣā*, of which he had two readable palm-leaf manuscripts which are preserved in the Whish collection at the Royal Asiatic Society, London–-Manuscript Nos.124 and 138. In order to highlight the importance of this work, *Yuktibhāṣā*, and perhaps with the premonition that he might not be around to explain its significance, Whish inscribed a note on the verso of the first blank palm-leaf of Manuscript No. 124, which is reproduced below:

"The Tantra-Sangraha Valiya Bhāshā rather the Yukti-Bhāshā containing the demonstration of the authors infinite series in the Tantra Sangrahah–-Charles M Whish"


The Tantra - sangraha Valiya Bhāshā;
rather the Yukti-Bhāshā containing the
demonstration of the authors infinite series
in the Tantra Sangrahah —

Royal Asiatic Society of Great Britain and Ireland

This important note of Whish seems to have been missed (or overlooked) by all the scholars who made any list or catalogue of the Whish Collection of manuscripts at the Royal Asiatic Society in London in the 19th as well as the 20th century. We shall return to this issue later in this article.

David Pingree's explanation as to why Whish's article was ignored

In a lecture delivered at the Department of History of Science of the Harvard University in 1990, which was later published in *Isis* in 1992, the renowned American historian of science, David Pingree, argued that Mādhava's achievements in demonstrating various infinite series, though very notable, should not be confused with any "discovery of the calculus" as they were based on "geometrical and algebraic arguments". He also offered an explanation as to why Whish's article was totally ignored by modern Western scholarship for over a century,

"presumably at first because they could not admit that an Indian discovered the calculus, but later because no one read anymore the *Transactions of the Royal Asiatic Society*, in which Whish's article was published."[11][p.562]:

… Hellenophilia… This results in the attitude that it is the task of the historian not to study the whole of a science within its cultural context, but to attempt to discover within the science elements similar to elements of modern Western science. One example I can give you relates to the Indian Mādhava's demonstration, in about 1400 A.D., of the infinite power series of trigonometrical functions using geometrical and algebraic arguments. When this was first described in English by Charles Whish, in the 1830s, it was heralded as the Indians' discovery of the calculus. This claim and Mādhava's achievements were ignored by Western historians, presumably at first because they could not admit that an Indian discovered the calculus, but later because no one read anymore the *Transactions of the Royal Asiatic Society*, in which Whish's article was published. The matter resurfaced in the 1950s, and now we have the Sanskrit texts properly edited, and we understand the clever way that Mādhava derived the series *without* the calculus; but many historians still find it impossible to conceive of the problem and its solution in terms of anything other than the calculus and proclaim that the calculus is what Mādhava found. In this case, the elegance and brilliance of Mādhava's mathematics are being distorted as they are buried under the current mathematical solution to a problem to which he discovered an alternate and powerful solution.

As we shall discuss below, contrary to the claims made by Pingree, Whish's paper was, in fact, widely noted and referred to during the decade (1832–1842) after the paper was read in 1832. It was such wide notice, and the fact that the *Transactions of the Royal Asiatic Society* was regarded as a highly reputed journal on such matters, which seems to have alarmed the leading lights of the British academic establishment such as Augustus De Morgan, and scholar administrators such as Charles P. Brown, so much that they thought it necessary to publicly denounce the Kerala work on infinite series as a "clever hoax". This seems to have led to a veritable consensus among the European Indologists and historians of mathematics to completely ignore the work of the Kerala school for the next hundred years or so. The important manuscripts of Malayalam works (such as the *Yuktibhāṣā*), collected by Whish and deposited in the library of the Royal Asiatic Society since 1836, were not even listed or catalogued; and reputed Indologists such as Albrecht Weber, George Thibaut and renowned historians of mathematics such as David Smith chose not to discuss the work of the Kerala school even while referring to the articles of Whish in their scholarly writings.

Initial response to Whish's papers (1832–1842)

As we noted earlier, Whish's paper "On the Hindu Quadrature of the Circle", was read at the meeting of the Royal Asiatic Society held in London on December 15, 1832. The proceedings of the meeting are described as follows in the January 1833 issue of the *Asiatic Journal and Monthly Register* [12]:

The general meeting was held on the 15th of December, at the usual hour; Andrew Macklew, Esq., member of council in the chair. … A paper by C. M. Whish, Esq., of the Madras civil service, communicated by the Madras Auxiliary Asiatic Society, and entitled "On the Hindu Quadrature of the Circle," was read. The materials for this paper are principally drawn from four *Sāstras*, entitled *Tantra Sangraha*, *Yuktabhāshā* (a commentary on the first), *Karana Paddhatih* and *Sadratnamālah*. Mr. Whish, in the first place, quotes several rules for finding the proportion of a circumference to a given diameter, extracted from the works of Aryabhata,

… and other authorities, which exhibit approximations so wonderfully correct, that European mathematicians who seek for such proportion in the doctrine of fluxions, or the more tedious continual bisection of an arc, must admire the means by which the Hindus have been able to extend the proportion to so great a length. He then proceeds to show that a system of fluxions peculiar to these authors alone among the Hindus has been followed by them in establishing their quadratures of the circle, and by the same method the sines, cosines etc. are found with the greatest accuracy. This position is illustrated by eight different infinite series, taken from the *Sāstras* above-named, the date of the first of which he proves to be A.D. 1608.[9] Mr. Whish concludes this essay with a simple and curious proof of the forty-seventh proposition of the first book of Euclid, extracted from the *Yuktabhāshā*, (the second *Sāstra* above mentioned), and which, he observes, was probably the form by which Pythagoras discovered this celebrated problem. The thanks of the Society were ordered to be returned to Mr. Whish for this communication, and the meeting adjourned to the 5th of January 1833.

Similar summaries of the paper of Whish were carried also in various other journals in 1832/33 [13]. It may be noted that all this reporting was done even prior to the publication of the article in 1834. Incidentally, Whish's 1827 article "On the Alphabetical Notation of the Hindus", was translated into French and published in the *Journal Asiatique* in 1835 [14].

One of the earliest references to the 1834 article of Whish was made by the leading German Indologist, Christian Lassen, in a paper on the systems of numerations used in Indian mathematics in 1837 [15]. An important bibliography of Sanskrit literature compiled by the German scholar Friedrich Adelung and published in 1837 [16], listed Warren's *Kālasan˙n˙kalita*, Whish's article of 1834 and the four major Kerala works (śāstras) mentioned therein. More importantly, Whish's article was cited in a volume published in 1841 on the development of calculus in Europe in the 17th century. Stephen Peter Rigaud and Stephen Jordan Rigaud in their *Correspondence of Scientific men of Seventeenth Century* [17] referred to the fact that a certain approximation to $\pi\pi$ mentioned in a letter of John Wallis to John Collins, dated September 1, 1767, was the same as the one attributed to *Tantrasan˙n˙graha* in Whish's article [17][pp.590–591]:

In the third volume of the Transactions of the Royal Asiatic Society of Great Britain and Ireland (1834), there is a very curious paper of Mr. Whish on the Hindu quadrature of the circle: and at p.511 he quotes a passage from Tantra Sangraha which gives the ratio of 33215 to 104348 for that of the diameter to the circumference of a circle. He does not mention Wallis's other numbers, which however are not quite so close.

Augustus De Morgan dismisses Kerala work as a "clever hoax" (1843)

The kind of response that the paper of Whish–-and the Kerala work on infinite series that was discussed therein–-seems to have received, both in the Indological and mathematical circles, does seem to have seriously disturbed the high priests of British academia. Indeed their response was not much different from what was first pronounced (though in private correspondence) by George Hyne in 1825, namely that these results must have been recently received by the Indians from certain European sources. Now, the same pronouncement is made publicly by one of Britain's leading mathematicians, Augustus De Morgan (1806–71),[10] in a review article on Indian Astronomy and Mathematics published in the widely read *Penny Cyclopaedia* (brought out by The Society for the Diffusion of Useful Knowledge) in 1843 [18]. In this review (which carries the strange title "Viga Ganita", purportedly referring to Indian algebra, *Bījagaṇita*), De Morgan declares at the outset that [18][pp.318–319]:

We have referred to this article all matters which relate to the astronomical and arithmetical science of the Hindus, partly because there is not enough to be said on the subject to make it worthwhile to distribute it under heads in a work like the present, and partly because it was desirable to defer the article in question as long as possible, in the hope that some further investigation of the points on which we are to write might make its appearance. … We cannot pretend to the knowledge of Oriental matters which is necessary to form the most positive judgement…

De Morgan devotes nearly five pages of his eight-page article to the issues surrounding the antiquity and development of Indian astronomy by reviewing the works of Guillaume Le Gentil, Jean-Sylvain Bailly, John Playfair, and the debate between Henry Colebrooke and John Bentley, and concludes that, "The astronomy of Hindus would have had little interest, but for their arithmetic and algebra" [18][p.323]. De Morgan then gives a cursory outline of the *Līlāvatī* and *Bījagaṇita* of Bhāskara and makes a brief mention of Indian trigonometry following Delambre. In his concluding section, De Morgan states [18][p.325]:

The modern state of science in India is generally reported, from Le Gentil downwards, to be at a low ebb… In this, the assertion of Le Gentil, Davis and Bentley, uncontradicted by Colebrooke, we should have been disposed implicitly to rely upon, had it not been for a paper published in our own day (*Mem. Asiatic Soc.*, vol. iii) by Mr. Whish of the Madras Civil Establishment, `On the Hindu Quadrature of the Circle.' If anyone whose name was not known to us had simply published this paper on his own authority, we should have had some suspicion that it was neither more nor less than a clever hoax; but it is published, with the Sanscrit at full length, in the Memoirs of the Society which is best able to judge of its contents.

Augustus De Morgan Wikimedia Commons

De Morgan is clear that but for the fact that the paper of Whish has appeared in the "Memoirs of the Society which is best able to judge of its contents", he would have right away dismissed the paper as a "clever hoax". De Morgan goes on to provide a brief introduction to the three texts *Tantrasan˙n˙graha* (which he wrongly dates to 1608), *Karaṇapaddhati* and *Sadratnamāla*, and then presents the various series for $\pi\pi$ as given in the paper of Whish. He, however, makes no mention of the *Yuktibhāṣā*, or the claim of Whish that the demonstrations of the series for $\pi\pi$, sine and cosine are presented in that text. Concluding his discussion on Whish's paper, De Morgan declares that it is "a staggering assertion that works written in 1608 and 1733 give series which not only were not known in Europe at those periods, but must have been discovered by something answering to the integral calculus". The fact that, according to Whish, these series are not known outside Malabar "even to this day", makes it very likely that "they were introduced from Europe at a comparatively recent period". He, therefore, suggests that several of these series must have been communicated to the Indians by the French

astronomer Le Gentil, who was in "the southern parts of the Deccan in 1769", and that the Catholic missionaries "may have given the rest" [18][pp. 325–326]:

… it is rather a staggering assertion that works written in 1608 and 1733 give series which not only were not known in Europe at those periods, but must have been discovered by something answering to the integral calculus. On this point we must suspend our opinion till these works are published, with such an account of them as may command belief in their being the genuine productions of the time at which they profess to have been written: but if we grant them existence at all, they are enough to show that there has been, in Malabar at least, knowledge enough to enable the possessors to avail themselves of the results of modern European mathematics. Mr. Whish asserts that the invention of these series originated in Malabar, and that they are not `even to this day known to the eastwards of the range of Ghauts which divides that country from the countries of Madura, Coimbatore, and Mysore'. This by itself seems to us to favour the presumption that they were introduced from Europe at a comparatively recent period. We rather suspect that Le Gentil who was in the southern parts of the Deccan in 1769, and who used every means to ingratiate himself that he might procure a knowledge of the native astronomical processes, in which, much against the will of the Brahmins, he succeeded by the aid of a Tamul Christian–-may have been the communicator of several of these series, and that Catholic missionaries may have given the rest.

There does not seem to be any further debate on the paper of Whish, or on the Kerala work on infinite series, after this devastating review in the *Penny Cyclopaedia*. Perhaps that was what was intended by De Morgan in the first place, since, without any ado, he removed all references to the paper of Whish when his article on "Viga Ganita" was reprinted fifteen years later in the *English Cyclopaedia* in 1858 [19]. Incidentally, the comprehensive bibliography of secondary sources on Indian astronomy and mathematics in the volumes of *Census of Exact Sciences in Sanskrit* compiled by David Pingree does not include any reference to either of these articles on Viga Ganita. De Morgan is better known to the historians of Indian mathematics through his patronage of the work of Ramchundra which he got republished in London in 1859 along with a glowing preface [20][pp. iii–xviii].

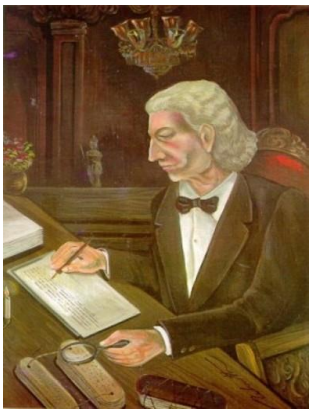
Charles Brown castigates Whish's paper as a "Fraudulent Document" (1863)

The British scholar administrator in South India, Charles Phillip Brown (1798–1884),[11] wrote a monograph on *Carnatic Chronology* in 1863, which has a section "On Some Fraudulent Documents". Brown refers to the article on Viga Ganita in the *Penny Encyclopaedia*, which "surmised" that the verses cited by Whish seemed to be "forgeries". Brown, in fact, claims to have identified the Brahmin who had composed these verses attributed to *Tantrasan˙n˙graha* in the article of Whish. The following is an extract from Brown's diatribe against the paper of Whish, and even the text *Tantrasan˙n˙graha* itself [21]:

Some papers printed in the Proceedings of the Royal Asiatic Society, by Mr. C.M. Whish, (who died in the year 1833) evince his skill in Mathematical investigation and in Sanscrit scholarship. The article *Viga Ganita*, in the Penny Cyclopaedia, and pages 325, 326 of the same volume, having mentioned Mr. Whish's discoveries, with a surmise that they may be forgeries, I offer the following description of their origin.

I have reason to believe that Mr. Whish himself was the author. This belief is founded on the information given to me by Visvambhara Sastri, one of the tutors at Madras College.

This learned brahman had been intimately acquainted with Mr. Whish, and assured me in the year 1840, that the Sanscrit stanzas conveying mathematical rules were constructed under Mr. Whish's guidance in A.D. 1820, by Ayya Sastri, a professor of Sanscrit law, who, after being long employed in the College, became a Law Officer in the Court of Sudr Udalut. Ayya Sastri was a free thinker and derided all religions. From time to time, while framing these verses, he shewed them to my informant, who laughs at the credulity of the English in believing them ancient.



Charles Phillip Brown (1798-1884) Wikimedia Commons (Telugu)

That the verses cited by Mr. Whish are apocryphal may easily be shewn. In the third volume of the Transactions, p. 509, we have a rule regarding the proportion that the diameter bears to the circumference of a circle. But this is quoted "from the Tantra Sangraham". Now this name denotes a "Summary of Magic." In the East India House, in the Madras College, and in Mr. Whish's own collection of books at the Royal Asiatic Society's Library, there are many books on the Tantras: but no one of them is named the Tantra Sangraham. Nor does this name occur in the Tantra Rajam, in the commentary on the Sarada Tilacam, or other similar treatises, which quote Tantrica authorities. …

In the Kala Sancalita, page 93, is a note by Mr. Hyne, who states that Mr. Whish ultimately replied that "he had reasons to believe the rules discovered to be modern: observing that not one who used the rules could demonstrate them."

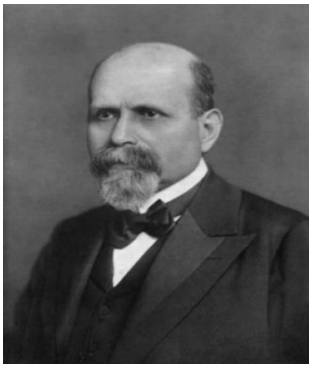
Mr. Whish had a valuable collection of Sanscrit manuscripts: which was presented by his brother to the Royal Asiatic Society's Library in London. Being in the Tamil, Malayalam and Telugu character they probably will be useless in England. They should be sent to the Government Library at Madras.

The Malayalam manuscripts in the Whish Collection at the Royal Asiatic Society

As we noted earlier, J.C. Whish, the brother of Charles Whish donated his entire collection comprising nearly 300 valuable manuscripts in Sanskrit and Malayalam to the Royal Asiatic Society, London, at its meeting on July 2, 1836. The proceedings of the meeting include the following account of this donation [22]:

From Mr. J.C. Whish, Esq. A large quantity of MSS. on palm leaves, principally in Sanskrit, written in the Malayalam character, and comprising the Vedas, and other religious,

philosophical, historical, and miscellaneous works of the Hindūs; also a quantity of similar MSS. on paper. The whole being the collection of his brother, Mr. C.M. Whish, of the Madras Civil Service, deceased.



Moriz Winternitz (1863 – 1937)

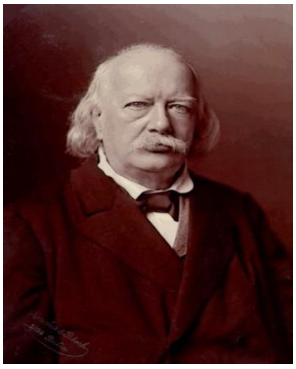
As we can see, the proceedings do not mention anything about manuscripts relating to astronomy or mathematics, which were perhaps included among the miscellaneous works. In any case, since scholars like De Morgan had declared that "we must suspend our opinion [on Whish's paper] till these works are published", it was to be hoped that some of the high scholars of Indology or history of mathematics in England would take the initiative of getting these manuscripts catalogued and studied. Nothing of that sort happened, and in 1863 we see Charles Brown demanding that these manuscripts may be sent to Madras, since "being in the Tamil, Malayalam and Telugu character, they will probably be useless in England".

In the meeting of the Royal Asiatic Society held on December 16, 1867, Reinhold Rost, the then Secretary of the Society, gave an account of the Whish collection of Sanskrit manuscripts. We present the following extract from a report of the meeting [23]:

Dr. Rost, the Secretary, gave an account of the Whish Collection of Sanskrit MSS. These manuscripts were formerly in the possession of Mr. C.M. Whish, of the Madras Civil Service, author of several valuable articles on Hindū astronomy, and on his death, in 1836, were presented to the Royal Asiatic Society by his brother, Mr. J.C. Whish, lately deceased. They are about 300 in number, three-fourths of which are written on palm leaves, the others on paper. With regard to the writing, nearly two hundred are in Grantham, and the remainder, including all the paper manuscripts, in the Malayalam character. One only of the MSS. is in Telugu characters, and none in ancient or modern Canarese: the collection may so far then be said to represent the Sanskrit literature of the Tamil and Malabar Brahmins only; while, as to the subjects, astronomical works preponderate to a large extent. However, the other branches of Hindu literature appear to have been by no means neglected, the collection embracing some rare gems in almost every department. Pending the preparation of a descriptive catalogue of the MSS., Dr. Rost desired to draw attention to the following works. In Vedic literature: the Ṛgveda, text of ashṭakas, 5 to 8, and portions of the 1st and 2nd ashṭakas with Sāyaṇa's commentary … In Mathematics and Astronomy: part of the Gārga-siddhānta; the Sūryasiddhānta, with Parameśvara's commentary; Bṛihaspati's Muhūrtavidhāna; the Tantrasaṅṅgraha; the Laghu-Ārya-siddhānta of the elder Āryabhaṭa (three copies, one with Parameśvara's commentary, the Bhaṭadīpikā).

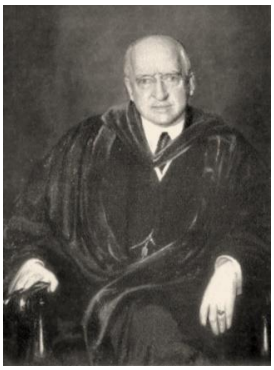
In the above summary, Rost refers to only the Sanskrit manuscripts collected by Whish, whom he refers to as the "author of several valuable articles on Hindu astronomy". As regards the works of the Kerala school, only two commentaries of Parameśvara and the *Tantrasaṅgraha* are mentioned, with no further details regarding the commentaries etc., of the latter.

In 1890, the Royal Asiatic Society published a list of titles of the Sanskrit manuscripts in the Whish collection [24]. Finally, in 1902, the Society published a descriptive catalogue of the Sanskrit Manuscripts of the Whish Collection prepared by the famous Austrian Indologist, Moriz Winternitz [25]. In his preface to the catalogue, Winternitz refers to the rare and valuable manuscripts in this collection pertaining to the Vedas, Mahabharata etc., but makes no mention of astronomy or mathematics. The catalogue later lists 400 works distributed among 215 manuscripts–-of which about 25 works are listed under astronomy and astrology. Apart from the two commentaries of Parameśvara, the only work of the Kerala school listed is "*Kriyākalāpa* of Tantrasaṅgraha" (Manuscript No. 134, which actually contains the Sanskrit commentary *Laghuvivṛti* of Śaṅkara on *Tantrasaṅgraha*)–- which is listed both under "Astronomy and Astrology" as well as "Tantra".



Albrecht Weber (1825-1901) Wikimedia Commons

Indeed, there is no reference to *Yuktibhāṣā* in the entire catalogue of Winternitz. Since many of the Sanskrit manuscripts were also in the Malayalam script, it would have taken little additional effort to catalogue the Malayalam works in the Whish collection. From the current digitized archives of the Royal Asiatic Society today, we see that there are only 13 manuscripts of Malayalam works in the entire Whish collection [26]. Had these also been included in the descriptive catalogue, then the world of modern scholarship would have come to know the details of the two palm-leaf manuscripts of *Yuktibhāṣā*, referred to earlier, including the important note inscribed by Whish in the Manuscript No.124, that it is a manuscript of "Yukti-Bhāshā containing the demonstration of the … infinite series in the Tantra Sangrahah". We can see no reason for omitting the Malayalam works from the descriptive catalogue of 1902, except for the extreme reluctance that prevailed among the European Indologists to rake up the issues raised in Whish's paper on the Kerala work on infinite series and their demonstrations which were supposed to be contained in the Malayalam work *Yuktibhāṣā*.



David Eugene Smith (1860-1944) Wikimedia Commons

During 1904–1920, Winternitz published what was reputed to be the most comprehensive history of Indian literature, in three volumes. The second part of the third volume devotes about 200 pages to "The Scientific Literature" (Die Wissenschaftliche Litteratur), of which about 23 pages are devoted to astronomy, astrology and mathematics [27][pp.555–577]. However, Winternitz makes no reference to the paper of Whish or to any of the Kerala works on mathematics or astronomy–-even those that were included in his catalogue–-in his survey of the literature on mathematics and astronomy.

Suppression of all reference to Kerala work on infinite series (1850–1925)

It appears that by the 1850s there had emerged a strong consensus among the European Indologists and historians of mathematics and astronomy, not to make any reference to the Kerala work on the infinite series for $\pi\pi$ and the sine and cosine functions, even while they made a reference to the papers of Whish for one reason or the other. The articles of Whish were indeed well known to the prominent European Indologists, as may be seen from the fact that in 1863, Albrecht Weber cited the 1827 article of Whish (on the Indian systems of numeration) in order to explain the Kaṭapayādi system of numeration in his review of a new edition of the Rāmāyaṇa published from Bombay [28]. Both the 1827 and 1834 articles of Whish were again brought to the notice of the Indological community in 1865 by the well known Indian scholar, Bhau Daji, in his article on the date and works of Āryabhaṭa and others, which was published in the *Journal of the Royal Asiatic Society* [29].

During 1860–1899, the other 1827 article of Whish, on the Greek origin of the Hindu Zodiac, was frequently cited in several studies of Indian astronomy, such as the popular translation of the *Sūryasiddhānta* by Ebenezer Burgess and William Whitney [30], the review article by John Burgess of the European studies of Indian astronomy in the 18th and 19th centuries [31][pp.746–750], and the scholarly monograph of George Thibaut (in German) on Indian Astronomy, Astrology and Mathematics [32]. None of these works makes any mention of the other two articles of Whish which discuss the Kerala work on the infinite series for $\pi\pi$–-notwithstanding the fact that these issues were germane to the subject of their discussions.

Even more intriguing is the fact that the renowned historian of mathematics, David Eugene Smith cites the 1834 article of Whish, without giving any account of its contents, in a famous article of his in 1913. Smith again refers to the article of Whish in his seminal two-volume history of mathematics completed in 1925. The 1913 article is on the geometry of the Hindus, and it is published in the very first issue of the *Isis* [33]. This article contains some discussion of the work of Āryabhaṭa, Brahmagupta, Mahāvīra and Bhāskara, who is referred to as the "last

of the well-known geometers of India" [33][p.198]. Among the eleven references given at the end of the paper, we see the article of Whish "On the Hindu Quadrature of the Circle". However, there is no citation made to the article in the entire text.

Bhau Daji (1824–1874) Wikimedia Commons

The story becomes even more curious when we come to the second volume of Smith's *History of Mathematics* published in 1925 [34]. In the chapter on geometry, there is a small section on the squaring of the circle, which includes a discussion of the "Hindu values of ππ". Here, Smith mentions the approximate values given by Āryabhaṭa, Brahmagupta and Bhāskara. Again, there is a reference to Whish's article in a footnote, which is somewhat more explicit than the citation made in the 1913 paper, and states [34][p.309]:
On the general subject of the Hindu quadratures see C.M. Whish, *On the Hindu Quadrature of the Circle*, a paper read before the Madras Literary Society, December 15, 1832.

However, there is no discussion of the contents of Whish's paper or any mention of the Kerala work on infinite series for ππ in this comprehensive work on the history of mathematics by David Smith. This is all the more surprising because, as is well-known, Smith was the co-author with Louis Karpinski of the famous book on *The Hindu-Arabic Numerals* published in 1911. It seems that Smith is shying away from entering into any discussion on the issue of the work of the Kerala school. However, by prominently citing the 1834 article of Whish in his 1913 paper and the 1925 book, Smith was perhaps giving a lead to the Indian scholars to take up this issue for further investigation.

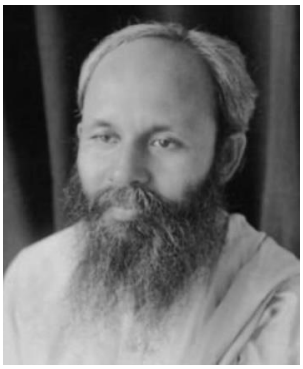
The resurrection of the work of the Kerala School (1926–1952)

Datta and Singh initiate the modern Indian scholarly study (1926–1936)

The reference to Whish's article, made in Smith's *History of Mathematics*, was promptly noted by Bibhutibhushan Datta (1888–1958), who had embarked on an intensive study of the source-works of Indian mathematics soon after completing his Ph.D. in applied mathematics from the Calcutta University in the early 1920s. In his 1926 paper "On the Hindu Values of ππ", Datta refers to Smith's book, and goes on to present a much more comprehensive account of the different approximations to ππ found in the Indian mathematical and astronomical works [35]. The final section of Datta's paper was on "Values in Series", where he states [35][p.41]:

The Hindus also obtained some interesting series as values of ππ. For these, and in fact for all the results taken from *Tantra-san˙n˙graha, Karaṇa-paddhati* and *Sadratnamāla*, I am indebted

to Whish's paper, "*On the Hindu Quadrature of the Circle*". These books are still in MSS. And I have had no access to them.



Bibhutibhushan Datta (1888-1958) Wikimedia Commons

Datta gives a summary account of all the series mentioned in Whish's paper, for the first time, after a lapse of over eighty years from the time when they were last mentioned in the 1843 article of De Morgan.

As is well known, the monumental work of Datta and his collaborator Avadhesh Narayan Singh (1901–1954) on the *History of Hindu Mathematics* was published in two parts in 1935 and 1938. In the preface to the first volume, the authors mention that there would be a third part, which would contain "the history of geometry, trigonometry, calculus and various other topics such as magic squares, theory of series and permutations and combinations" [36]. The section on series of this proposed third volume was later edited and published by Kripa Shankar Shukla in 1993 [37].

Avadesh Narayan Singh (1901-1954) **coutesy:** R.C. Gupta

While Datta had mentioned that he had no access to the manuscripts of the Kerala works in his 1926 article, several of these manuscripts seem to have been collected by him and Singh while working on their *History*. In 1936, Singh published an article on the use of series in India in the first issue of *Osiris*, where he declares at the very outset that [38][p.607]:
Evidence of the use of infinite series in India is found in the fifteenth century. The infinite series appear to have been obtained in connection with the evaluation of the trigonometrical functions and the computation of $\pi\pi$. Noteworthy achievements of the Hindu mathematicians in this theory are the expansions of SinxSinx, CosxCosx, and Tan−1xTan−1x in powers of xx, obtained long before they were known in Europe or anywhere else.

Singh discusses the arc-tangent series as given in the paper of Whish. He presents the reference to these verses as found in the manuscripts of *Karaṇapaddhati* and *Sadratnamāla*. He also gives, for the first time, a translation of the verses from *Karaṇapaddhati* which give the series for the sine and cosine functions, thereby fulfilling a part of the promise made by Whish over a hundred years ago in 1832. As we shall see below, the work *Karaṇapaddhati* was edited and published by K. Sambasiva Sastri in 1937.

The seminal work of C.T. Rajagopal and co-workers (1940–1952)

C.T. Rajagopal (1903-1978) **courtesy:** Indian National Science Academy

It was during the decade of 1940–1950 that the rest of the promises made in the article of Whish got fulfilled. This was mainly due to the seminal work of Cadambathur Thiruvenkatacharlu Rajagopal (1903–1978) (see [39] for an overview of his work) and his co-workers, who published a series of articles [40],[41],[42],[43],[44],[45],[46],[47],[48],[49] during 1940–1952[12] explaining, in modern mathematical terms, all the important demonstrations for the infinite series for $\pi\pi$, the sine and the cosine functions, as given in the Malayalam text *Yuktibhāṣā*. Mukunda Marar and Rajagopal state right at the beginning of their publication of 19[44], titled "On the Hindu Quadrature of the Circle", that [41][pp.65–66]:

This paper is a sequel to an article bearing the same title contributed more than a hundred years ago to the *Transactions of the Royal Asiatic Society of Great Britain and Ireland*, by Charles M. Whish of the Hon. East India Company's Civil Service in the Madras Establishment.[13] The article of Whish has come to be accepted as one of our chief sources of information concerning Hindu achievements in "circle squaring", but the questions it raises with regard to the date of these achievements have still to be answered. If the date given by Whish is accepted, the first significant Hindu investigations of $\pi\pi$ … must have preceded by some two centuries the efforts of Newton, Leibniz and Gregory in the same direction. Such a conclusion is no doubt provisional until we have established the antiquity as well as the indigenous origin of the four "śāstras" from which Whish draws his material: *Tantrasan˙n˙graha*, *Yukti-Bhāṣā*, *Karaṇa-Paddhati* and *Sadratnamāla*. Nevertheless, the existence of these mathematical works,[14] with a main theme which is supposed to be five hundred years old … must make one wary of any assertion that "after the time of Bhāskara (born A.D. 1114), no Indian mathematical work of historical value or interest is known".[15] …

… Our purpose, in what follows, is not to offer a solution to these difficulties, but bring them vividly to the fore. This we propose to do by filling in the details of a picture which Whish has given in outline–-an undertaking which Whish himself contemplated in a sequel to his essay. After enunciating the rules, embodied in Sanskrit ślokas, for expressing $\pi\pi$ in infinite series … he planned to publish the demonstrations of the rules as given in *Yukti-Bhāṣā* … But he died

before he could complete the task that he had set himself … In the hundred years from Whish's death until now, there would appear to have been but a single attempt to fill the gap viz., Mukunda Marar's essay, "Derivation of the value of ππ–-How the ancients did it", in a recent number of the *Journal of the Cochin Teachers' Association*.[16]

The first section of our paper gives the geometrical proof, substantially as it stands in *Yukti-Bhāṣā*, of Gregory's series for the inverse tangent which is the origin of all the series for ππ in Whish's article. … Our second section is devoted exclusively to Gregory's series for the ππ and the method employed by the Hindu mathematicians, as it is set forth in the *Yukti-Bhāṣā*, to convert this alternating series into others more rapidly convergent and consequently more useful in evaluating ππ.

T.A. Sarasvati Amma (1918 – 2000) **courtesy:** R.C. Gupta

As has been noted in the above article of Marar and Rajagopal, Prince Rama Varma Tampuran seems to have played a crucial role in supporting the work of Rajagopal and co-workers, perhaps by providing the manuscripts of the Kerala works and by helping in their translation etc. Soon, Rama Varma Tampuran, in collaboration with Akhileswaraiyer, came up with a pioneering edition, along with detailed notes in Malayalam, of the mathematics part of *Yuktibhāṣā* in 1948.

When the contents of all these important publications were taken note of in the 1950s and 1960s, by well-known historians of mathematics such as J.E. Hoffmann (1953) [50], C.N. Srinivasiengar (1958, 1967) [51],[52], A.P. Yushkevich (1961, 1964) [53], D.T. Whiteside (1961, 1968) [54],[55], H.L.L. Busard (1962) [56], T.A. Sarasvati Amma (1963, 1979) [57],[58], A.K. Bag (1966, 1979) [59],[60] and M.E. Baron (1969) [61], one could clearly see the emergence of a really new chapter in the historiography of Indian Mathematics.[17]

Publication of the source-works of the Kerala school (1874–1952)

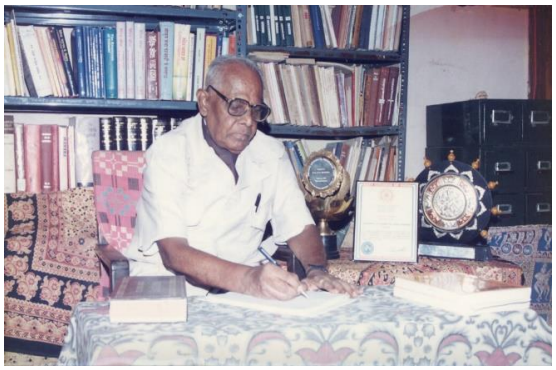
The first source-work of the Kerala school to be published was the *Bhaṭadīpikā* commentary of Parameśvara on the *Āryabhaṭīya* in 1874. This also happened to be the first edition of the *Āryabhaṭīya*, and was carried out by Hendrik Kern, based on a manuscript in the Whish collection and another one from Arthur Burnell. In 1898, the Kavanodayam Press in Calicut published the *Sadratnamālā* (c.1819) of Śan˙n˙karavarman (1774–1839) along with the author's auto-commentary. However, this pioneering edition does not seem to have led to any study or discussion of the contents of this seminal work. The fourth chapter of this work, titled *jyācāpādiprakaraṇam*, presents a detailed discussion of the relation between the circumference and diameter of a circle, as well as the relation between arcs and their Rsines.

In all, fifteen source-works of the Kerala school were published during 1874–1952. These included the *Gaṇitapāda* and *Kālakriyāpāda* portions of the *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, and the *Karaṇapaddhati* of Putumana Somayājī, which were all edited by K. Sambasiva Sastri during 1930s. As we noted earlier, the most important work published during 1874–1952 was the pioneering edition of the Gaṇita section of the *Yuktibhāṣā*, which was published along with detailed notes in Malayalam by Rama Varma Tampuran and Akhileswaraiyer in 1948.

The following is the list of the fifteen Kerala source-works published during 1874–1952:

1. *Āryabhaṭīya* with *Bhaṭadīpikā* of Parameśvara, Ed. by H. Kern, E. J. Brill, Leiden, 1874.
2. *Sadratnamāla* of Śaṅkaravarman with Auto-commentary, Kavanodayam Press, Kozhikode, 1898.
3. *Goladīpikā* of Parameśvara, Ed. by T. Ganapati Sastri, Trivandrum Sanskrit Series 49, Trivandrum, 1916.
4. *Prakīrṇasaṅgraha* of Putumana Somayājī, included in *Pañcabodhaṃ-Bhāṣā* by Kanippayyoor Sankaran Nambutirippad, Kunnamkulam, 1927.
5. *Pañcabodha* of Putumana Somayājī, included in *Jytotiṣaśāstrasubodhinī*, Vol. II, by Punnasseri Nilakantha Sarma, Trissur, 1929.
6. *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somayājī, Vol. I, *Gaṇitapāda*, Ed. by K. Sambasiva Sastri, Trivandrum Sanskrit Series 13, Trivandrum, 1930.
7. *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somayājī, Vol. II, *Kālakriyāpāda*, Ed. by K. Sambasiva Sastri, Trivandrum Sanskrit Series 22, Trivandrum, 1931.
8. *Karaṇapaddhati* of Putumana Somayājī, Ed. by K. Sambasiva Sastri, Trivandrum Sanskrit Series 126, Trivandrum, 1937.
9. *Mahābhāskarīya* with *Dīpikā* of Parameśvara, Ed. by B.D. Apte, Anandasram Sanskrit Series 126, Pune, 1945.
10. *Laghubhāskarīya* with *Vyākhyā* of Parameśvara, Ed. by B.D. Apte, Anandasram Sanskrit Series 128, Pune, 1946.
11. *Candravākya* of Vararuci, Ed. by C. Kunhan Raja, Adyar Library, Chennai, 1948.
12. *Haricarita* of Parameśvara, Ed. by V. Krishnamacharya, Adyar Library, Chennai, 1948.
13. *Yuktibhāṣā* of Jyeṣṭhadeva, the *Gaṇita* Part Ed. with Detailed Notes in Malayalam by Rama Varma Tampuran and A.R. Akhileswaraiyer, Mangalodayam Press, Trissur, 1948.
14. *Laghubhāskarīya* with *Vivaraṇa* of Śaṅkarānārāyaṇa, Ed. by P.K. Narayana Pillai, Trivandrum Sanskrit Series 162, Trivandrum, 1949.
15. *Laghumānasa* of Muñjāla with *Vyākhyā* of Parameśvara, Ed. by B.D. Apte, Anandasram Sanskrit Series 126, Pune, 1952.

In spite of the publication of these 15 major source-works during the period 1874–1952, and the landmark studies of Datta and Singh and Rajagopal and co-workers, there still remained huge gaps in our understanding of the history and achievements of the Kerala school. There was little clarity about the period and works of most of the illustrious astronomers of this school, starting from the legendary Mādhava himself. The authorship of even a seminal text such as the *Yuktibhāṣā* was not known. A large number of texts of Mādhava, Parameśvara, Nīlakaṇṭha, Śaṅkara and Acyuta were yet to be located and published. While there was some clarity on the contributions of the Kerala school in developing the infinite series and approximations for $\pi\pi$ and the sine and cosine functions, there was no awareness of the great contributions of this school in the field of astronomy.



K.V. Sarma, at home, at work **courtesy:** M.D. Srinivas

The Monumental Contribution of K.V. Sarma (1943–2005)

Krishna Venkateswara Sarma (1919–2005) was born in Chengannur, presently in the Alappuzha District of Kerala. After completing a BSc, he obtained a Master's degree in Sanskrit in 1942 from the Maharaja College of Arts, Thiruvananthapuram. He worked in the Manuscripts section of the Kerala University Oriental Research Institute during 1943–1951, and with the New Catalogus Catalogorum Project and the Department of Sanskrit in the University of Madras during 1951–1962. During 1962–1983, Sarma was associated with the Vishvesvaranand Institute of Sanskrit and Indological Studies at Hoshiarpur, affiliated with the Panjab University. Returning to Chennai, he founded the Sree Sarada Education Society and Research Centre, which has become the Prof. K.V Sarma Research Foundation after Sarma's demise in 2005. During his very active and prolific research career spanning over sixty years, K.V. Sarma authored a large number of books and articles on various aspects of Indian Studies (see [64] for more details). Here, we shall focus on his monumental contribution in the field of astronomy and mathematics, especially the work of the Kerala school.

A comprehensive survey of the history of the Kerala school

As we noted earlier, notwithstanding the publication of about 15 major source-works during the period 1874–1952 (that we listed above), there were huge gaps in our understanding of the history and achievements of the Kerala school. This may be seen for instance from the titles of two landmark papers of Sarma which appeared in 1958. The first one is titled "Date of Mādhava: A Little-Known Indian Astronomer" [65]. It summarises some of the important verses and results in mathematics and astronomy attributed to Mādhava in the *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, and the then unpublished commentary *Kriyākramakarī* on *Līlāvatī*. The article determines, for the first time, the period of Mādhava as 1350–1410, by considering the epochs mentioned in his work *Veṇvāroha*, as also the epoch (1431) of the work *Dṛggaṇita* of his student Parameśvara.

The other article is titled "Jyeṣṭhadeva and his Identification as the Author of *Yuktibhāṣā*" [66]. There was still a lot of confusion about the name and identity of the author of *Yuktibhāṣā*, even 125 years after the work was first mentioned by Whish in 1832. Sarma identified Jyeṣṭhadeva as the author of *Yuktibhāṣā* by locating a statement found in Manuscript No. 755 of Kerala University Manuscripts Library, which contained a work called *vikṣepacalanavāsanā*–-which was later published by Sarma as a part of the compilation entitled *Gaṇitayuktayaḥ* in 1979. The statement, "*jyeṣṭhadevo'pi bhāṣāyāṃ nādhikaṃ kiñciduktavān*", clearly showed that Jyeṣṭhadeva was the author of "*bhāṣā*" or *Yuktibhāṣā*. Further confirmation was obtained in a

long passage found in Manuscript No. 9886 in the Oriental Institute Baroda. This is a Malayalam commentary on *Sūryasiddhānta*, which mentions that Jyeṣṭhadeva, like Nīlakaṇṭha, was the pupil of Dāmodara the son of Parameśvara, and that he was also the author of *Yuktibhāṣā*.

Sarma's tireless efforts at compiling a detailed bibliography of the works of the Kerala school resulted in the publication in 1972 of the first ever comprehensive landmark survey of the work of Kerala school, *A History of the Kerala School of Hindu Astronomy* [67]. In this pioneering survey, K.V. Sarma listed around 400 works on mathematics and astronomy. Sarma also noted that of these, there were 211 Karaṇa texts, 34 Tantra texts and 6 Siddhānta texts. More than 150 works happened to be in Malayalam. Thirty years later, in 2002, Sarma published a survey of the *Science Texts in Sanskrit in the Manuscripts Repositories of Kerala and Tamil Nadu*, where he listed over 450 works in Sanskrit on mathematics and astronomy from Kerala [68].

Publication of editions and translations of the source-works (1953–2008)

With his expertise in Malayalam and Sanskrit, and his familiarity with the manuscript collections in Kerala and elsewhere, K.V. Sarma had an uncanny ability to identify important unpublished works. He embarked on the daunting task of editing and publishing them, right from the early 1950s. He brought out an edition and translation of *Rāśigolasphuṭanīti* of Acyuta Piṣāraṭi in 1953. Sarma was able to locate manuscripts of the famous seventh-century text on the Parahita system, the *Grahacāranibandhana* of Haridatta, which he brought out in 1954. In 1955, he edited and translated the *Siddhāntadarpaṇa* of Nīlakaṇṭha Somayājī. The next year 1956, saw the first-ever publication of a work of Mādhava, when Sarma published his celebrated work presenting an improved vākya method for computing the motion of the Moon, *Veṇvāroha*, along with the Malayalam commentary of Acyuta Piṣāraṭi. As an appendix to this edition, Sarma also included an edition of the accurate *Candravākya* of Mādhava.

During the decade 1957–66, Sarma published five works of Mādhava's disciple, Parameśvara, including *Goladīpikā* with the author's auto-commentary, and the famous karaṇa work of the Dṛk System, *Dṛggaṇita*, composed in c.1431. During 1970–1979, Sarma brought out the *Sphuṭacandrāpti* of Mādhava; the seminal works *Golasāra*, *Candracchāyāgaṇita* with auto-commentary, *Siddhāntadarpaṇa-vyākhya* and *Jyotirmīmāṃsā* of Nīlakaṇṭha; and *Sphuṭanirṇayatantra* of Acyuta. In 1979, he published a compilation *Gaṇitayuktayaḥ*, comprising twenty-seven short tracts, presenting yuktis (demonstrations) for various computational procedures in astronomy. Yuktis or demonstrations were also the central concern of two extensive commentaries, *Kriyākramakarī* and *Yuktidīpikā* of Śaṅkara Vāriyar, on *Līlāvatī* and *Tantrasaṅgraha* respectively, which were published by Sarma in 1975 and 1977. These Sanskrit commentaries are in fact based on the seminal Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva. From the 1980s onwards, Sarma had been working on a critical edition and translation of the *Yuktibhāṣā*. He asked K. Ramasubramanian, M.D. Srinivas and M.S. Sriram to prepare detailed explanatory mathematical notes for this work in English. He had the satisfaction to oversee the finalisation of the press copy of the first volume just prior to his demise in 2005. The two-volume work was finally published in 2008. Sarma also published a Sanskrit version of *Yuktibhāṣā*, based on a manuscript in GOML Madras, in 2004.

In all, during the period 1953–2005, Sarma brought out pioneering editions of nineteen source-works of the Kerala school, apart from the compilation *Gaṇitayuktayaḥ* comprising twenty-seven short tracts. Following is the list of the works of the Kerala school published by Sarma:

1. *Rāśigolasphuṭanīti* of Acyuta Piṣāraṭi, Ed. and Tr. by K.V. Sarma, Adyar Library, Madras, 1953. Rev. Ed. with Appendices, Vishvesvaranand Institute, Hoshiarpur, 1977.
2. *Grahacāranibandhana* of Haridatta, Ed. by K.V. Sarma, K.S.R. Institute, Madras, 1954.
3. *Siddhāntadarpaṇa* of Nīlakaṇṭha Somayājī, Ed. and Tr. by K.V. Sarma, Adyar Library, Madras, 1955. Rev. Ed. with the Auto-commentary of Nīlakaṇṭha Somayājī, Vishvesvaranand Institute, Hoshiarpur, 1976.
4. *Veṇvāroha* of Mādhava with the Malayalam Commentary of Acyuta Piṣāraṭi, Ed. by K.V. Sarma, Sanskrit College, Tripunithura, 1956.
5. *Goladīpikā* of Parameśvara, Ed. with Auto-commentary and Tr. by K.V. Sarma, Adyar Library, Madras, 1957.
6. *Grahaṇāṣṭaka* of Parameśvara, Ed. by K.V. Sarma, K.S.R. Institute, Madras, 1959.
7. *Dṛggaṇita* of Parameśvara, Ed. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1963.
8. *Grahaṇamaṇḍana* of Parameśvara, Ed. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1965.
9. *Grahaṇanyāyadīpikā* of Parameśvara, Ed. and Tr. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1966.
10. *Golasāra* of Nīlakaṇṭha Somayājī, Ed. and Tr. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1970.
11. *Sphuṭacandrāpti* of Mādhava, Ed. & Tr. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1973.
12. *Sphuṭanirṇayatantra* of Acyuta Piṣāraṭi, Ed. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1974.
13. *Līlāvatī* with the Commentary *Kriyākramakarī* of Śaṅkara Vāriyar, Ed. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1975.
14. *Candracchāyāgaṇita* of Nīlakaṇṭha Somayājī, Ed. with Auto-commentary by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1976.
15. *Tantrasaṅgraha* with the Commentary *Yuktidīpikā* of Śaṅkara Vāriyar, Ed. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1977.
16. *Jyotirmīmāṃsā* of Nīlakaṇṭha Somayājī, Ed. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1977.
17. *Gaṇitayuktayaḥ* (Collection of Twenty-seven Tracts), Ed. by K.V. Sarma, Vishvesvaranand Institute, Hoshiarpur, 1979.
18. *Sadratnamāla* of Śaṅkaravarman, Ed. by K.V. Sarma, INSA, New Delhi, 2001.
19. *Gaṇitayuktibhāṣā* (Sanskrit Version), Ed. by K.V. Sarma, IIAS, Shimla, 2004.
20. *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva, Ed. & Tr. by K.V. Sarma, with Notes by K. Ramasubramanian, M. S. Sriram and M. D. Srinivas, 2 Vols., Hindustan Book Agency, New Delhi, 2008. Rep. Springer, New York, 2009.

Apart from the above series of publications by Sarma, another twelve works of the Kerala school were published during 1953–2008. Eight of them were published during 1953–58 itself, and these included the very important editions, by Suranad Kunjan Pillai, of the *Golapāda* part of the *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha and the *Tantrasaṅgraha* of Nīlakaṇṭha with the commentary *Laghuvivṛti* of Śaṅkara; the editions of *Karaṇapaddhati*, firstly with detailed notes in Malayalam by P.K. Koru, and with two Malayalam commentaries by S.K. Nayar; the edition of the *Siddhāntadīpikā* super-commentary of Parameśvara on *Mahābhāskarīya* by T.S. Kuppanna Sastri; and, an edition of the entire text of *Yuktibhāṣā* brought out by the GOML, Madras.

The following is the list of the twelve source-works published during 1953–2008, in addition to the ones edited by Sarma during this period:

1. *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva, Ed. by T. Chandrasekharan, GOML, Chennai, 1953.
2. *Karaṇapaddhati* of Putumana Somayājī, Ed. with Malayalam Notes by P.K. Koru, Cherp, 1953.
3. *Uparāgaviṃśati* of Acyuta Piṣāraṭi, Ed. by Rama Varma Tampuran, Trippunithura, 1954. Rev. Ed. with Malayalam Commentary by K. A. Poduval, Sanskrit College, Tripunithura, nd.
4. *Karaṇapaddhati* of Putumana Somayājī, Ed. with Two Malayalam Commentaries by S.K. Nayar, GOML, Madras, 1956.
5. *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somayājī, Vol. III, *Golapāda*, Ed. by Suranad Kunjan Pillai, Trivandrum Sanskrit Series 185, Trivandrum, 1957.
6. *Sūryasiddhānta* with the *Vivaraṇa* of Parameśvara, Ed. with Notes by K. S. Shukla, Lucknow University, Lucknow, 1957.
7. *Mahābhāskarīya* with the *Bhāṣya* of Govindasvāmin and the super-commentary *Siddhāntadīpikā* of Parameśvara, Ed. with notes by T.S. Kuppanna Sastri, K.S.R. Institute, Chennai, 1957.
8. *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī, Ed. with the *Laghuvivṛti* of Śaṅkara Vāriyar, by Suranad Kunjan Pillai, Trivandrum Sanskrit Series 188, Trivandrum, 1958.
9. *Karaṇottama* of Acyuta Piṣāraṭi, Ed. with Auto-commentary by Raghavan Pillai, Trivandrum Sanskrit Series 213, Trivandrum, 1964.
10. *Karaṇāmṛta* of Citrabhānu, Ed. by V. Narayanan Namboodiri, Trivandrum Sanskrit Series 240, Trivandrum, 1975.
11. *Mahājyānayanaprakāra*, Ed. and Tr. by D. Gold and D. Pingree, *Historia Scientiarum*, 42, 1991, pp.49–65.
12. *Pañcabodha*, Ed. by Nataraja Pillai, Sreevidya Niketanam, Trivandrum, 1994.

Thus, with the lead given by K.V. Sarma, the period 1953–2008 indeed saw a spate of publications of thirty-one source-works of the Kerala school apart from a compilation including 27 small tracts.

Other publications of K.V. Sarma

It is important to note that, in addition to bringing out works of the Kerala school, Sarma also brought out landmark editions and translations of the *Vedāṅgajyotiṣa*, *Āryabhaṭīya*, *Pañcasiddhāntikā* and *Bṛhatsaṃhitā*, and also of *Vākyakaraṇa*, the basic text of the vākya system. Some of this work was done in collaboration with his reputed colleagues T.S. Kuppanna Sastri (1900–1982) and Kripa Shankara Shukla (1918–2007). These studies, especially of *Āryabhaṭīya* and *Vākyakaraṇa* are of crucial importance in understanding the later developments initiated by the Kerala school.

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AL PHYSICS, UNIVERSITY OF MADRAS
A contented K.V. Sarma at the Tantrasangraha@500 event in Chennai **courtesy:** M.D. Srinivas

In collaboration with another reputed historian of Indian science, B.V. Subbarayappa (1925–2019), Sarma prepared a valuable *Source Book* which presented important passages from various source-works of Indian Astronomy along with translations, with all the passages being arranged theme-wise [68]. Sarma also gave a series of lectures on observational astronomy in ancient and medieval periods, which was published as a monograph by the Calicut University [69].

Sarma was a prolific writer of articles, both scholarly and popular, on all aspects of Indian Studies, especially Jyotiṣa, Gaṇita and manuscriptology. Some of his important articles have been compiled in two volumes, edited by Siniruddha Dash [70],[71]. Sarma also wrote meticulous summaries on the work of the great astronomers and mathematicians of India for various encyclopaedias. In particular, he contributed a large number of articles, especially on the savants of the Kerala school, for the first two editions of the *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*, edited by Helaine Selin and published by Springer in 1997 and 2008. Some of these articles have been reproduced in the recent edition of the *Encyclopaedia* in 2016 [72].[18]

The Legacy of K.V. Sarma

In discussing the legacy of K.V. Sarma, we shall focus on mainly three issues that lie at the heart of much of his work on the Kerala school's astronomy and mathematics. The first is his concern with bringing out editions and translations of works that present yuktis or demonstrations of the various results and procedures in astronomy and mathematics. While C.T. Rajagopal and co-workers and others (such as T.A. Sarasvati Amma, A.K. Bag and R.C. Gupta) did pioneering work in explaining the demonstrations discussed in *Yuktibhāṣā*, it was the painstaking effort of Sarma which resulted in the publication of the Sanskrit commentaries of Śan˙n˙kara Vāriyar (*Kriyākramakarī* and *Yuktidīpikā*), and more crucially the preparation of the critical edition of *Yuktibhāṣā* of Jyeṣṭhadeva, along with English translation and notes, which ensured that all the demonstrations worked out in the Kerala tradition became widely accessible in their original form, along with translations and detailed mathematical explanations. This has made it possible for scholars to appreciate the rigour and sophistication of the Kerala school's work on infinite series. It is now well established that the proofs found in this 16th century Kerala work are more like the modern proofs (with careful consideration of the orders of magnitudes and the limiting procedures) in comparison to the proofs given by the great European savants when they rediscovered these series in the 17th century. The volume *Gaṇitayuktayaḥ* edited by Sarma contains twenty-seven tracts presenting yuktis for various

astronomical procedures. Most of these tracts are yet to be translated and studied. Amidst all his scholarly endeavours, Sarma was particularly keen to publish texts which give us first-hand knowledge of the way of thinking and working of the Indian astronomers and mathematicians.

The second concern of Sarma was to look for texts which introduced novel procedures or models in planetary theory, computation of eclipses, spherical astronomy and so on. The earliest publications of Sarma such as the *Rāśigolasphuṭanīti* of Acyuta Piṣāraṭi and *Veṇvāroha* of Mādhava were important texts which gave a better method for the `reduction to ecliptic' and an improved vākya method for the computation of the lunar longitude. In collaboration with Kuppanna Sastri, Sarma brought out the 13th-century work, *Vākyakaraṇa*, which happens to be the basic text for understanding the vākya system of computations.

While working on the edition and translation of *Yuktibhāṣā*, Sarma noticed that, in the astronomy part of the text, Jyeṣṭhadeva was actually discussing a new planetary model as formulated by Nīlakaṇṭha. He presented this finding in an important paper on *Yuktibhāṣā* that he wrote with S. Hariharan in 1991 [73][p.193]:

Chapter VIII [of *Yuktibhāṣā*] deals with the planetary theory… There is a new concept in the treatment of the epicycles as compared to the *Āryabhaṭīya* and other works. In a nutshell, the planetary theory broadly is like this. The earth is the centre and the sun and the moon go around the earth. As for other planets, with the earth as the centre, the *ś īghra* goes round the earth with the mean motion of the sun. The mean planet moves on a circle with the *śīghra* as the centre. The true planet is on the *mandocca* circle with the mean planet as its centre. Alternatively, the last two circles can be interchanged. This theory is advocated by Nīlakaṇṭha in his commentary on *Āryabhaṭīya* and practically all later Kerala authors have followed suit. In fact, Nīlakaṇṭha tries to say that it was the view of Āryabhaṭa also. If *ś īghra* is identified with the sun itself then this agrees broadly with the modern theory with the positions of earth and sun reversed. In fact, the western astronomer Tycho Brahe (1546–1601) appears to have adopted a similar theory.

This revised planetary model was presented in detail by Nīlakaṇṭha in his major works *Tantrasan˙n˙graha*, *Golasāra*, *Siddhāntadarpaṇa* and its auto-commentary, *Āryabhaṭīyabhāṣya* and a small tract entitled *Grahasphuṭānayane Vikṣepavāsanā* (which is a part of the compilation *Gaṇitayuktayaḥ* edited by Sarma). The essential features of this model, as presented in the above works, were brought to light in a paper by K. Ramasubramanian, M.D. Srinivas and M.S. Sriram in 1994 [74]. Considerable work has now been done on this revised planetary model of Nīlakaṇṭha and its adoption by the later Kerala astronomers (for a recent overview see [75][pp.487-535]). Thus, it is now well established that the Kerala school came up with two of the major innovations, namely the discovery of calculus and of non-geocentric planetary models, more than a century before they were to become hallmarks of the European scientific revolution.

A third issue, that was repeatedly highlighted by K.V. Sarma, has been the importance given to the constant checking and updating of the theories and models in the Indian astronomical tradition using continued and accurate observations. While this was emphasised in several ancient texts, this was particularly highlighted in the works of Parameśvara, Nīlakaṇṭha and Jyeṣṭhadeva. Sarma had written about this in his editions of the works of Parameśvara, but he was indeed excited to chance upon a manuscript of the seminal text of Nīlakaṇṭha, the *Jyotirmīmāṃsā*, which discusses at length the crucial role of continuous parīkṣā, or examination of the theories by observation, and the consequent process of revision and bringing

out newer and newer karaṇas, or computational manuals, that are accurate for the period under consideration. Sarma has discussed this issue extensively, along with elaborate quotations from the source-works, in his "Source Book", the book on "Observational Astronomy", and in several articles that he wrote on the subject [68],[69],[70].

Continuing the Legacy of the Great Savants (2005–2021)

Finally, we briefly touch upon the recent work on the contributions of the Kerala school. This is largely based on the substantial corpus of source-works of the Kerala school that have been edited and published, as also the scholarly studies that have been published in the last century. Thus, much of this work may be said to be in continuation with the seminal contributions of the great savants such as Śan˙n˙karavarman, Charles Whish, B. Datta and A.N. Singh, K. Sambasiva Sastri, C.T. Rajagopal, Rama Varma Tampuran, Suranad Kunjan Pillai and K.V. Sarma and their illustrious colleagues.

Recent decades have seen the publication of a large number of scholarly articles and books, which deal with various aspects of the work of the Kerala school (see, for instance [72, 75-85], and the references cited therein). Even if we restrict ourselves to only the books published in English, we find that there are more than thirty-five books and monographs published in the last twenty-five years which include at least a section devoted to the discussion of the work of the Kerala school. The number of scholarly articles would indeed be much larger. From this vast literature, we shall present two recent assessments of the work of the Kerala school in order to illustrate how the discourse has changed from what we saw in the nineteenth century. The renowned mathematician David Mumford has characterised the work of the Kerala school as "the crown jewel of Indian mathematics", in his 2010 review of Kim Plofker's book on *Mathematics in India* [86][p.389]:

Chapter 7 of Plofker's book is devoted to the crown jewel of Indian mathematics, the work of the Kerala school. Kerala is a narrow fertile strip between the mountains and the Arabian Sea along the southwest coast of India. Here, in a number of small villages, supported by the Maharaja of Calicut, an amazing dynasty of mathematicians and astronomers lived and thrived. A large proportion of their results were attributed by later writers to the founder of this school, Mādhava of Sangamagrama, who lived from approximately 1350 to 1425. It seems fair to me to compare him with Newton and Leibniz. The high points of their mathematical work were the discoveries of the power series expansions of arctangent, sine, and cosine. By a marvellous and unique happenstance, there survives an informal exposition of these results with full derivations, written in Malayalam, the vernacular of Kerala, by Jyeṣṭhadeva perhaps about 1540. This book, the *Gaṇita-Yukti-Bhāṣā*, has only very recently been translated into English with an extensive commentary. As a result, this book gives a unique insight into Indian methods. Simply put, these are recursion, induction, and careful passage to the limit.

Roy Wagner, who has written extensively on the foundations of mathematics and also on the work of the Kerala school, refers to the Kerala mathematics of the 14–16 centuries as "the most advanced mathematical culture of its time throughout the world", in a 2019 publication (which is the proceedings of a conference on the foundations of mathematics in 2016) [87][pp.390–391]:

Turning back to our case study: Kerala mathematics in the fourteenth to the sixteenth century. This culture was the most advanced mathematical culture of its time throughout the world. Its breakthroughs include power series like approximations of trigonometric functions and the

value of pi … Some historians of mathematics claimed that Indian mathematics had no interest in proofs … This position has been refuted, as discussions of proofs and justification have been documented already in early classical Sanskrit mathematics … Part of the problem is that the texts known to western scholars were mostly succinct summaries of algorithms in verse. But in India, these highly elliptic texts were accompanied by oral teaching and written commentaries, which included detailed justifications … Within this tradition, the most highly elaborate proofs that survive in Indian sources come from Kerala. … Indian proofs did not start with axiomatic foundations. Indian proofs could rely on evidence elicited from observing a diagram or the use of analogies to physical situations. But while many Indian logical schools preferred observation over inference, this tendency does not reduce Indian proofs to inductive generalizations–-a good proof had to give compelling reasons that would convince the community of specialists, not only show that a certain mathematical technique works by means of examples.

While it is heartening to see that there is a growing literature on the work of the Kerala school, the situation is somewhat disappointing when we take stock of the number of source-works or their translations that have been published in the last decade and a half. We list below the few source-works that have been published since 2008:

1. *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī, Ed. & Tr. with Notes by K. Ramasubramanian and M.S. Sriram, Hindustan Book Agency, New Delhi, 2011. Rep. Springer, New York, 2011.
2. *Sadratnamāla* of Śaṅkaravarman, Ed. & Tr. with Notes by S. Madhavan, K.S.R. Institute, Chennai, 2011.
3. *Madhyamānayanaprakāra* of Mādhava, Ed. & Tr. by U.K.V. Sarma, R. Venkateswara Pai and K. Ramasubramanian, *Ind. Jour. Hist. Sc.*, 46(1), 2011, pp. T1-T29.
4. *Karaṇapaddhati* of Putumana Somayājī, Ed. & Tr. with Notes by R. Venkateswara Pai, K. Ramasubramanian, M.S. Sriram and M.D. Srinivas, Hindustan Book Agency, New Delhi, 2017. Rep. Springer, Singapore, 2017.

Currently, there are several important source-works of the Kerala school–-even some works composed by legendary figures such as Mādhava, Parameśvara, Nīlakaṇṭha, Jyeṣṭhadeva, Putumana Somayājī, Śaṅkara and Acyuta–-which are yet to be edited and published. Further, even among the fifty major works (and about thirty small tracts) which have been published in the last 150 years (1874–2021), only about a dozen have been translated; and even among them, there are only about half a dozen works which have been studied in depth to explicate their full technical content.

It will be befitting to the legacy of the great savants mentioned above, that our young scholars imbibe their great dedication, perseverance and love and respect for our tradition, and carry forward their mission to ensure that the entire corpus of works of the Kerala school is rigorously and comprehensively studied and brought to light. In this way, they would ensure that the great achievements of the Kerala school of Astronomy and Mathematics are not only accorded their due place in the history of science, but also serve as a source of ideas and inspiration for a resurgent Indian science in our times.

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Footnotes

1. George Hyne arrived in India in 1817 and was appointed as an Assistant Surgeon in 1820. Hyne seems to have become an important member of the Madras Literary Society, which was founded in 1818. In 1826, Hyne was appointed by the Madras Government as the Secretary of the newly formed Committee of Public Instruction, though he passed away within a few months thereafter. ↩
2. The reference here is to the 1790 article of William Jones [8][p.302]. ↩
3. The Royal Asiatic Society of Great Britain and Ireland was the premier learned society founded to further the investigation of "science, literature and arts in relation to Asia". It was started in 1823 at the initiative of the leading Indologist of England, Henry Thomas Colebrooke (1765–1837), who also served as its Founder Director during 1823–37. The King consented to be its patron and the Royal Charter was granted to the Society in 1824. The Asiatic Societies at Calcutta, Bombay etc., were made its associate

Societies. The first volume of its *Transactions* was published in 1827. The third volume of the *Transactions* seems to have been issued in three parts in 1831, 1833 and 1834 and the consolidated volume bears the year 1835 on its title page. ↩

4. The Collections of Colonel Colin Mackenzie (1754–1821), Dr. Leyden and Mr. C. P. Brown constitute a large part of the vast collection of manuscripts preserved in Government Oriental Manuscripts Library (GOML), Madras, which consists of manuscripts of works in literature, history, philosophy and science, written in South Indian and Oriental languages, and of Kaifiyats and inscriptions, found in many places belonging to different periods. ↩
5. From this passage it appears that Whish might have had an occasion to meet Śan˙n˙karavarman (1774–1839) who was the brother of the Rāja of Kaṭattanād and the author of *Sadratnamāla*, which was composed in Kali 4921 or 1819 CE.↩
6. The semantics of the word `vulgar' has itself changed since the last two hundred years. Back when Whish used it, the word meant as something that referred to, or belonged to, the masses and the common people of the land. ↩
7. Despite the clear statement made by Whish that *Tantrasan˙n˙graha* was composed in 4600 Kali or around 1500 CE, many later studies cite Whish as attributing this text to 1608.. ↩
8. The proof of the theorem of Pythagoras given by Whish, however, seems to differ significantly from the proof that is found in all the manuscripts of *Yuktibhāṣā*. ↩
9. This is the first instance of misquoting Whish on the date of the *Tantrasan˙n˙graha*, which is more or less correctly given in the original paper as Kali 4,600 or around A.D. 1500.↩
10. De Morgan's father and grandfather served as British officials in India and Augustus himself was born in Madurai. He was educated in Cambridge and later worked as a Professor of Mathematics in London University. ↩
11. Charles Brown was the son of an English missionary in Calcutta. He joined the East India Company's service in 1817 and became well known for his proficiency in the Telugu language and literature. After returning to England in 1854, he worked in London University as a Professor of Telugu. ↩
12. The last two articles [48][49] were published much later in 1978 and 1986. ↩
13. Here is a footnote in the original giving the reference to the article of Whish. Regarding the four śāstras mentioned in the title of Whish's article, it is noted that: "These works, with the exception of the third [*Karaṇa-Paddhati*], are available only in manuscript. Our study of the relevant portions of the works and the literature relating to the works has been helped not a little by the devoted labours of Prince Rāma Varma (the 12th prince of Cochin) and Mr. K. Balagangadharan, one of our colleagues.". ↩
14. Here, there is a footnote in the original giving references to the 1937 edition of the *Karaṇa-Paddhati* and some of the available manuscripts of the other three works. ↩
15. The reference here is to G. R. Kaye, *Indian Mathematics*, Calcutta, 1915, p.24. ↩
16. This is a reference to the 1940 publication of K. Mukunda Marar [40]. ↩
17. This does not, however, mean that all the later historical accounts of mathematics (or even specialised works dealing with the development of infinite series or calculus) do take due note of the Kerala contributions wherever required. For instance, the scholarly account of *The Historical Development of Calculus* by Charles Edwards (1978) [62], or the more recent specialised treatise on *The Rise and Development of the Theory of Infinite Series* by Giovanni Ferraro (2008) [63], make no reference at all to the contributions of the Kerala school. ↩
18. K.V. Sarma contributed thirty-four important articles for the second edition of the *Encyclopaedia* which appeared in 2008, three years after his demise. In the third edition

of the *Encyclopaedia*, published in 2016, sixteen of Sarma's articles have been retained as such. However, we find that fifteen of the scholarly articles contributed by Sarma have been replaced by inaccurate and sloppy articles written by someone who has no scholarly publications to his credit either in astronomy or mathematics, or in the history of science. This is particularly distressing as the *Encyclopaedia* is otherwise a scholarly publication, and such casual attitude is not displayed in the choice of various contributions pertaining to China, West Asia or other cultures and civilisations. The editor and the publishers are best advised to replace these articles with the earlier articles of K.V. Sarma (the edition is also accessible on the internet), at least till such time that they organise new articles on these topics written by reputed and competent scholars. ↩

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Part II

WHO WAS MĀDHAVA?

Paper 2

Vatakkumkur Rajarajavarma Raja (1938)

Samgamagrāma Mādhavan (in Malayalam)

Extracts from

Vatakkumkur Rajarajavarma Raja's *Keraḷīya Saṃskṛta Sāhitya Caritram* (History of Sanskrit Literature of Kerala) Volume I (in Malayalam) published by Kamalalaya Book Depot, Thiruvananthapuram, 1938, p. 315

and

Vatakkumkur Rajarajavarma Raja's *Keraḷīya Saṃskṛta Sāhitya Caritram* (History of Sanskrit Literature of Kerala) Volume III (in Malayalam) published by Mangalodayam Press, Thrissur, 1950, pp. 388–90.

Vatakkumkur Rajarajavarma Raja

Vadakkumkur Rajarajavarma Raja (1891–1970) was a Sanskrit-Malayalam scholar. He has written several great epics, short poems and deep interpretations, all based on stories appearing in classical Sanskrit puranic and other texts. He is known as a great poet, biographer, literary critic, researcher and also as a scientist. He is also the author of a six-volume history of Sanskrit Literature of Kerala.

Source: A scanned copy of the article was obtained from the library attached to the Kerala Sahithya Academy, curtsey the authorities of the Academy, on 10 January 2025.

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Malayalam article

Extracts from

Vatakkumkur Rajarajavarma Raja,
"**Keraliya Samskritha Sahitya Charitram**" **Vol. I**,
Kamalalaya Book Depot, Thiruvananthapuram, ME 1113 (AD 1938), p. 315

and

Vatakkumkur Rajarajavarma Raja,
"**Keraliya Samskritha Sahitya Charitram**" **Vol. III**,
Mangalodayam, Thrissur, ME 1125 (AD 1950), pp. 388-390

For bibliographic details, see: K. V. Sarma, "A History of Kerala School of Hindu Astronomy", Vishveshvaranand Institute, Hoshiarpur, 1972, p. xii

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സങ്ഗ്മഗ്രാമമാധവൻ.

സങ്ഗമഗ്രാമം ഇരിങ്ങാലക്കുടയാകയാൽ ആ ദേശത്തുകാരനാണ് സങ്ഗമഗ്രാമമാധവൻ എന്നു് വിശ്വസിക്കാം. ഇദ്ദേഹം ഒരു വാരിയരാണെന്നു് ഊഹം തോന്നുന്നു.

"സങ്ഗമഗ്രാമജോ മാധവഃ പുനരത്യാസന്നാം പരിധിസങ്ഖ്യാം ഉക്കവാൻ:—

വിബുധനേത്രഗജാഹി ഹതാശന ത്രിഗുണവേദഭവാരണബാഹവഃ
നവതിഖർവ്വമിതേ വ്രതിവിസ്മരേ പരിധിമാനമിദം ജഗദുർബുധാഃ.

ഇതി.

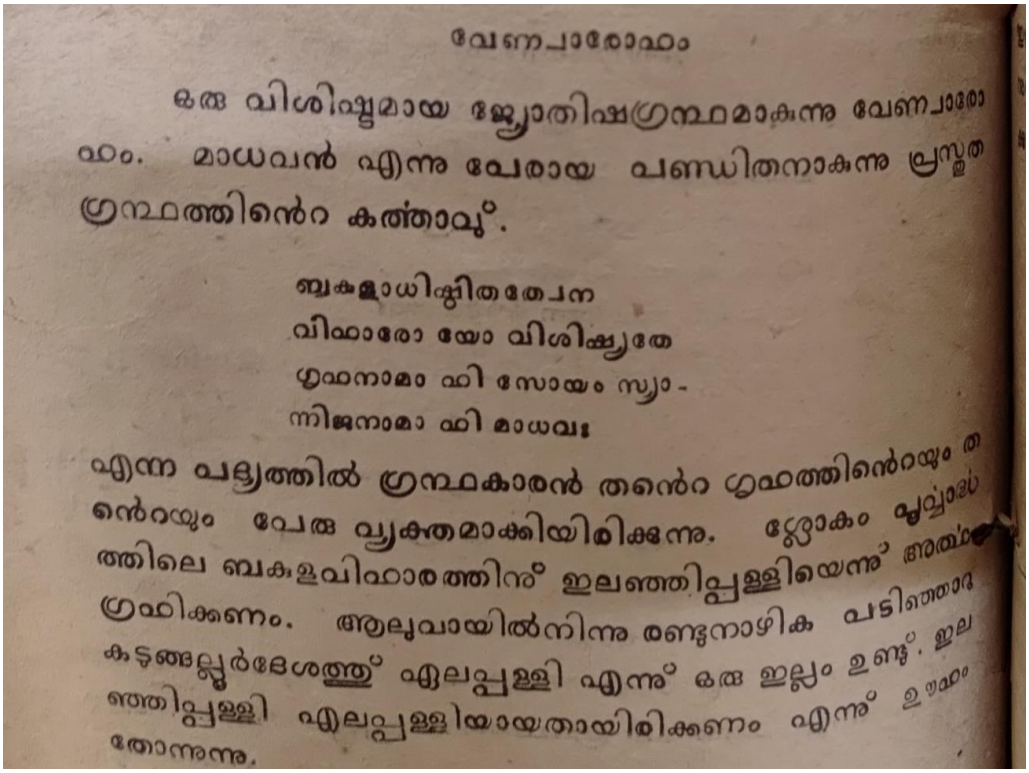
എന്നു് പത്താമത്തെ സൂത്രത്തിന്റേയും,

"തദനന്തരം പുനസ്സദ്വിഷയം വസന്തതിലകം സങ്ഗമഗ്രാമമാധവനിർമ്മിതം പദ്യം ച ശ്രുതം യഥാ—

ജീവേ പരസ്പരനിജേതരമൌഢ്വികാഭ്യാ—
മഭ്യസ്യ വിസ്മൃതി ഗുണേന വിഭജ്യമാനേ
അന്യോന്യയോഗവിരഹാനുഗുണേ ഭവേതാം
യദ്വാ സ്വലംബകൃതി ഭേദപദീകൃതേ ദ്വേ.

ഇതി.

എന്നു് പന്ത്രണ്ടാം സൂത്രത്തിന്റേയും ഭാഷ്യത്തിൽ പ്രസ്തുത മാധവന്റെ മതത്തെ പ്രദർശിപ്പിച്ചിരിക്കുന്നു. ജ്യോതിശ്ശാസ്ത്രത്തിൽ അസാധാരണവൈദുഷ്യം സമ്പാദിച്ചിരിക്കുന്ന ഒരു മഹാനും ബഹുകർത്താവും ആണ് ഈ മാധവൻ എന്ന വിചാരിക്കേണ്ടതാകുന്നു.



വേണപാരോഹം

ഒരു വിശിഷ്ടമായ ജ്യോതിഷഗ്രന്ഥമാകുന്നു വേണപാരോ
ഹം. മാധവൻ എന്നു പേരായ പണ്ഡിതനാകുന്നു പ്രസ്തുത
ഗ്രന്ഥത്തിന്റെ കർത്താവു്.

ഭകുളാധിഷ്ഠിതതേപന
വിഹാരോ യോ വിശിഷ്യതേ
ഗ്രഹനാമാ ഹി സോയം സ്യാ-
ന്നിജനാമാ ഹി മാധവഃ

എന്ന പദ്യത്തിൽ ഗ്രന്ഥകാരൻ തന്റെ ഗ്രഹത്തിന്റെയും ത
ന്റെയും പേരു വ്യക്തമാക്കിയിരിക്കുന്നു. ശ്ലോകം പൂർവ്വാർ
ത്തിലെ ഭകുളവിഹാരത്തിനു് ഇലഞ്ഞിപ്പള്ളിയെന്നു് അർത്ഥം
ഗ്രഹിക്കണം. ആലുവായിൽനിന്നു രണ്ടുനാഴിക വടിഞ്ഞാറു
കുടുങ്ങല്ലൂർദേശത്തു് ഏലപ്പള്ളി എന്നു് ഒരു ഇല്ലം ഉണ്ടു്. ഇല
ഞ്ഞിപ്പള്ളി ഏലപ്പള്ളിയായതായിരിക്കണം എന്നു് ഊഹം
തോന്നുന്നു.

* മാധവവിരചിതമായ വേണാരോഹത്തിനു് അച്യുതൻ ഒരു ഭാഷാവ്യാഖ്യാനം നിർമ്മിച്ചിട്ടുണ്ടെന്നു മുൻപു പറഞ്ഞിട്ടുണ്ടു്. ആഴവാഞ്ചേരി തമ്പ്രാക്കളുടെ ആജ്ഞയനുസരിച്ചാണു് ആ വ്യാഖ്യാനം നിർമ്മിച്ചതു്. ഭാഷാവ്യാഖ്യാനകർത്താവായ അച്യുതൻ തൃക്കണ്ടിയൂർ അച്യുതപ്പിഷാരടിയാകണമെന്നു തോന്നുന്നു. കൊല്ലവർഷം ൯൭൦-നും ൭൭൦-നും മദ്ധ്യം മാധവന്റെ ജീവിതകാലമെന്നു വിശ്വസിക്കാം.

വേണാരോഹം ഇപ്രകാരം തുടങ്ങുന്നു.

ശ്രിയേ ഭവതു മേ ദേവശ്ശിവശ്ശീതാംശുശേഖരഃ
തഥൈവ തത്സുതോപ്യസ്മ വിഘ്നോ വിഘ്നോപശാന്തയേ
അവിചാരകൃതം വാചാമചാതുഷ്മപോഹതു
ജാഗ്രതീ രസനാഗ്രേ മേ ഗിരാമപ്യധിദേവതാ
പ്രഭാകരാദയസ്സർവ്വേ പ്രണതാന്തഃപ്രഭാകരാഃ
ദിവി ഗ്രഹാഃ പ്രസീദന്തു ത്രിലോകാനന്ദിവിഗ്രഹാഃ

ഇപ്രകാരം വിഷയാനുസരണം ഇഷ്ടദേവതാവന്ദനം ചെയ്തതിനുശേഷം ഗ്രന്ഥത്തിലെ പ്രതിപാദ്യവസ്തുവിനെ സൂചിപ്പിക്കുന്നു.

വിധുവാക്യൃധ്രുവൈക്യേന തിഥിനക്ഷത്രലക്ഷണേ
പ്രാഗേവ തദുപായേഷു കഥിതേഷ്വപി കൈശ്ചന
ഇഹ സൂത്രഃ പ്രകീർണ്ണാനാം പ്രക്രിയാണാമസംഗ്രഹാൽ
ഉപേയുഷാം ബഹൂൻ മോഹാൻ ബഹ്വനാമവലോകനാൽ
താ ഏവൈകത്ര സംഗൃഹ്യ ഗദിതും വിസ്തരം വിനാ
തത്സാധനേഷു ചൈകൈകാൻ ഭേദാൻ ദർശയിതും കചിൽ
കചനോപായമേവാദ്യം ലഗ്നം വക്തും തദീരിതാൻ
ഉത്തമപ്യുപരം കാപി സ്ഥൂലസൂക്ഷ്മതയോഭയോഃ

* കേ. സം. സാ. ച. രണ്ടാംഭാഗം ഉപോദ്ഘാതം പുറം നോക്കുക.

മാധവൻതാൻ ചമച്ചുള്ള
വേണാരോഹത്തിനച്യുതൻ
ഭാഷാവ്യാഖ്യാനമുണ്ടാക്കി
നേതൃനാരായണാജ്ഞയാ

അപി തന്മാത്രമേവൈകമേകാനാം പ്രതിബോധിതം
അദ്വിഷ്ടമേവ വാസ്പഷ്ടം ദ്വിഷ്ടാം ദ്വേഷ്ടേന ചേതസാ
അമുമേവാം ശ്രുമദ്ധ്യാനമപരം തദനുഷംഗികം
അത ഏവാധികം സൈക്ഷ്ണ്യമവബോധയിതും വിധേഃ
അയം ജനോഽയ യതതേ ധിയം ജനയതേളശീം
പണ്ഡിതാഃ ഖണ്ഡിതാശേഷപരപക്ഷപരിഗ്രഹാഃ

ഈ പദ്യങ്ങളിൽനിന്നു പ്രസ്തുതഗ്രന്ഥത്തിന്റെ പ്രതിപാദ്യ സ്വഭാവം വിശദമാകും. ഒരു പ്രസിദ്ധഗ്രന്ഥമാണു് വേണ്ടാ രോഹം. ഞാൻ പരിശോധിച്ച ഗ്രന്ഥത്തിൽ ഒൻപതു പദ്യങ്ങൾ മാത്രമേ എഴുതിക്കാണുന്നുള്ളു.

Paper 3

P. K. Koru (1944)

Vṛttaparidhiyilūṭe (in Malayalam)

Mangalodayam Magazine (Malayalam) dated July–August 1944, pp. 187–190.

P. K. Koru

P. K. Koru (15 January 1890 – 16 January 1968) was an academician and a politician. As an academician he started his career as a teacher in Government Brennen College of Teacher Education, Thalasseri and later moved to Sastri College in South Africa. As a politician, he represented the Guruvayur Constituency in the first Legislative Assembly of Kerala. He had deep knowledge in the traditional astronomy of India.

Source: A scanned copy of the article was obtained from the library attached to the Ramavarma Appan Thampuran Smarakam, Ayyanthol, Thrissur, Kerala, curtsey the authorities of the Smarakam, on 7 January 2025.

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Malayalam article

Vṛttaparidhiyilūṭe

P. K. Koru

Mangalodayam vol. 20, Karkkitakam 1119 (July-August 1944)

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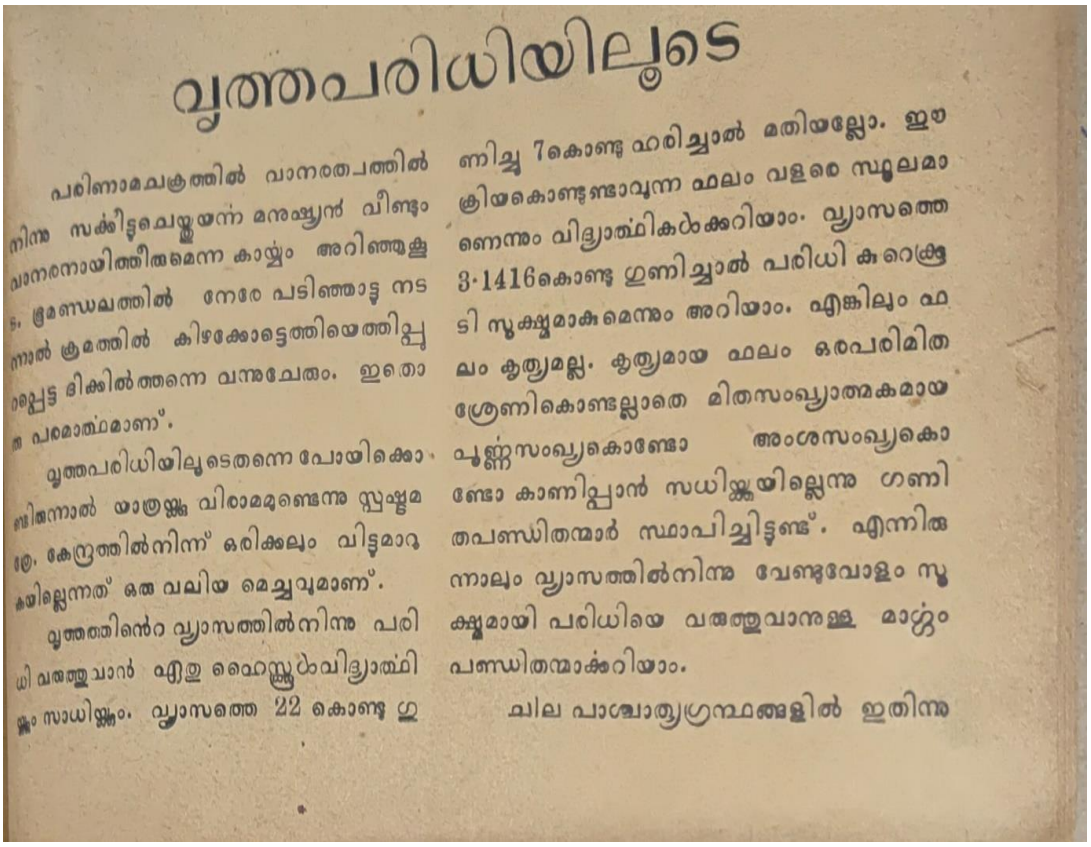
വൃത്തപരിധിയിലൂടെ

പി. കെ. കോരു

മംഗളോദയം, 20, 1119 കർക്കിടകം (1944 ജൂലായ്-ആഗസ്റ്റ്)
പേജ് 187-190



വൃത്തപരിധിയിലൂടെ

പരിണാമചക്രത്തിൽ വാനരത്വത്തിൽ നിന്നു സക്കീട്ടുചെയ്തയന്ന മനുഷ്യൻ വീണ്ടും വാനരനായിത്തീരുമെന്ന കാര്യം അറിഞ്ഞുള്ള ഭ്രമണ്ഡലത്തിൽ നേരേ പടിഞ്ഞാട്ടു നടന്നാൽ ക്രമത്തിൽ കിഴക്കോട്ടെത്തിയെത്തിപ്പെട്ടെ ദിക്കിൽത്തന്നെ വന്നുചേരും. ഇതൊരു പരമാർത്ഥമാണ്.

വൃത്തപരിധിയിലൂടെതന്നെ പോയിക്കൊണ്ടിരുന്നാൽ യാത്രയ്ക്കു വിരാമമുണ്ടെന്നു സ്പഷ്ടമാണ്. കേന്ദ്രത്തിൽനിന്ന് ഒരിക്കലും വിട്ടുമാറുകയില്ലെന്നത് ഒരു വലിയ മെച്ചവുമാണ്.

വൃത്തത്തിന്റെ വ്യാസത്തിൽനിന്നു പരിധി വരയ്ക്കുവാൻ ഏതു ഹൈസ്കൂൾവിദ്യാർത്ഥിക്കും സാധിക്കും. വ്യാസത്തെ 22 കൊണ്ടു ഗുണിച്ചു 7കൊണ്ടു ഹരിച്ചാൽ മതിയല്ലോ. ഈ ക്രിയകൊണ്ടുണ്ടാവുന്ന ഫലം വളരെ സ്ഥൂലമാണെന്നും വിദ്യാർത്ഥികൾക്കറിയാം. വ്യാസത്തെ 3·1416കൊണ്ടു ഗുണിച്ചാൽ പരിധി കുറെക്കൂടി സൂക്ഷ്മമാകുമെന്നും അറിയാം. എങ്കിലും ഫലം കൃത്യമല്ല. കൃത്യമായ ഫലം ഒരപരിമിത ശ്രേണികൊണ്ടല്ലാതെ മിതസംഖ്യാത്മകമായ പൂർണ്ണസംഖ്യകൊണ്ടോ അംശസംഖ്യകൊണ്ടോ കാണിപ്പാൻ സധിയ്ക്കയില്ലെന്നു ഗണിതപണ്ഡിതന്മാർ സ്ഥാപിച്ചിട്ടുണ്ട്. എന്നിരുന്നാലും വ്യാസത്തിൽനിന്നു വേണ്ടുവോളം സൂക്ഷ്മമായി പരിധിയെ വരയ്ക്കുവാനുള്ള മാർഗ്ഗം പണ്ഡിതന്മാർക്കറിയാം.

ചില പാശ്ചാത്യഗ്രന്ഥങ്ങളിൽ ഇതിനു

ള്ള ഒരു ചലിതമാഗ്രത്തെ സ്കോട്ട്ലാണ്ടിലെ ഗണിതശാസ്ത്രവിദഗ്ധനായിരുന്ന ജെയിംസ് ഗ്രഗറിയാണ് ആദ്യം കണ്ടെത്തിയതെന്നു പറഞ്ഞിട്ടുണ്ട്. ഗ്രഗറി ക്രൈസ്തവാബ്ദം 1638-ൽ ജനിക്കുകയും 1675-ൽ മരിക്കുകയും ചെയ്തു. ചില൪ പരിധ്യാനയനത്തിനുള്ള സൂത്രം ഗ്രഗറിയല്ല ആദ്യം മനസ്സിലാക്കിയത്, ജ൪മ്മനിയിലെ പണ്ഡിതാഗ്രേസരൻ ലീബ്നിറ്റ്സ് ആയിരുന്നുവെന്നുംപറയുന്നു. ലീബ്നിറ്റ്സ് 1646-നും 1716-നും ഇടയിലാണ് ജീവിച്ചിരുന്നത്. ഈ മാഗ്രം ആ൪ ആദ്യം കണ്ടെത്തിയതായാലും ഗണിതശാസ്ത്രത്തിൽ ഇതിനുള്ള ഒരു മാന്യമായ ഒരു സ്ഥാനമുണ്ട്. കണ്ടുപിടിച്ച ശാസ്ത്രജ്ഞനെ സംബന്ധിച്ച് അദ്ദേഹത്തിന്റെ രാജ്യക്കാ൪ക്ക് അഭിമാനിക്കുവാനും ലോകർക്ക കൃതജ്ഞതാപൂ൪വ്വം പ്രശംസിക്കുവാനും ധാരാളം അവകാശവുമുണ്ട്.

ഗണിതശാസ്ത്രത്തിൽ പരിധ്യാനയനം പോലെയുള്ള വിദ്യകൾക്കുള്ള സ്ഥാനം യവനപണ്ഡിതനായ ആ൪ക്കിമീഡിസ്സിനെക്കുറിച്ചുള്ള ഒരു പഴയ ഇതിഹാസത്തിൽനിന്നു ഗ്രഹിക്കാം. ആ൪ക്കിമീഡിസ്സ് അതിസമ൪ത്ഥനായിരുന്നു. സിസ്സിൽ താൻ വസിച്ചിരുന്ന ഉറമ്പുപട്ടണമായ സൈരാക്യൂസ് നഗരത്തെ ആക്രമിക്കുവാനായി പട്ടാളക്കാരോടുകൂടി ശത്രുനൗകകൾ നഗരത്തിനടുത്ത കടലിൽ അണിനിരന്നപ്പോൾ അദ്ദേഹം പ്രത്യേകതരം കണ്ണാടികളിലൂടെ സൂര്യരശ്മികളെ കപ്പലുകളിൽ കേന്ദ്രീകരിച്ച് അവയെ തീപ്പിടിപ്പിച്ചുവത്രേ. ഇദ്ദേഹം ഈജിപ്റ്റിലും മറ്റും സഞ്ചരിച്ചു കൃഷിക്കാക്കും മറ്റു തൊഴിലാളികൾക്കും ഉപകാരപ്രദമായ അനവധി യന്ത്രങ്ങൾ നി൪മ്മിച്ചിരുന്നു. ഇതെല്ലാം അറിഞ്ഞത് അദ്ദേഹത്തിന്റെ സ്നേഹിതന്മാ൪ ആ യന്ത്രങ്ങളെ വിവരിക്കുന്ന ഒരു ഗ്രന്ഥം എഴുതണമെന്ന് അദ്ദേഹത്തോടാവശ്യപ്പെട്ടപ്പോൾ, അതെല്ലാം വെറും കുട്ടികളികൾ മാത്രമാണെന്നും തന്റെ ശ്രദ്ധയും ബഹുമാനവും ശുദ്ധഗണിതത്തിൽ അർപ്പിച്ചിരിയ്ക്കുന്നുവെന്നും മറുപടി പറഞ്ഞു. അദ്ദേഹം മരിയ്ക്കുന്നതിനു മുമ്പു തന്റെ ബന്ധുജനങ്ങളെ പറഞ്ഞേല്പിച്ച കാര്യം ഒരു മാതൃകയായിരുന്നു: തന്റെ ശവകുടീരത്തിന്മേൽ താൻ കണ്ടുപിടിച്ച ഗോളച്ഛേദനയനമാഗ്രത്തെ വിശദീകരിയ്ക്കുന്ന ഒരു ര്രേഖികയും അതിനുകീഴ് കൃത്യമായി വെയ്ക്കാവുന്ന ഒരു ഗോളവും കൊത്തിയുണ്ടാക്കണമെന്നായിരുന്നു അത്. ഈ വിധത്തിലാണ് പൂജ്യപാദന്മാരായ ഗണിതപണ്ഡിതന്മാ൪ ഗോളച്ഛേദനയനം, പരിധ്യാനയനം ഈ തരത്തിലുള്ള വിദ്യകളെ ആദരിച്ചുപോന്നതും ഇന്നും ആദരിയ്ക്കുന്നതും.

പരിധ്യാനയനപഥത്തിന്റെ പ്രഥമദ്രഷ്ടാവു സ്കോട്ട്ലാണ്ടുകാരനോ ജ൪മ്മൻകാരനോ അല്ല, ഒരു കേരളീയനാണെന്നാണ് ഈ ലേഖനകർത്താവായ കേരളീയനു വാദിയ്ക്കുവാനുള്ളത്. ലേഖകന് ഇതു കുറെ അധികം കാലം സംശയത്തിൽ ഇരുന്നു. കരണപദ്ധതിയെന്ന ജ്യോതിഷഗ്രന്ഥത്തിൽ പരിധ്യാനയനത്തിന് ആദ്യമായിപ്പുറത്തിട്ടുള്ള മാഗ്രം ഇതാണ്:

''വ്യാസാച്ചതുർഘ്നാൽ ബഹുശഃപ്രഥമസ്ഥാൽ
ത്രിപഞ്ചസപ്താദ്യയുഗാഹതാനി
വ്യാസേ ചതുർഘ്നേ ക്രമശസ്ത്വണം സപം
കുര്യാൽ തദാ സ്യാൽ പരിധിഃ സൂസൂക്ഷ്മഃ''

ഇതിന്റെ സാരംകൂടിപ്പുറയാം. വൃത്തവ്യാസത്തെ നാലുകൊണ്ടു ഗുണിച്ചു പലേടത്തുംവെച്ച് അവയെ ക്രമേണ 1, 3, 5 തുടങ്ങിയ ഓജസംഖ്യകളെക്കൊണ്ടു ഹരിച്ചു ഫലങ്ങളെ ഉണ്ടാക്കി അവയിൽ ഒന്നാമത്തേത്, മൂന്നാമത്തേത് മുതലായവ ധനമായും മറ്റേവ ഋണമായും ഒന്നിച്ചു സംകലനം ചെയ്താൽ സൂക്ഷ്മമായ പരിധിയുണ്ടാകും. ഇതുതന്നെയാണ് ഗ്രഗറിയുടെ പരിധ്യാനയനശ്രേണി: $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots\ldots$

കരണപദ്ധതി സുമാ൪ 250 കൊല്ലം മുമ്പായിരിയ്ക്കണം നി൪മ്മിച്ചതെന്നു ശ്രീ. കെ. എസ്. നമ്പൂതിരിപ്പാട് ജാതകാദേശത്തിന്റെ

മുഖവുരയിൽ 20കൊല്ലം മുമ്പ് എഴുതിക്കണ്ടു. അതു ശരിയാണെങ്കിൽ കരണപദ്ധതി 1670 നടുത്തായിരിയ്ക്കണമല്ലോ നിമ്മിച്ചിരിയ്ക്കുക. ഇതു ഗ്രഗറിയുടേയും ലീബ്നിറ്റ്സിന്റേയും യൌവനകാലംതന്നെ. പക്ഷേ യൂറോപ്പിൽനിന്നു വില മതിച്ച ഒരു രത്നം ഭാരതഖണ്ഡത്തിൽ വന്നുചേരുവാൻ അക്കാലത്തെ സ്ഥിതിയ്ക്കു വളരെ താമസം വേണം. ആ കാലം ഇവിടെ കാണുന്നില്ല. എന്നുമാത്രമല്ല, കരണപദ്ധതിയുടെ കർത്താവായ ചോമാതിരി ഇതുകൊണ്ടു കൈകായ്പും ചെയ്തിട്ടുള്ളതു കാണുമ്പോൾ അത് അദ്ദേഹത്തിന്റെയോ അദ്ദേഹത്തിന്റെ പൂർവ്വന്മാരുടെയോ കയ്യിൽ എത്തീട്ടു കാലം കുറെ ആയി എന്നു തോന്നിപ്പോകും. ഇതിൽനിന്ന് ഉറയ്ക്കേണ്ടതു ചോമാതിരി ഈ വിഷയത്തിൽ യൂറോപ്പിനു കടപ്പെട്ടിട്ടില്ല എന്നതന്നെയാണ്. പക്ഷേ ചോമാതിരിയുടെ ഫലം തിട്ടമായിപ്പറയുവാൻ പ്രയാസമായിരുന്നു. കരണപദ്ധതിവ്യാഖ്യാനത്തിൽ ഒരിടത്തു കൊടുത്തിട്ടുള്ള അഹർഗണസംഖ്യയായ 'മജോസുരസ്ഥം സകുൽ' (1772786) എന്നതിൽനിന്നു 1752നു മുമ്പാണ് ആ ഗ്രന്ഥം നിർമ്മിയത് എന്നേ പറവാൻ പാടുള്ളൂ.*

യുക്തിഭാഷ എന്ന മലയാളഭാഷയിൽ എഴുതിട്ടുള്ള ഗണിതഗ്രന്ഥത്തിൽ ഇതേ സൂത്രം തന്നെ സയുക്തികം പ്രതിപാദിച്ചിരിയ്ക്കുന്നു. ഈ ഗ്രന്ഥത്തിന്റെ ഭാഷയ്ക്കു നന്നേ പഴക്കം തോന്നും. ഭാഷാഭിജ്ഞന്മാർ ഒരു മൂന്നൂറുകൊല്ലം പഴക്കം കാണുമെന്ന് അഭിപ്രായപ്പെടുന്നുണ്ട്. ഇതു ലേഖകന്ന് അത്ഭുദകരമായിതോന്നി. എങ്കിലും ഈ അഭിപ്രായം ഉറപ്പിപ്പാൻ പാടില്ലാത്തതായിത്തോന്നി. യുക്തിഭാഷയുടെ സമാപ്തിപ്രസ്താവനയിൽ "ആലേഖി യുക്തിഭാഷാ വിപ്രേണ ബ്രഹ്മദത്തസംജ്ഞേന 'യേ ഗോളപഥസ്ഥാസ്സ്യുഃ' കലിരഹിതാശ്ശോധയന്ത്വസ്മേ." എന്നതിൽ യുക്തിഭാഷ എഴുതിയ—ഗ്രന്ഥം നിമ്മിച്ചതോ പകർത്തിയതോ എന്ന സംശയം അപ്പഴും തീർന്നില്ല—കാലത്തെ കലിദിനസംഖ്യ 'യേ ഗോളപഥസ്ഥാസ്സ്യുഃ' (1771931) എന്നു കൊടുത്തിട്ടുണ്ട്. അതു ക്രൈസ്തവാബ്ദം 1670 വരെയുള്ള പഴക്കം മാത്രമേ തരുന്നുള്ളൂ.

കേരളത്തിൽ അമൂല്യഗ്രന്ഥങ്ങൾ ഇന്നും അച്ചടിയ്ക്കാതെ കോവിലകങ്ങളിലേയും മനകളിലേയും ഗ്രന്ഥപ്പുരകളിൽ സ്വസ്ഥമായി മറഞ്ഞുകിടപ്പുണ്ടെന്നുള്ളതു പരിതാപകരമായ വാസ്തവമാണ്. ഇന്നു കേരളത്തിൽ നടപ്പുള്ളതും പ്രസിദ്ധിയുള്ളതും ആയ ജ്യോതിഷതന്ത്രത്തിലെ കരണഗ്രന്ഥങ്ങളുടെ മൂലഗ്രന്ഥമായ തന്ത്രസംഗ്രഹം അവയിൽ ഒന്നാണ്. ഈ ഗ്രന്ഥം ഒന്നു നോക്കുവാൻ ലേഖകന്ന് ഇടയായി. അതിലും പ്രകൃതവിഷയമായ പരിധ്യാന്യനമാനത്തെ പ്രസ്താവിയ്ക്കുന്നതിന്റെ യുക്തി ആചാര്യന്നു നല്ലവണ്ണം അറിഞ്ഞിരുന്നു എന്ന നിലയിൽ അതിനെസ്സംബന്ധിച്ചു മറ്റു വിഷയങ്ങളെ പ്രതിപാദിയ്ക്കുകയും ചെയ്തിട്ടുണ്ട്. പ്രസ്താവം ഇപ്രകാരമാണ്.

വ്യാസേ വ്യാരിധി(4)നിഹിതേരൂപ(1)ഏതേ വ്യാസസാഗര(4)ഭിഹതേ ശി(ശ)ശരാ(5)ദി വിഷമസംഖ്യാ ഭക്തമൂണം സ്വം പൃഥക്രമാൽ കുര്യ്യാൽ * * * * * സൂക്ഷ്മപരിധിസ്സ്യാൽ ബാഹുകൃതേ ചാ ഹരണാതോതിസൂക്ഷ്മശ്ച.

ഇതിന്റെ സാരവും കരണപദ്ധതിയിൽ പറഞ്ഞിട്ടുള്ളതുതന്നെയാണ്. തന്ത്രസംഗ്രഹത്തെക്കുറിച്ചുള്ള ഭക്തി മാത്രം പോരല്ലോ അതിന്റെ നിമ്മാണകാലം നിശ്ചയിയ്ക്കുവാൻ.

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* ഒരു ഭാരതീയവർഷത്തിന്നു 365·2587 സമാനമെന്നും മേടം 1-ാംനു ഉൾപ്പെട്ട ക്രൈസ്തവാബ്ദത്തോടു 3101 ചേർത്താൽ ആയതു കഴിഞ്ഞ കലിവർഷസംഖ്യയായെന്നും ഓർത്താൽ ഇതിലുള്ള കണക്കുകൾ എളുപ്പം പരിശോധിയ്ക്കാം. ഭാരതീയഗ്രന്ഥകാരന്മാർ തിഥി കൊടുക്കുക കഴിഞ്ഞ കലിവർഷത്തിലെ ദിനസമൂഹസംഖ്യകൊണ്ടാണ്.

ഭാഗ്യവശാൽ ഗ്രന്ഥാരംഭകാലവും സമാപ്തികാലവും 'ഹേ വിഷ്ണോ നിഹിതം കുൽസ്തം' (1680548) എന്നും 'ലക്ഷ്മീശനിഹിതജ്യാനെ' (1680553) എന്നും ഉള്ള കലിദിനങ്ങളെക്കൊണ്ടു ഗ്രന്ഥത്തിൽത്തന്നെ പറഞ്ഞിരിയ്ക്കുന്നു. ഈ കാലം ക്രൈസ്തവാബ്ദം 1500 ലാണെന്നു കാണാം. ഇതിൽനിന്നും ഈ വിഷയത്തിൽ കുറെക്കൂടി ദൃഢമായ തെളിവു കിട്ടുന്നു. വൃത്തപരിധ്യാനയനമാഗ്ഗത്തിന്റെ പ്രഥമദൃഷ്ടാവ് പാശ്ചാത്യനല്ല കേരളീയൻതന്നെ. ഒരു സംശയംകൂടി വരാം, ഭാരതത്തിലെ ഇതരദേശങ്ങളിൽ ഒരാളായിക്കൂടേ എന്ന്. അങ്ങനെയായിരുന്നുവെങ്കിൽ ആ ദേശങ്ങളിൽ നിന്നിച്ച ഗ്രന്ഥങ്ങളിൽ ചിലതിൽ ഇതു കാണേണ്ടിയിരുന്നു. അങ്ങനെ കാണുന്നില്ല. പൂർവ്വസിദ്ധാന്തങ്ങളിലോ മറ്റോ ഉണ്ടായിരുന്നുവെങ്കിൽ സൂത്രങ്ങളിൽ കണ്ണുള്ള ആര്യഭടൻ, ശാസ്ത്രവ്യവസ്ഥയിൻ ശ്രദ്ധയുള്ള ഭാസ്കരൻ മുതലായ ആചാര്യന്മാർ അതു കാണാതെ പോകുവാൻ വഴിയില്ല. ആര്യഭടനും ഭാസ്കരനും പറയുന്ന വൃത്തപരിധ്യാനയനമാഗ്ഗങ്ങളെ അറിവാൻ വേറെ വഴികൾ ഉണ്ട്; അവയെ അവർ പ്രതിപാദിയ്ക്കുന്നുമുണ്ട്. ജ്യാനയനാദികമ്മങ്ങളിൽ അവർ അതിസമർത്ഥന്മാരായിരുന്നു എങ്കിലും പരിധ്യാനയനം അവക്കറിഞ്ഞിരുന്നില്ലെന്നു വിചാരിപ്പാനേ വഴി കാണുന്നുള്ളൂ. അതിനാൽ പരിധ്യാനയനത്തിന്റെ സൂക്ഷ്മാഗ്ഗം ഒന്നാമതായി കണ്ടുപിടിച്ചവൻ എന്ന കാര്യമായ സ്ഥാനം ഒരു കേരളീയന്നുതന്നെ നല്കേണ്ടതാണ്.

ഇവിടെ ദുഃഖകരമായ ഒരു പ്രശ്നം ഉത്ഭവിയ്ക്കുന്നു: ഇത്ര നിപുണന്മാരായിരുന്ന കേരളീയരുടെ മഹത്പം എങ്ങനെ മങ്ങിമാഞ്ഞു മറഞ്ഞുപോയി എന്ന്. അതിനുള്ള കാരണം തൊണ്ണൂറ് ശതമാനം അഭ്യസ്തവിദ്യരായ കേരളീയർ ഇന്നും അഭിമാനിയ്ക്കുന്ന ശിഥിലസംസ്കാരത്തിന്റെ പ്രചാരവും കേരളത്തിലെ അനാശാസ്യമായ ആചാരങ്ങളുമാണെന്ന് ഈ ലേഖകനു തോന്നുന്നു.

പി. കെ. കോരു, എം.എ., എൽ.ടി.

Paper 4

Ramavarma (Maru) Thampuran (1944)

"Vṛttaparidhiyilūṭe" (in Malayalam)

Mangalodayam Magazine (Malayalam) dated September–October 1944, pp. 252–256.

Ramavarma (Maru) Thampuran

Ramavarma (Maru) Thampuran (1894–1974) was a great scholar with deep knowledge in traditional Indian astronomy. It was Ramavarma (Maru) Thampuran who, along with A. R. Akhileswara Aiyer, published for the first time Jyeṣṭhadeva's Malayalam *Yuktibhāṣā* (Part I) with notes and explanations in modern mathematical notations in the year 1948.

Source: A scanned copy of the article was obtained from the library attached to the Ramavarma Appan Thampuran Smarakam, Ayyanthol, Thrissur, Kerala, curtsey the authorities of the Smarakam, on 7 January 2025.

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Malayalam article

“Vṛttaparidhiyilūṭe”

Ramavarma Maru Tampuran

Mangalodayam vol. 20, Kanni 1120 (September-October 1944)

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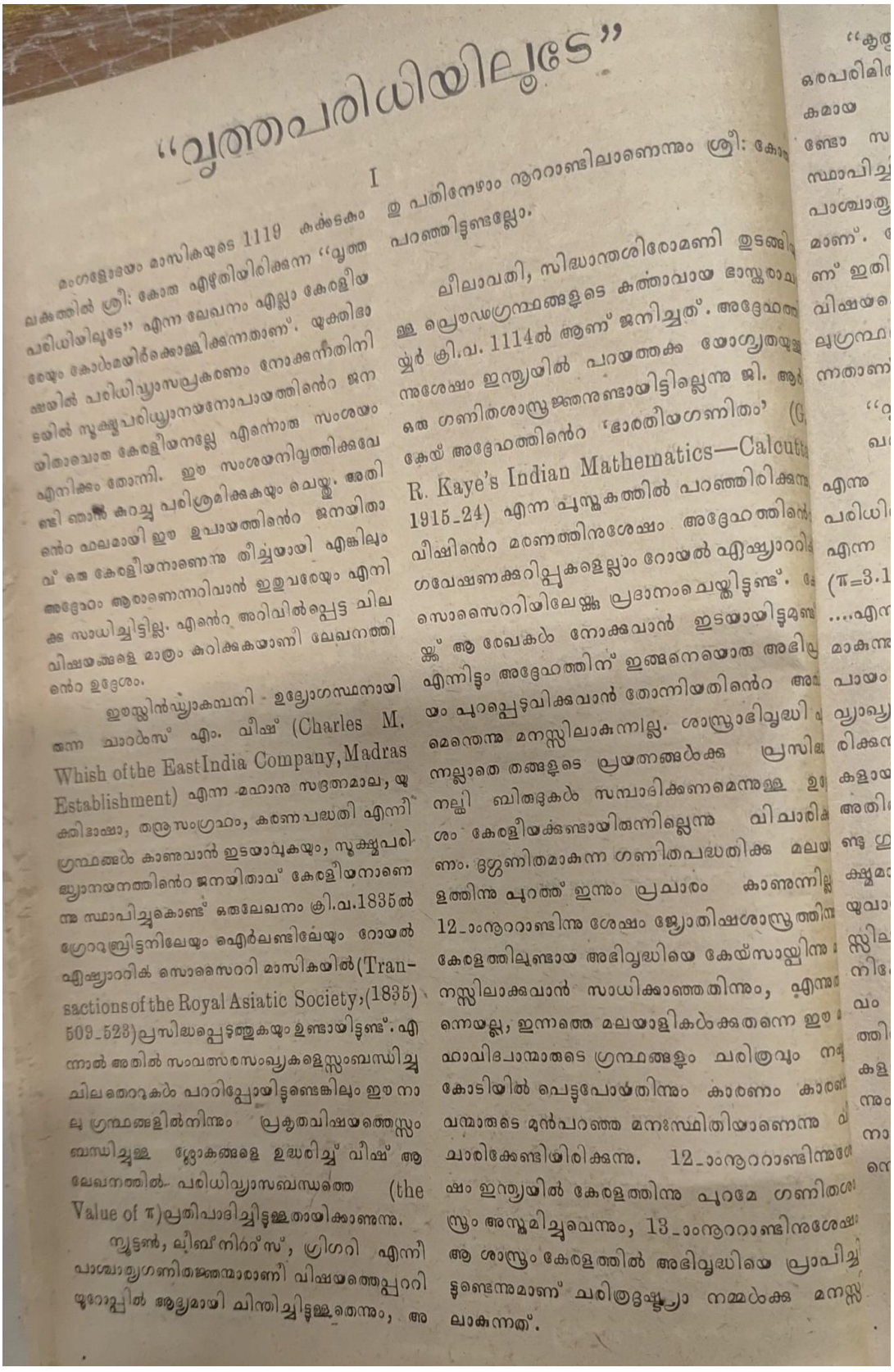
“വൃത്തപരിധിയിലൂടെ”

രാമവർമ്മ മരുത്തമ്പുരാൻ

മംഗളോദയം, 20, 1120 കന്നി (1944 സെപ്തംബർ - ഒക്ടോബർ)
പേജ് 252-256



"വൃത്തപരിധിയിലൂടേ"

I

മംഗളോദയം മാസികയുടെ 1119 കർക്കടകം ലക്കത്തിൽ ശ്രീ: കോത എഴുതിയിരിക്കുന്ന "വൃത്ത പരിധിയിലൂടേ" എന്ന ലേഖനം എല്ലാ കേരളീയരേയും കോൾമയിർക്കൊള്ളിക്കുന്നതാണ്. യുക്തിഭദ്രയിൽ പരിധിവ്യാസപ്രകരണം നോക്കുന്നീതിനിടയിൽ സൂക്ഷ്മപരിധ്യാനയനോപായത്തിന്റെ ജനയിതാവായ കേരളീയനല്ലേ എന്നൊരു സംശയം എനിക്കും തോന്നി. ഈ സംശയനിവൃത്തിക്കുവേണ്ടി ഞാൻ കുറച്ചു പരിശ്രമിക്കുകയും ചെയ്തു. അതിന്റെ ഫലമായി ഈ ഉപായത്തിന്റെ ജനയിതാവ് ഒരു കേരളീയനാണെന്നു തീച്ചയായി എങ്കിലും അദ്ദേഹം ആരാണെന്നറിവാൻ ഇതുവരേയും എനിക്കു സാധിച്ചിട്ടില്ല. എന്റെ അറിവിൽപ്പെട്ട ചില വിഷയങ്ങളെ മാത്രം കുറിക്കുകയാണീ ലേഖനത്തിന്റെ ഉദ്ദേശം.

ഈസ്റ്റിൻഡ്യാകമ്പനി - ഉദ്യോഗസ്ഥനായിരുന്ന ചാൾസ് എം. വീഷ് (Charles M. Whish of the East India Company, Madras Establishment) എന്ന മഹാനു സദ്ഗമാല, യുക്തിഭാഷാ, തന്ത്രസംഗ്രഹം, കരണപദ്ധതി എന്നീ ഗ്രന്ഥങ്ങൾ കാണുവാൻ ഇടയാവുകയും, സൂക്ഷ്മപരിജ്ഞാനയനത്തിന്റെ ജനയിതാവ് കേരളീയനാണെന്നു സ്ഥാപിച്ചുകൊണ്ട് ഒരുലേഖനം ക്രി.വ.1835ൽ ഗ്രേറ്റ്ബ്രിട്ടനിലേയും ഐർലണ്ടിലേയും റോയൽ ഏഷ്യാറ്റിക് സൊസൈറ്റി മാസികയിൽ (Transactions of the Royal Asiatic Society, (1835) 509-523) പ്രസിദ്ധപ്പെടുത്തുകയും ഉണ്ടായിട്ടുണ്ട്. എന്നാൽ അതിൽ സംവത്സരസംഖ്യകളെസ്സംബന്ധിച്ച ചിലതെറ്റുകൾ പറ്റിപ്പോയിട്ടുണ്ടെങ്കിലും ഈ നാലു ഗ്രന്ഥങ്ങളിൽനിന്നും പ്രകൃതവിഷയത്തെസ്സംബന്ധിച്ചുള്ള ശ്ലോകങ്ങളെ ഉദ്ധരിച്ച് വീഷ് ആ ലേഖനത്തിൽ പരിധിവ്യാസബന്ധത്തെ (the Value of π) പ്രതിപാദിച്ചിട്ടുള്ളതായിക്കാണുന്നു.

ന്യൂട്ടൻ, ലീബ്നിറ്റ്സ്, ഗ്രിഗറി എന്നീ പാശ്ചാത്യഗണിതജ്ഞന്മാരാണീ വിഷയത്തെപ്പറ്റി യൂറോപ്പിൽ ആദ്യമായി ചിന്തിച്ചിട്ടുള്ളതെന്നും, അതു പതിനേഴാം നൂറ്റാണ്ടിലാണെന്നും ശ്രീ: കോത പറഞ്ഞിട്ടുണ്ടല്ലോ.

ലീലാവതി, സിദ്ധാന്തശിരോമണി തുടങ്ങിയുള്ള പ്രൌഢഗ്രന്ഥങ്ങളുടെ കർത്താവായ ഭാസ്കരാചാര്യർ ക്രി.വ. 1114ൽ ആണ് ജനിച്ചത്. അദ്ദേഹത്തിനുശേഷം ഇന്ത്യയിൽ പറയത്തക്ക യോഗ്യതയുള്ള ഒരു ഗണിതശാസ്ത്രജ്ഞനുണ്ടായിട്ടില്ലെന്നു ജി. ആർ കേയ് അദ്ദേഹത്തിന്റെ 'ഭാരതീയഗണിതം' (G. R. Kaye's Indian Mathematics—Calcutta 1915-24) എന്ന പുസ്തകത്തിൽ പറഞ്ഞിരിക്കുന്നു. വീഷിന്റെ മരണത്തിനുശേഷം അദ്ദേഹത്തിന്റെ ഗവേഷണക്കുറിപ്പുകളെല്ലാം റോയൽ ഏഷ്യാറ്റിക് സൊസൈറ്റിയിലേയ്ക്കു പ്രദാനംചെയ്തിട്ടുണ്ട്. ആ രേഖകൾ നോക്കുവാൻ ഇടയായിട്ടുണ്ട്. എന്നിട്ടും അദ്ദേഹത്തിന് ഇങ്ങനെയൊരു അഭിപ്രായം പുറപ്പെടുവിക്കുവാൻ തോന്നിയതിന്റെ അടിസ്ഥാനമെന്തെന്നു മനസ്സിലാകുന്നില്ല. ശാസ്ത്രാഭിവൃദ്ധി ഉണ്ടാകുന്നുള്ളതെ തങ്ങളുടെ പ്രയത്നങ്ങൾക്കു പ്രസിദ്ധി നല്കി ബിരുദങ്ങൾ സമ്പാദിക്കണമെന്നുള്ള ഉദ്ദേശം കേരളീയർക്കുണ്ടായിരുന്നില്ലെന്നു വിചാരിക്കണം. ഉല്കൃഷ്ടഗണിതമാകുന്ന ഗണിതപദ്ധതിക്കു മലയാളത്തിനു പുറത്ത് ഇന്നും പ്രചാരം കാണുന്നില്ല. 12-ാംനൂറ്റാണ്ടിനു ശേഷം ജ്യോതിഷശാസ്ത്രത്തിനു കേരളത്തിലുണ്ടായ അഭിവൃദ്ധിയെ കേയ്സായ്പിനു മനസ്സിലാക്കുവാൻ സാധിക്കാഞ്ഞതിനും, എന്നുമാത്രമല്ല, ഇന്നത്തെ മലയാളികൾക്കുതന്നെ ഈ മഹാവിദ്വാന്മാരുടെ ഗ്രന്ഥങ്ങളും ചരിത്രവും നശിച്ചുകോടിയിൽ പെട്ടുപോയതിനും കാരണം കാരണവന്മാരുടെ മുൻപറഞ്ഞ മനഃസ്ഥിതിയാണെന്നു വിചാരിക്കേണ്ടിയിരിക്കുന്നു. 12-ാംനൂറ്റാണ്ടിനുശേഷം ഇന്ത്യയിൽ കേരളത്തിനു പുറമേ ഗണിതശാസ്ത്രം അസ്തമിച്ചുവെന്നും, 13-ാംനൂറ്റാണ്ടിനുശേഷം ആ ശാസ്ത്രം കേരളത്തിൽ അഭിവൃദ്ധിയെ പ്രാപിച്ചിട്ടുണ്ടെന്നുമാണ് ചരിത്രദൃഷ്ട്യാ നമ്മൾക്കു മനസ്സിലാകുന്നത്.

"കൃത്യമായ ഫലം (പരിധ്യാനയനത്തിങ്കൽ) ഒരപരിമിതശ്രേണികൊണ്ടല്ലാതെ മിതസംഖ്യാത്മകമായ പൂർണ്ണസംഖ്യകൊണ്ടോ അംശസംഖ്യകൊണ്ടോ സാധിക്കയില്ലെന്നു ഗണിതപണ്ഡിതന്മാർ സ്ഥാപിച്ചിട്ടുണ്ട്." ഈ അപരിമിതശ്രേണിയെ പാശ്ചാത്യന്മാർ കണ്ടുപിടിച്ചതു 17-ാംനൂറ്റാണ്ടിലുമാണ്. കേരളീയർ ഏകദേശം ഏഴു കാലത്തിലാണ് ഇതിനെ മനസ്സിലാക്കിയതെന്നറിവാൻ, ഈ വിഷയത്തെ ഉപപാദിച്ചിരിക്കുന്ന മൻപറഞ്ഞ നാലുഗ്രന്ഥങ്ങളുടെ കാലം നിർണ്ണയിച്ചാൽ മതിയാകുന്നതാണ്. അതിനു നമുക്കു ശ്രമിക്കാം.

"വ്യാസേഭനന്ദാഗ്നിഹതേ വിഭക്തേ
ഖഭാണസൂര്യൈഃ പരിധിസ്സസൂക്ഷ്മഃ"

എന്നു ലീലാവതിയിൽ ഭാസ്കരാചാര്യർ വ്യാസപരിധിസംബന്ധത്തെ പറഞ്ഞിരിക്കുന്നു. 1250 എന്ന വ്യാസത്തിനു പരിധി 3927 എന്ന്. (π=3.14160.) [ശരിയായിട്ടുള്ള ബന്ധം 3.15159 ....എന്നിരിക്കുമ്പോൾ] ഇത് ഏറെക്കുറെ സ്ഥൂലമാകുന്നു. ഈ ബന്ധത്തെ നിശ്ചയിക്കുവാനുള്ള ഉപായം ഗണേശൻ അദ്ദേഹത്തിന്റെ ലീലാവതീവ്യാഖ്യാനത്തിൽ പറഞ്ഞിട്ടുണ്ട്. 1250 വ്യാസമായിരിക്കുന്ന ഒരു വൃത്തത്തിന്റെ അകത്തു 384 സമഭുജകളായിരിക്കുന്ന ഒരു ക്ഷേത്രത്തെ ഉണ്ടാക്കിയാൽ അതിൽ ഓരോ ഭുജയുടെ പരിമാണത്തെ 384 കൊണ്ടു ഗുണിച്ചാൽ 3927 ആകുന്ന പരിധി കിട്ടും. സൂക്ഷ്മമായിരിക്കുന്ന പരിധിവ്യാസബന്ധത്തെ അറിയുവാൻ ഉതകുന്ന ശ്രേണിയെ ഭാസ്കരാചാര്യർ മനസ്സിലാക്കിയിരുന്നില്ലെന്നാണ് ഇതുകൊണ്ട് അനുമാനിക്കേണ്ടത്. അതുകൊണ്ട് ഈ ശ്രേണിയുടെ ഉത്ഭവം 12-ാംനൂറ്റാണ്ടിനുശേഷമാണ്. അക്കാലത്തിന്നിപ്പുറം കേരളീയേതരന്മാരായ പ്രതിഭാശാലികളായ ഗണിതജ്ഞന്മാർ ഇന്ത്യയിലുണ്ടായിട്ടില്ലെന്നും ചരിത്രകാരന്മാർ പറഞ്ഞിരിക്കുന്നു. അതിനാൽ ഈ ശ്രേണിയുടെ ഉത്ഭവം കേരളത്തിൽത്തന്നെയാണെന്നു നിശ്ചയിക്കണം.

വ്യാസേ വാരിധിനിഹതേ
രൂപഘ്നേ വ്യാസസാഗരാഭിഹതേ
ശിശരാദി വിഷമസംഖ്യാ
ക്ഷേത്രമുണം സപം വൃത്തകുമാൽ കൃത്സ്നാൽ.

തസ്യാ ഉത്ഭവഗതായാ
സമസംഖ്യം തദ്ദളം ത്രിണോനേ സ്യാൽ
തദ്വർഗ്ഗോ രൂപയുതോ
ഹാരോ വ്യാസാബ്ധിഘ്നാതതഃ പ്രഗുണൽ.

താഭ്യാമാപ്തം സപമൂനേ
കൃതേ ധനേ ശോധനീയഞ്ച കരണീയം
സൂക്ഷ്മഃ പരിധിസ്സ സ്യാൽ
ബഹുകൃതേപാ ഹരണതോഽതിസൂക്ഷ്മഃ.

തന്ത്രസംഗ്രഹത്തിലെ ഈ ശ്ലോകം എനിക്കാണും ശ്രീ. കോരു ഉദ്ധരിച്ചിട്ടുണ്ടല്ലോ. ഇതിന്റെ പൂർവ്വഭാഗം മൂലശ്രേണി [പരിധി=4 വ്യാസം$\times\frac{1}{3}+(\frac{1}{5}-\frac{1}{7}....)$]യെ വിവരിച്ചിട്ടുണ്ട്. ഈ ക്രിയകൊണ്ടു പരിധി വളരെ കൂടുകയാണെങ്കിൽ സൂക്ഷ്മമായ ഒരു പരിധിയെ ലഭിക്കുവാൻ വളരെ സമയം വ്യയം ചെയ്യേണ്ടിവരും. തൽപരിഹാരത്തിന്നായി ഉത്തരഭാഗംകൊണ്ട് ഒരു സംസ്കാരത്തെയാണ് പറയുന്നത്. എളുപ്പത്തിന്നായിക്കൊണ്ടു വേറേയും പല ശ്രേണികളും ഈ ഗ്രന്ഥത്തിൽത്തന്നെ പറഞ്ഞിട്ടുണ്ട്. മൂലശ്രേണിയും, സംസ്കാരങ്ങളും ഇവയിൽനിന്നുത്ഭവിച്ച മറ്റു ശ്രേണികളും യുക്തിഭാഷയിൽ സയുക്തികം പ്രതിപാദിക്കപ്പെട്ടിട്ടുമുണ്ട്. യുക്തിഭാഷാ തന്ത്രസംഗ്രഹത്തെ അനുസരിച്ചാണെഴുതിയിട്ടുള്ളതെന്നു തൽഗ്രന്ഥകാരൻതന്നെ പറഞ്ഞിരിക്കുന്നു.

ഈ ശ്രേണികളിൽ പലതും സദ്ഗണമാലയിലും ഉദ്ധരിച്ചുകാണുന്നുണ്ട്. ഈ ഗ്രന്ഥത്തിന്റെ കർത്താവു പോർളാതിരി (കടത്തനാട്) രാജകുടുംബത്തിലെ അംഗമായ ശങ്കരവർമ്മയാണെന്നും നിർമ്മാണകാലകലി 'ഖോകാംബേ സിദ്ധസേവ്യേ' എന്നാണെന്നും ഗ്രന്ഥത്തിൽ പറഞ്ഞിട്ടുണ്ട്. അപ്പോൾ നിർമ്മാണം ക്രി. വ. 1819-ലാണ്. യുക്തിഭാഷയുടെ കർത്താവിന്റെ പേർ ബ്രഹ്മദത്തൻനമ്പൂതിരി എന്നാണെന്നും നിർമ്മാണകാലകലി 'യേ ഗോളപഥസ്ഥാസ്യുഃ' എന്നാണെന്നും അതിൽ കൊടുത്തിട്ടുണ്ട്. അദ്ദേഹത്തിന്റെ ഇല്ലപ്പേർ 'പറങ്ങോട്ട്' എന്നാണെന്നു 'നമ്പൂതിരിമാർ' എന്ന പുസ്തകത്തിൽ കാണുന്നു: ഇതിന്റെ നിർമ്മാണകാലം ക്രി. വ. 1750.

ആര്യഭടീയം ഭാഷ്യത്തിന്റെ ഗ്രന്ഥകർത്താവു കേളല്ലൂർ നീലകണ്ഠസോമയാജിപ്പാടാണെന്നു ഗ്ര

ന്ഥത്തിൽനിന്നുതന്നെ വ്യക്തമാകുന്നുണ്ട്. ഭാഷ്യത്തിൽ പലേടത്തും 'തന്ത്രസംഗ്രഹേ മയാപ്യുക്തം' എന്നു കാണുന്നതുകൊണ്ടു തന്ത്രസംഗ്രഹത്തിന്റെ കർത്താവും കേളല്ലൂർ സോമയാജിപ്പാടാണെന്നു തീച്ചയായി. ഇദ്ദേഹത്തിന്റെ ജനനം കലിവർഷം 4545-ൽ (അതായതു ക്രി. വ. 1443-ൽ) ആണെന്ന് അദ്ദേഹത്തിന്റെ സിദ്ധാന്തദർപ്പണത്തിൽനിന്നും തെളിയുന്നു. തന്ത്രസംഗ്രഹത്തിന്റെ നിർമ്മാണകാലകലി—ഹേ വിഷ്ണോ നിഹിതം കൽസ്നം—എന്ന്. അതായതു ക്രി. വ. 1500-ൽ. പതിനഞ്ചാം നൂറാണ്ടിൽതന്നെ ഈ ശ്രേണികൾക്കും സംസ്കാരങ്ങൾക്കും കേരളത്തിൽ പ്രചാരം സിദ്ധിച്ചു എന്നതു തീച്ചതന്നെ. സോമയാജിപ്പാട് സാക്ഷാൽ ദ്രഗ്ഗണിതകർത്താവായ ആലത്തൂർഗ്രാമത്തിൽ വടശ്ശേരി എന്ന ഇല്ലത്തു പരമേശ്വരാചാര്യരുടെ മകൻ ദാമോദരൻനമ്പൂതിരിയുടെ ശിഷ്യനാണെന്നും ആര്യഭടീയം ഭാഷ്യത്തിൽനിന്നും തെളിയുന്നുണ്ട്. ഈ പരമേശ്വരാചാര്യരുടെ കാലം—ഏവം ദ്രഗ്ഗണിതം ശാകേ ത്രിഷ്വവിശ്വമിതേ കൃതം—എന്നതിൽനിന്നു മനസ്സിലാക്കാം. അതു ശാകവർഷം 1353,—അതായതു ക്രി. വ. 1430. 55 കൊല്ലത്തെ പരിക്ഷണത്തിന്റെ ഫലമായിട്ടാണ് ഈ ഗ്രന്ഥം അദ്ദേഹം നിർമ്മിച്ചതെന്നും പറയുന്നുണ്ട്.

കരണപദ്ധതിയിലും ഈ ശ്രേണികളിൽ ചിലവ ഉദ്ധരിച്ചു കാണുന്നുണ്ട്. കരണപദ്ധതിക്ക് ഏകദേശം 250 കൊല്ലത്തെ പഴക്കമുണ്ടെന്നു 20 കൊല്ലം മുമ്പു ശ്രീ. കെ. എസ്. നമ്പൂതിരിപ്പാട് ഒരു മുഖവുരയിൽ എഴുതിയിട്ടുള്ളതായി ശ്രീ. കോര പറയുന്നു. അപ്പോൾ അതിന്റെ നിർമ്മാണകാലം ഏകദേശം ക്രി. വ. 1670-ലായിരിക്കണമെന്നും പറഞ്ഞിട്ടുണ്ട്. കരണപദ്ധതി വ്യാഖ്യാനത്തിൽ 'തേജോ സൂരസ്ഥം സകൃൽ' എന്നൊരു അഹർഗണസംഖ്യ (1772786—അതായതു ക്രി.വ.1752-ൽ) ഉള്ളതുകൊണ്ട് 1752-നു മുമ്പാണ് ഈ ഗ്രന്ഥം എഴുതിയതെന്നും വിചാരിക്കണമെന്നു ശ്രീ. കോര പറയുന്നു. ഗ്രന്ഥത്തിൽ ഒടുവിലത്തെ ശ്ലോകത്തിൽ കാണുന്ന 'ഗണിതമേതൽ സമ്യഗാലോക്യ' (1765653—അതായതു ക്രി. വ. 1733) എന്നതു നിർമ്മാണകാലകലിയാണെന്നു വീഷ് അഭിപ്രായപ്പെട്ടിട്ടുണ്ട്. ഇവിടെ തേജോസൂരസ്ഥം സകൃൽ എന്നതു വ്യാഖ്യാതാവിന്റെ കാലത്തെ ഒരു കലി എന്നല്ലാതെ കരണപദ്ധതിനിർമ്മാണകാലമായിട്ടു യാതൊരു ബന്ധവുമില്ലെന്നാണ് എനിക്കു തോന്നുന്നത്. കലിയെ നിർദ്ദേശിക്കുന്ന വാക്യങ്ങളുടെ ഒടുവിൽ കലി അവധായക ശബ്ദങ്ങളെ ഗണിതഗ്രന്ഥകർത്താക്കൾ സാധാരണയായി ഉപയോഗിച്ചുകാണാറുണ്ട്. 'ഗണിതമേതൽ സമ്യഗാലോക്യ' എന്നതിൽ ആവക ശബ്ദമൊന്നും കാണുന്നില്ല.

തന്നെപ്പറ്റി യാതൊന്നും വിട്ടുകൊടുക്കാത്ത ഒരു മനഃസ്ഥിതിയാണു കരണപദ്ധതികർത്താവിനു കാണുന്നത്. അദ്ദേഹത്തെപ്പറ്റി ഏതാനും കാര്യങ്ങൾ അറിവു കിട്ടുന്നത് ഒടുക്കത്തെ ശ്ലോകത്തിൽനിന്നാണ്.

ഇതി ശിവപുരനാമഗ്രാമജഃ കോപി യജ്വാ
കിമപി കരണപദ്ധത്യാഖ്യയം തന്ത്രമുച്ചൈഃ
വ്യധിതഗണിതമേതൽ സമ്യഗാലോക്യ സന്തഃ
കഥിതമിഹ വിദന്തഃ സന്തു സന്തോഷവന്തഃ.

ഗ്രന്ഥകർത്താവു ശിവപുരം ഗ്രാമ (തൃശ്ശിവപേരൂർ ഗ്രാമം) ത്തിലെ ഒരു സോമയാജിപ്പാടാണെന്നു ക്രമേ കാണുന്നുള്ളൂ. കരണപദ്ധതിയുടെ ഒരു ഭാഷ്യവ്യാഖ്യാനത്തിൽ ഇങ്ങനെ ഒരു ശ്ലോകം കാണുന്നു.

നൂതനഗൃഹസോമസുതാ
രചിതായാഃ കരണപദ്ധതേർവ്വിഭാഷാ
ഭാഷാം വിലിഖതി കശ്ചിൽ
ബാലാനാം ബോധനാർത്ഥമല്പധിയാം.

ഗ്രന്ഥകർത്താവിന്റെ പേർ നൂതനഗൃഹ സോമത്ത് (പുതുമന ചോമാതിരിപ്പാട്) എന്നാണെന്നു കാണുന്നു. ശ്രീ. വടക്കുംകൂർ രാജരാജവർമ്മരാജാവു ദ്ദേഹത്തിന്റെ കേരളീയസംസ്കൃതസാഹിത്യചരിത്രത്തിന്റെ രണ്ടാംഭാഗത്തിൽ ഗോവിന്ദഭട്ടന്റെ ഗണിതചികാഗ്രന്ഥത്തിൽനിന്ന് ഒരു ശ്ലോകമെടുത്തു കാണിച്ചിട്ടുണ്ട്.

നവീനവിപിനേ മഹീമഖഭുജാംമണിസ്സോമയാ-
ജ്യുദാരഗണകോത്ര യസ്സമവേച്ച തേനാമുനാ
വ്യലേഖി സുദൃഗ്ഭത്തമാ കരണപദ്ധതിസ്സംസ്കൃതാ
ത്രിപഞ്ചശിഖിഭൂമിതപ്രഥിതശാകസംവത്സരേ.

കരണപദ്ധതികർത്താവു പുതുവന (പുതുമന) സോമാതിരിയാണെന്നും ഗ്രന്ഥം ശാകവർഷം 1353-ൽ [illegible] പ്പിക്കപ്പെട്ടുവെന്നും പറഞ്ഞിരിക്കുന്നു. ഇതു വിശ്വസിക്കത്തക്കതാണെന്നാണെന്റെ അഭിപ്രായം.

ദ്ദേഹവും പരമേശ്വരാചാര്യരും സമകാലീനന്മാരാണെന്നൊരു ഐതിഹ്യവുണ്ട്. കരണപദ്ധതിയിൽ പരമേശ്വരാചാര്യരുടെ ദൃക്കരണങ്ങളെപ്പറ്റി പറയുന്നതായിട്ടു ഞാൻ ഓർക്കുന്നില്ല. ഗണിതപരിഷ്കരണമാർഗ്ഗത്തെപ്പറ്റി പറയുന്നുമുണ്ട്.

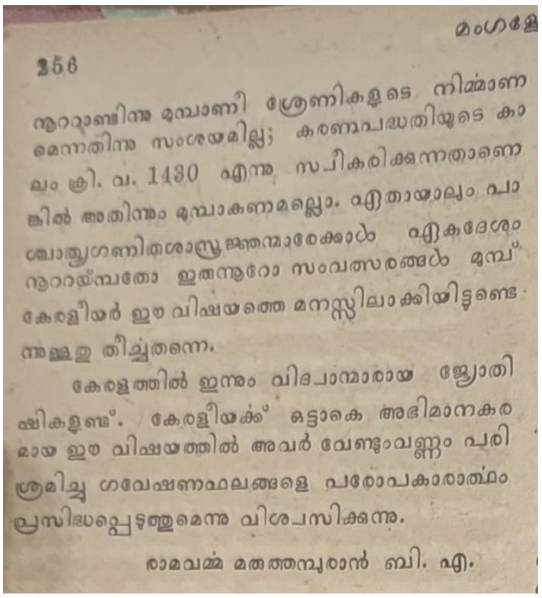
ഉല്ലണിതവും ഈ വർഷത്തിൽത്തന്നെയാണ (അതായത് ക്രി. വ. 1430) ണ്ടാക്കിയതെന്നു മുമ്പു പറഞ്ഞിട്ടുണ്ടല്ലോ. അങ്ങനെയാണെങ്കിൽ ഉല്ലണിതകർത്താവായ പരമേശ്വരാചാര്യർക്കും ഈ ശ്രേണികളെപ്പറ്റി അറിവുണ്ടായിരിക്കണം. പരമേശ്വരാചാര്യരുടെ ഭടദീപിക എന്ന ആര്യഭടീയം വ്യാഖ്യാനത്തിൽ ഈ വിഷയമൊന്നും കാണുന്നില്ല. ഭടദീപിക ആര്യഭടീയം ഭാഷ്യംപോലെ സവിസ്തരമായ ഒരു ഗ്രന്ഥമല്ലാത്തതുകൊണ്ടായിരിക്കാം. ഉല്ലണിതം ഗ്രന്ഥം ഇതുവരെയും കാണുവാൻ സാധിച്ചിട്ടില്ല. ആചാര്യരുടെ ലീലാവതീവ്യാഖ്യാനം ഭാസ്കരീയം വ്യാഖ്യാനം മുതലായ മറ്റു ഗ്രന്ഥങ്ങളും പരിശോധിക്കേണ്ടതായിട്ടാണിരിക്കുന്നത്. ഗോവിന്ദഭട്ടന്റെ വാക്യം സ്വീകരിക്കുന്നതായാൽ കേരളത്തിൽ പരിധ്യാനയനോപായം ക്രി. വ. 1430നു മുമ്പ് ഉണ്ടായിരുന്നുവെന്നു വിചാരിക്കണം.

ഈ ലേഖനം അവസാനിപ്പിക്കുന്നതിനു മുമ്പ് എന്റെ ഒരഭിപ്രായത്തെക്കൂടി വിദ്വജ്ജനസമക്ഷം അവതരിപ്പിച്ചുകൊള്ളട്ടെ. ആ മതം സൂക്ഷ്മമോ അസ്ഥൂലമോ എന്ന് അവർ തന്നെ തീർച്ചയാക്കട്ടെ. കേരളത്തിൽ സംഗ്രാമമാധവൻ എന്നൊരതിപ്രസിദ്ധനായ വിദ്വച്ചിരോമണി ഉണ്ടായതായിട്ടു പലരും കേട്ടിരിക്കാം. അദ്ദേഹം ആരാണെന്നോ അദ്ദേഹത്തിന്റെ ഗ്രന്ഥങ്ങൾ ഏതെല്ലാമാണെന്നോ വല്ലവരും ചോദിക്കുകയാണെങ്കിൽ കേരളീയരായ നമുക്കു തലതാഴ്ത്തുകതന്നെ നിവൃത്തിയുള്ളൂ. അദ്ദേഹം ഇരിങ്ങാലക്കുടെ തെക്കെവാരിയത്തെ ഒരംഗമാണെന്നു ശ്രീ. വടക്കുംകൂർ രാജരാജവർമ്മരാജാ അഭിപ്രായപ്പെട്ടിട്ടുണ്ട്. ആധുനികന്മാർ അദ്ദേഹത്തിന്റെ ഗ്രന്ഥങ്ങൾ കണ്ടതായിട്ട് എനിക്കറിവില്ല. എന്നാൽ അദ്ദേഹത്തിന്റെ വകതന്നെപ്പോട്ടകൾ അവിടവിടെയായി കാണുന്നുണ്ട്. അദ്ദേഹത്തിനെ വിദ്വൽകോടിയിലേയ്ക്കുയർത്തുവാൻ അവമാത്രം മതിയായവയാകുന്നു. ജീവപരസ്പരന്യായത്തിന്റെ (അതായത് [Sin (A+B)=Sin A cos B+cos A Sin B] മൂലകർത്താവായിട്ടാണ് അദ്ദേഹത്തിന്നു ഇന്നും കേരളത്തിൽ പ്രസിദ്ധി. പ്രകൃതവിഷയത്തിലുള്ള അദ്ദേഹത്തിന്റെ ചില വാക്യത്തെ ഇവിടെ ഉദ്ധരിക്കാം. പരിധിമാനത്തെ വളരെ സൂക്ഷ്മമായി അദ്ദേഹം നിർണ്ണയിച്ചിട്ടുണ്ട്. അത് ഇങ്ങനെയാണ്.

വിബുധനേത്രഗജാഹിഹുതാശന-
ത്രിഗുണവേദഭവാരണബാഹവഃ
നവനിഖർവ്വമിതേയുതവിസ്തരേ
പരിധിമാനമിദംജഗദുബ്ഭുധാഃ

900000000000 ($9\times10''$) എന്ന വ്യാസത്തിന്റെ പരിധി =2827433388233 എന്ന് ($\pi=3.14159265359$). ഈ ബന്ധത്തെ അനുസരിച്ച് അദ്ദേഹം 'ശ്രേഷ്ഠം നാമവരിഷ്യാനാം' എന്നു തുടങ്ങിയുള്ള ഇരുപത്തിനാലു തല്പരാദി മഹാജ്യാക്കളും ഉണ്ടാക്കിയിട്ടുണ്ട്. തല്പരാദി മഹാജ്യാനയനത്തിൽ എളുപ്പത്തിന്നായിക്കൊണ്ട് 'വിദ്വാംസ്സുന്നബലഃ' ഇത്യാദി അഞ്ചു തല്പരാദിവാക്യങ്ങളെ നിമ്മിച്ച് അവയിൽനിന്നു ജ്യാനയനോപായത്തെയും വിധിച്ചിട്ടുണ്ട്. ഇത്ര സൂക്ഷ്മമായിട്ട പരിധിയേയും മഹാജ്യാക്കളേയും ഉണ്ടാക്കണമെങ്കിൽ മൻപ്രസ്താവിച്ച അപരിമിതശ്രേണികളുടെ സഹായം കൂടാതെ നിവൃത്തിയില്ല എന്നു ഗണിതത്തിൽ കയ്യുവെച്ച ഏതൊരുത്തനും അറിയാവുന്നതാണ്. സംഗ്രാമമാധവൻ ഈ ശ്രേണികളെ ഉണ്ടാക്കുകയും സൂക്ഷ്മപരിധ്യാനയനത്തെയും തല്പരാദിമഹാജ്യാനയനത്തെയുംകൊണ്ട് അവയുടെ പ്രയോജനത്തെ കാണിക്കുകയുംചെയ്തു എന്നു വിചാരിക്കുന്നതിൽ അസാംഗത്യമുണ്ടെന്നു തോന്നുന്നില്ല.

ഈ മാധവൻ ഏതുകാലത്താണു ജീവിച്ചിരുന്നതെന്നു സ്പഷ്ടമായിട്ടു നിശ്ചയിക്കുവാൻ യാതൊരു മാർഗ്ഗവുമില്ല. ഈ പരിധിമാനവും തല്പരാദിമഹാജ്യാക്കളും വിദ്വാംസ്സുന്നബലഃ ഇത്യാദി വാക്യങ്ങളും മാധവോദിതങ്ങളായിട്ട കേളല്ലൂർ ചോമാതിരിപ്പാട്ട് അദ്ദേഹത്തിന്റെ ആര്യഭടീയം ഭാഷ്യത്തിൽ ഉദ്ധരിച്ചിട്ടുണ്ട്. വിദ്വാംസ്സുന്നബലഃ ഇത്യാദി വാക്യങ്ങൾ കരണപദ്ധതിയിൽ ചേർത്തും കാണുന്നുണ്ട്. അതുകൊണ്ട് മാധവൻ കരണപദ്ധതികർത്താവിന്റെയും കേളല്ലൂർ ചോമാതിരിപ്പാടിന്റേയും പൂർവ്വകാലീനനാണെന്നു സിദ്ധമായി. ക്രി. വ. പതിനഞ്ചാം



മംഗളോ

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നൂറുണ്ടിനു മുമ്പാണീ ശ്രേണികളുടെ നിർമ്മാണ
മെന്നതിനു സംശയമില്ല; കരണപദ്ധതിയുടെ കാ
ലം ക്രി. വ. 1430 എന്നു സ്ഥിരീകരിക്കുന്നതാണെ
ങ്കിൽ അതിനും മുമ്പാകുമല്ലൊ. എതായാലും പാ
ശ്ചാത്യഗണിതശാസ്ത്രജ്ഞന്മാരേക്കാൾ എകദേശം
നൂറായ്പതോ ഇരുനൂറോ സംവത്സരങ്ങൾ മുമ്പ്
കേരളീയർ ഈ വിഷയത്തെ മനസ്സിലാക്കിയിട്ടുണ്ടെ
ന്നുള്ളതു തീച്ചയത്രെ.

കേരളത്തിൽ ഇന്നും വിദ്വാന്മാരായ ജ്യോതി
ഷികളുണ്ട്. കേരളീയർക്ക് ഒട്ടാകെ അഭിമാനകര
മായ ഈ വിഷയത്തിൽ അവർ വേണ്ടുംവണ്ണം പരി
ശ്രമിച്ചു ഗവേഷണഫലങ്ങളെ പരോപകാരാർത്ഥം
പ്രസിദ്ധപ്പെടുത്തുമെന്നു വിശ്വസിക്കുന്നു.

രാമവർമ്മ മരുത്തമ്പുരാൻ ബി. എ.

Paper 5

K. Kunjunni Raja (1944)

Samgamagrāma Mādhavan (in Malayalam)

Kairali Magazine (Malayalam), December 1944, 250–252.

K. Kunjunni Raja

K. Kunjunni Raja (1920–2005) had his education at Thrissur, Chennai and London. After working as a lecturer in various government colleges in Kerala, he joined Department of Sanskrit, Madras University in 1951. While at Chennai, he was closely associated with the activities of the Adyar Library. He was also associated with the academic activities of several institutions like Central Sahithya Akademy, Rashtriya Sanskrit Sansthan, and Universities. He was prolific writer having authored about 30 books and more than 150 articles in English and Malayalam.

Source: A scanned copy of the article was obtained from the library attached to the Ramavarma Appan Thampuran Smarakam, Ayyanthol, Thrissur, Kerala, curtsey the authorities of the Smarakam, on 7 January 2025.

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Malayalam article

Samgamagrāma Mādhavan

K. Kunjunni Raja

Kairali vol. 29, no. 9, December 1944 – January 1945

pages 250-252

സംഗമഗ്രാമമാധവൻ

കെ. കുഞ്ഞുണ്ണി രാജാ

കൈരളി 29, ലക്കം 9, 1120 ധനു
(ഡിസംബർ 1944 – ജനുവരി 1945)
പേജ് 250-252

സംഗമഗ്രാമ മാധവൻ

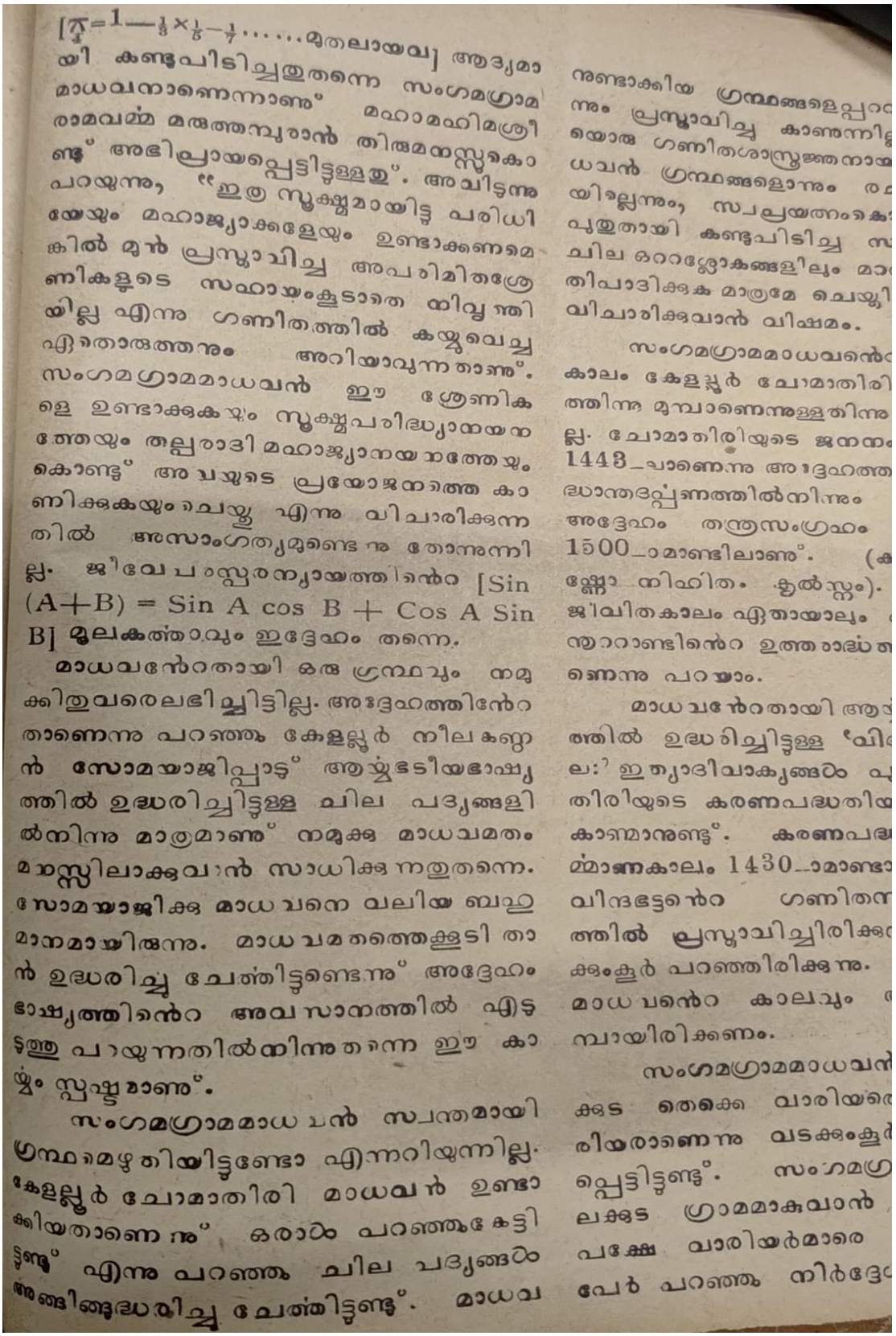
(കെ. കുഞ്ഞുണ്ണി രാജാ എം. എ.)

കേരളീയ ഗണിതശാസ്ത്രജ്ഞന്മാരിൽ പ്രധാനമായൊരു സ്ഥാനമർഹിക്കുന്ന ആളാണ് സംഗമഗ്രാമമാധവൻ. ഈ അദ്ദേഹത്തിന്റെ ജീവിതകാലത്തെപ്പറ്റിയോ കൃതികളെപ്പറ്റിയോ ആയ വിവരങ്ങൾ ഇനിയും ലഭിക്കേണ്ടിയിരിക്കുന്നതേ ഉള്ളൂ. കേരളജ്യോതിഷത്തിൽ സോമയാജിപ്പാട്ട് ആയ്ക്കുടി ശങ്കുണ്ണിയാശാൻ മാധവമതത്തെ അത്യാദരപൂർവ്വം ഒട്ടുകൂടി അവിടവിടെ ഉദ്ധരിച്ചിട്ടുള്ളതിനാൽ ഈ മഹാന്റെ കാലത്തെപ്പറ്റി ഏകദേശമെന്നെങ്കിലും നാം അറിയാതെ പോകുമായിരുന്നു. കേരളീയ പണ്ഡിതന്മാരുടെ അസാമാന്യമായ അതിവിനയം നിമിത്തം ഗ്രന്ഥത്തിന്റെ പേരുപോലും ശാസ്ത്രത്തിന്റെ പേരായിട്ടുള്ള നാട്ടുപതിവിനും ഇതുതന്നെ കാരണമാണെന്നു തോന്നുന്നു.

പരിധിസംബന്ധത്തെ ഇത്രയധികം സൂക്ഷ്മമായി നിർണ്ണയിച്ചിട്ടുള്ള

"ചിബുധൈർനഗജാഹി ഹതാശനത്രിഗുണ വേദഭവാരണ ബാഹവഃ
നവനിഖർവ്വമിതേ വൃത്തവിസ്തരേ
പരിധിമാനമിദം ജഗദുർബ്ബുധാഃ"

എന്നതുപ്രകാരം 900000000000 (തൊണ്ണൂറായിരംകോടി) എന്ന വ്യാസത്തിനു് 2827433388233 എന്ന പരിധി [ചിബുധർ = 33, നേത്രം = 1, ഗജം = 8, അഹി=8, ഹുതാശനൻ=3, ത്രി=3, ഗുണം=3, വേദം=4, ഭം=27, വാരണം=8, ബാഹു=2]. അപ്പോൾ പരിധി വ്യാസം ബന്ധം (π) = 3 1415 92653 59. ഇതിനെ അനുസരിച്ചു് ഇരുപത്തിനാലു തല്പരാദി മഹാജ്യാക്കളും അദ്ദേഹം ഉണ്ടാക്കിയിട്ടുണ്ടു്. 'ചിദ്ഘനം സംഗുണബലം' എന്നു തുടങ്ങിയ അഞ്ചു തല്പരാദിവാക്യങ്ങൾ നിർമ്മിച്ചു് അവയിൽനിന്നു ജ്യാക്കൾ കണക്കാക്കുന്നതിനുള്ള ഉപായത്തെയും അദ്ദേഹം നിർദ്ദേശിച്ചിട്ടുണ്ടു്. പരിധി വ്യാസബന്ധത്തെപ്പറ്റിയുള്ള അപരിമിതശ്രേണികളും



[$\frac{\pi}{4} = 1 - \frac{1}{3} \times \frac{1}{5} - \frac{1}{7} \ldots\ldots$ മുതലായവ] ആദ്യമാ
യി കണ്ടുപിടിച്ചതുതന്നെ സംഗമഗ്രാമ
മാധവനാണെന്നാണു് മഹാമഹിമശ്രീ
രാമവർമ്മ മരുത്തമ്പുരാൻ തിരുമനസ്സുകൊ
ണ്ടു് അഭിപ്രായപ്പെട്ടിട്ടുള്ളതു്. അവിടുന്നു
പറയുന്നു, "ഇതു സൂക്ഷ്മമായിട്ടു പരിധി
യേയും മഹാജ്യാക്കളേയും ഉണ്ടാക്കണമെ
ങ്കിൽ മുൻ പ്രസ്താവിച്ച അപരിമിതശ്രേ
ണികളുടെ സഹായംകൂടാതെ നിവൃത്തി
യില്ല എന്നു ഗണിതത്തിൽ കയ്യുവെച്ച
ഏതൊരുത്തനും അറിയാവുന്നതാണു്.
സംഗമഗ്രാമമാധവൻ ഈ ശ്രേണിക
ളെ ഉണ്ടാക്കുകയും സൂക്ഷ്മപരിധ്യാനമ
ത്തേയും തല്പരാദി മഹാജ്യാനയനത്തേയും
കൊണ്ടു് അവയുടെ പ്രയോജനത്തെ കാ
ണിക്കുകയും ചെയ്തു എന്നു വിചാരിക്കുന്ന
തിൽ അസാംഗത്യമുണ്ടെന്നു തോന്നുന്നി
ല്ല. ജീവപാസ്പരന്യായത്തിന്റെ [Sin
(A+B) = Sin A cos B + Cos A Sin
B] മൂലകർത്താവും ഇദ്ദേഹം തന്നെ.

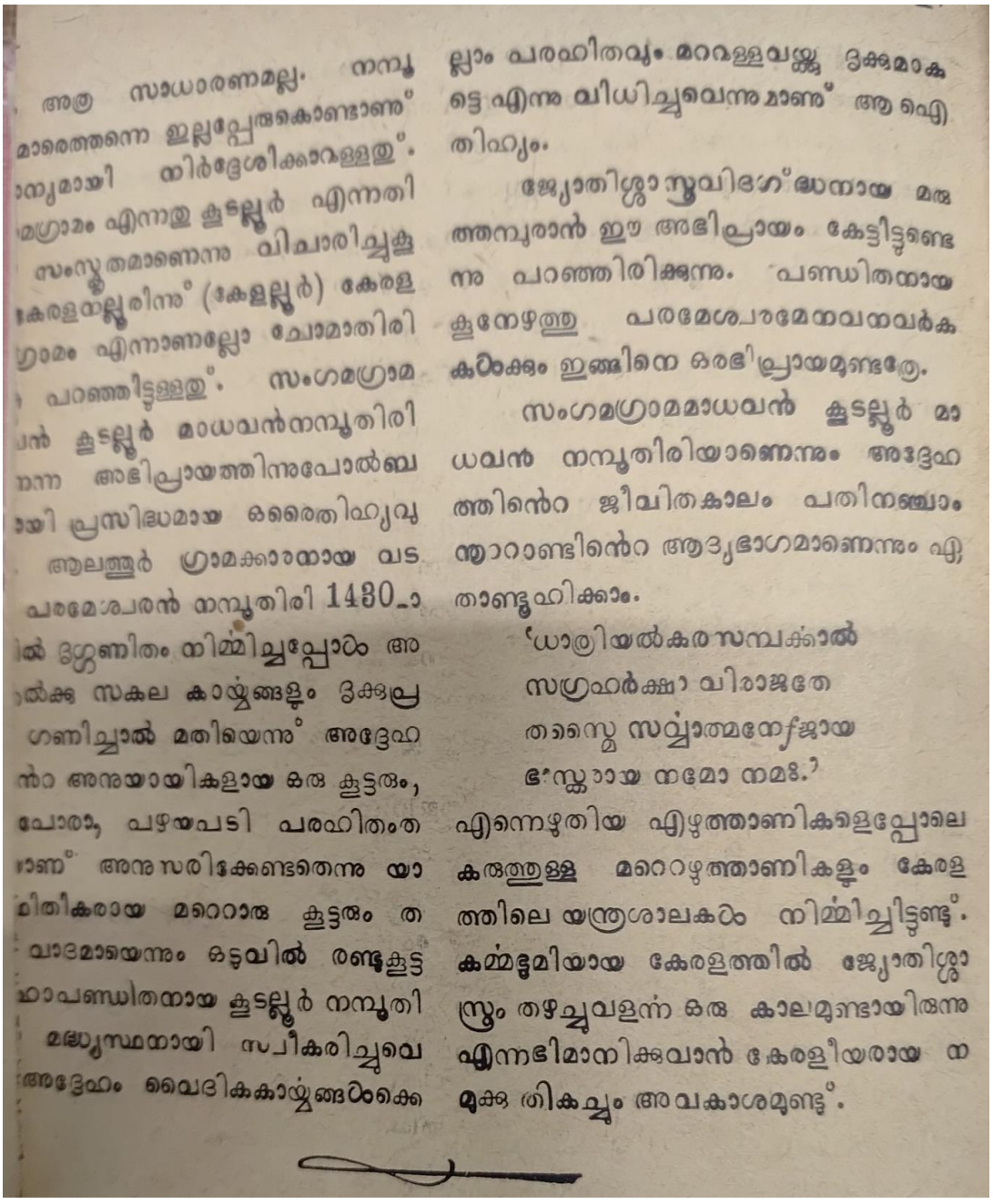
മാധവന്റേതായി ഒരു ഗ്രന്ഥവും നമു
ക്കിതുവരെ ലഭിച്ചിട്ടില്ല. അദ്ദേഹത്തിന്റേ
താണെന്നു പറഞ്ഞു കേളല്ലൂർ നീലകണ്ഠ
ൻ സോമയാജിപ്പാടു് ആര്യഭടീയഭാഷ്യ
ത്തിൽ ഉദ്ധരിച്ചിട്ടുള്ള ചില പദ്യങ്ങളി
ൽനിന്നു മാത്രമാണു് നമുക്കു മാധവമതം
മനസ്സിലാക്കുവാൻ സാധിക്കുന്നതുതന്നെ.
സോമയാജിക്കു മാധവനെ വലിയ ബഹു
മാനമായിരുന്നു. മാധവമതത്തെക്കുടി താ
ൻ ഉദ്ധരിച്ചു ചേർത്തിട്ടുണ്ടെന്നു് അദ്ദേഹം
ഭാഷ്യത്തിന്റെ അവസാനത്തിൽ എടു
ത്തു പറയുന്നതിൽനിന്നുതന്നെ ഈ കാ
ര്യം സ്പഷ്ടമാണു്.

സംഗമഗ്രാമമാധവൻ സ്വന്തമായി
ഗ്രന്ഥമെഴുതിയിട്ടുണ്ടോ എന്നറിയുന്നില്ല.
കേളല്ലൂർ ചോമാതിരി മാധവൻ ഉണ്ടാ
ക്കിയതാണെന്നു് കരാമ പറഞ്ഞുകേട്ടി
ട്ടുണ്ടു് എന്നു പറഞ്ഞു ചില പദ്യങ്ങൾ
അങ്ങിങ്ങുദ്ധരിച്ചു ചേർത്തിട്ടുണ്ടു്. മാധവ
നുണ്ടാക്കിയ ഗ്രന്ഥങ്ങളെപ്പറ
ന്നും പ്രസ്താവിച്ച കാണന്നിട്ട
യൊരു ഗണിതശാസ്ത്രജ്ഞനായ
ധവൻ ഗ്രന്ഥങ്ങളൊന്നും ര
യില്ലെന്നും, സപ്രയത്നംകൊ
പുതുതായി കണ്ടുപിടിച്ച സ
ചില ഒറ്റശ്ലോകങ്ങളിലും മാ
തിപാദിക്കുക മാത്രമേ ചെയ്തി
വിചാരിക്കുവാൻ വിഷമം.

സംഗമഗ്രാമമാധവന്റെ
കാലം കേളല്ലൂർ ചോമാതിരി
ത്തിനു മുമ്പാണെന്നുള്ളതിനു
ല്ല. ചോമാതിരിയുടെ ജനനം
1443–ാമാണെന്നു അദ്ദേഹത്ത
ഭാന്തദർപ്പണത്തിൽനിന്നും
അദ്ദേഹം തന്ത്രസംഗ്രഹം
1500–ാമാണ്ടിലാണു്. (ക
ഷ്ണോ നിഹിതം കുൽസ്നം).
ജീവിതകാലം ഏതായാലും
നൂറാണ്ടിന്റെ ഉത്തരാർദ്ധ
ണെന്നു പറയാം.

മാധവന്റേതായി ആര്യ
ത്തിൽ ഉദ്ധരിച്ചിട്ടുള്ള 'ഖില
ലഃ' ഇത്യാദിവാക്യങ്ങൾ പു
തിരിയുടെ കരണപദ്ധതിയ
കാണുന്നുണ്ടു്. കരണപദ്ധ
മ്മാണകാലം 1430–ാമാണ്ടോ
വിന്ദഭട്ടന്റെ ഗണിതസ
ത്തിൽ പ്രസ്താവിച്ചിരിക്കു
ക്കുംകൂർ പറഞ്ഞിരിക്കുന്നു.
മാധവന്റെ കാലവും
ന്വായിരിക്കണം.

സംഗമഗ്രാമമാധവൻ
ക്കുട തെക്കെ വാരിയരെ
രിയരാണെന്നു വടക്കുംകൂർ
പ്പെട്ടിട്ടുണ്ടു്. സംഗമഗ്രാ
ലക്കുട ഗ്രാമമാകുവാൻ
പക്ഷേ വാരിയർമാരെ
പേർ പറഞ്ഞു നിർദ്ദേ



അതു സാധാരണമല്ല. നമ്പൂ
മാരെത്തന്നെ ഇല്ലപ്പേരുകൊണ്ടാണ്
നുമായി നിർദ്ദേശിക്കാറുള്ളതു്.
മഗ്രാമം എന്നതു കൂടല്ലൂർ എന്നതി
സംസ്കൃതമാണെന്നു വിചാരിച്ചതു
കേരളമല്ലൂരിനു് (കേളല്ലൂർ) കേരള
ഗ്രാമം എന്നാണല്ലോ ചോമാതിരി
പറഞ്ഞിട്ടുള്ളതു്. സംഗമഗ്രാമ
ൻ കൂടല്ലൂർ മാധവൻനമ്പൂതിരി
ന്ന അഭിപ്രായത്തിനുപോൽബ
ായി പ്രസിദ്ധമായ ഒരൈതിഹ്യവു
ആലത്തൂർ ഗ്രാമക്കാരനായ വട
പരമേശ്വരൻ നമ്പൂതിരി 1430-ാ
ിൽ ദ്രഗ്ഗണിതം നിർമ്മിച്ചപ്പോൾ അ
ിൽക്കു സകല കായ്ങ്ങളും ദൃക്കുപ്ര
ഗണിച്ചാൽ മതിയെന്നു് അദ്ദേഹ
ൻറെ അനുയായികളായ ചില കൂട്ടരും,
പോരാ, പഴയപടി പരഹിതംത
ാണു് അനുസരിക്കേണ്ടതെന്നു യാ
ിതികരായ മറ്റൊരു കൂട്ടരും ത
വാദമായെന്നും ഒടുവിൽ രണ്ടുകൂട്ട
രും പണ്ഡിതനായ കൂടല്ലൂർ നമ്പൂതി
മദ്ധ്യസ്ഥനായി സ്വീകരിച്ചുവെ
അദ്ദേഹം വൈദികകായ്ങ്ങൾക്കെ
ല്ലാം പരഹിതവും മറവുള്ളവയ്ക്കു ദൃക്കുമാക
ട്ടെ എന്നു വിധിച്ചുവെന്നുമാണു് ആ ഐ
തിഹ്യം.

ജ്യോതിശ്ശാസ്ത്രവിഭാഗത്തിനായി മരു
ത്തനൂരാൻ ഈ അഭിപ്രായം കേട്ടിട്ടുണ്ടെ
ന്നു പറഞ്ഞിരിക്കുന്നു. പണ്ഡിതനായ
കുനേഴത്തു പരമേശ്വരമേനവനവർക
കും ഇങ്ങിനെ ഒരഭിപ്രായമുണ്ടത്രേ.

സംഗമഗ്രാമമാധവൻ കൂടല്ലൂർ മാ
ധവൻ നമ്പൂതിരിയാണെന്നും അദ്ദേഹ
ത്തിൻറെ ജീവിതകാലം പതിനഞ്ചാം
ന്നൂറാണ്ടിൻറെ ആദ്യഭാഗമാണെന്നും എ
താണ്ടൂഹിക്കാം.

'ധാത്രിയൽകര സമ്പക്കാൽ
സഗ്രഹർക്ഷാ വിരാജതേ
തസ്മൈ സവ്വാത്മനേ ജ്യായ
ഭാസ്കരായ നമോ നമഃ.'

എന്നെഴുതിയ എഴുത്താണികളെപ്പോലെ
കരുത്തുള്ള മറ്റൊഴുത്താണികളും കേരള
ത്തിലെ യന്ത്രശാലകളിൽ നിർമ്മിച്ചിട്ടുണ്ടു്.
കർമ്മഭൂമിയായ കേരളത്തിൽ ജ്യോതിശ്ശാ
സ്ത്രം തഴച്ചുവളർന്ന ഒരു കാലമുണ്ടായിരുന്ന
തിനഭിമാനിക്കുവാൻ കേരളീയരായ ന
മുക്കു തികച്ചും അവകാശമുണ്ടു്.

Paper 6

Ulloor S. Parameswara Aiyer (1954)

Iriññāṭappiḷḷi Mādhavan Nampūri (in Malayalam)

Extract from
Ulloor S. Parameswara Aiyer's *Keraḷa Sāhitya Caritram* (History of Kerala Literature) Volume 2 (in Malayalam) published by the Publications Department, Travancore University, Thiruvananthapuram in September 1954, pp. 98–100.

Ulloor S. Parameswara Aiyer

Ulloor S. Parameswara Iyer (1877–1949) was an Indian poet of Malayalam literature and a historian. He was one of the modern triumvirate poets of Kerala in the first half of the 20th century, along with Kumaran Asan and Vallathol Narayana Menon.

The most noted of his historical works was Kerala Sahitya Charitram, which narrates the history of Malayalam language, culture, and literature and the book was published by the University of Travancore in five volumes after the writer's death. The first three volumes were published respectively in 1953, 1954 and 1955. The last two volumes were published in 1957.

Source: Internet Archive

Back to Contents.

ഇരിഞ്ഞാടപ്പള്ളി മാധവൻനമ്പൂരി:—വിദ്യാമാധവ
നിൽനിന്നു ഭിന്നനായ ഈ ആചാര്യനെ 'സങ്ഗമഗ്രാമമാധവൻ' എന്ന
പേരിൽ അപ്പന്മാർ വ്യവഹരിക്കുന്നു. ഇരിഞ്ഞാലക്കുട തെക്കേടത്തു
വാരിയത്താരിൽ ഒരാളായിരുന്ന ഈ ദൈവജ്ഞൻ എന്നു ചിലർ
അഭ്യൂഹിക്കുന്നതു നിരാസ്പദമാണ്. സങ്ഗമഗ്രാമമെന്നു് ഇരിഞ്ഞാല
ക്കുട ഗ്രാമത്തിനും പേരുണ്ടു്. ഇരിഞ്ഞാലക്കുട തീവണ്ടിയാപ്പീസിനു
സമീപമുള്ള ഇരിഞ്ഞാടപ്പള്ളി ഇല്ലത്തിലെ ഒരങ്ഗമായിരുന്നു സങ്ഗമഗ്രാമ
മാധവൻ എന്നു് ഒരൈതിഹ്യമുള്ളതു വിശ്വസനീയമായി തോന്നുന്നു.
വടശ്ശേരി പരമേശ്വരൻനമ്പൂരി താൻ കണ്ടുപിടിച്ച ദൃഗ്ഗണിതപദ്ധതി
അദ്ദേഹത്തെ കാണിച്ചപ്പോൾ ഗണിതസ്കന്ധത്തിൽമാത്രം അതിനു പ്രയോഗം
നല്ലിയാൽ മതി എന്നു് അദ്ദേഹപ്രഭൃതികൾ അഭിപ്രായപ്പെട്ടുവത്രേ.
അതു ശരിയാണെങ്കിൽ വടശ്ശേരിക്കും മാനനീയനായ ഒരു ജ്യോതിർ
വിതായിരുന്ന അദ്ദേഹം എന്നു വന്നുകൂടുന്നു. വടശ്ശേരിയെ അപേക്ഷിച്ചു
ജ്യേഷ്ഠനായിരുന്നിരിക്കണം അദ്ദേഹം. വേണാരോഹാദിഗ്രഹണങ്ങ
ളിലെ പ്രണേതാവു് എന്ന നിലയിൽ ദൃഗ്ഗണിതക്കാർ അദ്ദേഹത്തെ
സാദരം സ്മരിക്കുന്നുണ്ടു്. കേളല്ലൂർ ചോമാതിരി അന്തർജനീയഭാഷ്യ
ത്തിൽ തന്നെപ്പറ്റി 'സൂര്യാച്ചന്ദ്രമസോർഗ്രഹണാദിഗണിതമെന്നാചാര്യകൃതസൂത്രാ
ണ്യേ' എന്നു നിർദ്ദേശിച്ചു് അതിനു് ഉപോൽബലകമായി "തദാ

നാരം പുനസ്തദിച്ഛയം വസന്തതിലകം സങ്ക്ഷേപാജ്ജാധവനിർമ്മിതം
ച ശ്രുത യഥാ

ജീവേ പരസ്പരനിജേതരദൈവികാഭ്യാ–
മന്യസ്യ വിസ്തൃതിഗുണാ വിജൃമാനേ
അന്യോന്യയോഗവിരഹാനുഗുണേ ഭവേതാം
യദ്വാ സപലംബകൃതഭേദപദീകൃതേ ദേ''

എന്നും മറ്റുള്ള വചനങ്ങളും ഉദ്ധരിക്കുന്നു. 'ജീവേ പരസ്പര'ന്യായത്തിന്റെ മൂലകർത്താവായിട്ടാണ് മാധവനെ കേരളത്തിൽ ഇന്നും പ്രസിദ്ധി എന്നും, പരിധിവ്യാസം ഗണിക്കുവാൻ ഉതകുന്ന ശ്രേണിയുടെ ഉത്ഭവം കേരളത്തിലായിരുന്നു എന്നും പരിധിമാനത്തെ കേരളീയർ അത്യന്തം സൂക്ഷ്മമായി നിർണ്ണയിച്ചിട്ടുണ്ടെന്നും അഭിജ്ഞന്മാർ പറയുന്നു.

വേണ്വാരോഹം:— അൻപത്തൊൻപതു ശ്ലോകങ്ങൾകൊണ്ടു ക്രിയാക്രമം വിവരിക്കുന്ന ഒരു കൃതിയാണ് വേണ്വാരോഹം. അതിനു തൃക്കണ്ടിയൂർ അച്യുതപ്പിഷാരടി ആഴ്ചാഞ്ചേരിതമ്പ്രാക്കളുടെ നിർദ്ദേശമനുസരിച്ചു ഗദ്യത്തിൽ ഒരു ഭാഷാവ്യാഖ്യാനവും രചിച്ചിട്ടുണ്ട്. താഴെക്കാണുന്ന ശ്ലോകങ്ങളും ഗ്രന്ഥാരംഭത്തിൽ ഉള്ളവയാണ്:

''ശ്രിയേ ഭവതു മേ ദേവഃശ്ശിവശ്ശീതാംശുശേഖരഃ
തമൈവ താവതോപ്യസ്മി വിഘ്നാ വിഘ്നോപശാന്തയേ.

അവിചാരകൃതം വാചാമചാതുര്യമപോഹതു
ജാഗ്രതീ രസനാഗ്രേ മേ ഗിരാമപ്യധിദേവതാ.

പ്രഭാകരാദയസ്സർവേ പ്രണതാന്തഃപ്രഭാകരാഃ
ദിവി ഗ്രഹാഃ പ്രസീദന്തു ത്രിലോകാനാദിവിഗ്രഹാഃ.

വരളം സംഖ്യാവിശേഷാംസ്തു വക്ഷ്യേസ്മന്മാത്രസൂചകൈഃ
പദ്യാനി രചയേ ദ്രയാംസ്ത്യജ്യ ജിഹ്മേതി നോ ജനാഃ.

ബകുളാധിഷ്ഠിതത്വേന വിഖ്യാതോ യോ വിശിഷ്യതേ;
ഗ്രാമോമവി സോയം സ്യാന്നിജനാമനി മാധവഃ.''

ഇവയിൽ ഒടുവിലത്തെ ശ്ലോകം പിഷാരടി വ്യാഖ്യാനിച്ചിരിക്കുന്നതു് ഇങ്ങനെയാണ്: ''യാതൊരു ഗ്രാമം ബകുളാധിഷ്ഠിതത്വംകൊണ്ടു വിശേഷിക്കപ്പെടുന്നതു്..... ബകുളം=ഇരഞ്ഞി; വിഖ്യാതം=പള്ളി. ഇരഞ്ഞി നിന്ന പള്ളി എന്നു് ഇല്ലപ്പേർ. തന്റെ നാമത്തിങ്കൽ മാധവൻ.'' ഇരഞ്ഞി നിന്ന പള്ളി അനാദികാലങ്ങളിൽ ഇരിഞ്ഞാടപ്പള്ളിയായി രൂപാന്തരപ്പെട്ടിരിക്കാം. ഒടുവിലത്തെ ശ്ലോകത്തിൽ ഗ്രന്ഥനിർമ്മല ശ്ലോകസംഖ്യ കാണിച്ചിട്ടുണ്ട്:

''ശ്ലോകൈരേകോനഷഷ്ട്യാതൈരഭിധായ ക്രിയാക്രമം
ഉദാഹരിഷ്യാമി സൂക്ഷ്മാണി വരളം വാക്യാന്യുപക്രമേ.''

വ്യാഖ്യാതാവിന്റെ ശ്ലോകമാണ് അടിയിൽ ചേർക്കുന്നതു്:

''മാധവൻതാൻ ചമച്ചുള്ള വേണുവാരോഹത്തിനച്യുതൻ
മോഹാവ്യാഖ്യാനമുണ്ടാക്കീ നേത്രനാരായണാഭിധയാ.''

കേളല്ലൂർ ചോമാതിരി ഉദ്ധരിക്കുന്ന വസന്തതിലകപദ്യം അടങ്ങിയ പൂതി മാധവന്റെ മറ്റൊരു വാങ്മയമായിരിക്കണമെന്നു തോന്നുന്നു.

Digitized By Kerala Sahitya Akademi

Paper 7

K. V. Sarma (1956)

Samgamagrāma Mādhavan (in Malayalam)

Mathrubhumi Weekly (Malayalam) dated 4 November 1956.

K. V. Sarma

K. V. Sarma (1919–2005) was born in Chengannur, presently in the Alappuzha District of Kerala. After obtaining a BSc degree, he secured a Master's degree in Sanskrit in 1942 from the Maharaja College of Arts, Thiruvananthapuram. He worked in the Manuscripts section of the Kerala University Oriental Research Institute during 1943–1951, and the Department of Sanskrit in the University of Madras during 1951–1962. During 1962–1983, Sarma was associated with the Vishvesvaranand Institute of Sanskrit and Indological Studies at Hoshiarpur, affiliated with the Panjab University. Returning to Chennai, he founded the Sree Sarada Education Society and Research Centre, which has become the Prof. K.V Sarma Research Foundation after Sarma's demise in 2005.

Source: A scanned copy of the article was obtained from the library attached to the Ramavarma Appan Thampuran Smarakam, Ayyanthol, Thrissur, Kerala, curtsey the authorities of the Smarakam, on 29 July 2024.

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സംഗമഗ്രാമമാധവൻ

കേരളീയജ്യോതിഷത്തിനു് അഭിമാനസ്തംഭമായി പരിലസിച്ച ജ്യോതിശ്ശാസ്ത്രജ്ഞൻ

കെ. വി. ശർമ്മ, എം. എ ; ബി. എസ്സ് സി.

ഭൂതത്തിലെ മറ്റു ഭാഗങ്ങളെ അപേക്ഷിച്ചു് ആര്യടേിയഗണിതം വേഗന്നി താഴ്ച്ച വളർന്നിട്ടില്ലെന്നും, ആര്യടേിയപദ്ധതിയെ അടിസ്ഥാനമാക്കി അനേകം ഗ്രന്ഥങ്ങളുണ്ടായിട്ടുള്ളതും കേരളക്കരയിലാകുന്നു. പരഹിതവും ടുക്കം ആര്യടേഗണിതത്തിൽ ചില പ്രധാനപരിഷ്കാരങ്ങളും ചെയ്തുണ്ടാക്കിയ ഗണിതപദ്ധതികളാണു്. ഈ പ്രധാനപരിഷ്കാരങ്ങളെ കൂടാതെ പല ലഘുകരണക്രിയകളും മറ്റു പരിഷ്കാരങ്ങളും കേരളീയാചാര്യന്മാർ ജ്യോതിശ്ശാസ്ത്രത്തിൽ ചെയ്തിട്ടുണ്ടു്. ഈ ആചാര്യന്മാരിൽ ഒരു മാന്യസ്ഥാനത്തിനവകാശിയാണു് സംഗമഗ്രാമമാധവൻ. ദ്യഗ്ഗണിതകർത്താവായ പരമേശ്വരന്റെ ഒരു ഗുരു കൂടിയായിരുന്ന ഇദ്ദേഹം. മാധവന്റെ കൃതികളെക്കുറിച്ചും കാലത്തെക്കുറിച്ചും അവിജ്ഞാതങ്ങളായ ചില വിവരങ്ങളും നല്കുകയാണീ ലേഖനത്തിന്റെ ഉദ്ദേശം.

ജ്യാഗണിതത്തിൽ ആവശ്യമായ 'ജീവാപരസ്പര' ന്യായത്തിന്റെ (Sin(A+B)=Sin A. CosB+ Cos A. Sin B) ഉപജ്ഞാതാവും 'ശ്ലോഷം നാമ വരിഷ്ഠാനാം' എന്നു തുടങ്ങിയ ഇരുപത്തിനാലു തല്പരാദി മഹാജ്യാവാക്യങ്ങളുടെ കർത്താവുമായിട്ടാണു് മാധവന്റെ പ്രസിദ്ധി. എന്നാൽ ഇതിനുപരിയായി അപരിമിതശ്രേണികം (Infinite Series) ഉപയോഗിച്ചു ഒരു വൃത്തത്തിന്റെ പരിധി സൂക്ഷ്മമായി നിർണ്ണയിച്ചുന്നതിനുള്ള മാർഗ്ഗം ഒന്നാമതായി കണ്ടുപിടിച്ച ഗണിതജ്ഞൻ കൂടിയാണു മാധവൻ എന്ന വസ്തുത പൊതുവെ അറിവുള്ളതല്ല. പത്തു പന്ത്രണ്ടു കൊല്ലം മുമ്പെഴുതിയ ഒരു ലേഖനത്തിൽ കൊച്ചി രാമവർമ്മ മഹത്തമനുരാൻ ഇക്കാര്യം അഭ്യൂഹിച്ചിട്ടുണ്ടു്. ("വൃത്തപരിധിയിലൂടെ", മംഗളോദയം, 20 (1119–20) 255). ഈ അഭ്യൂഹം ശരിയാണെന്നും മാധവൻ തന്നെയാണു് ഈ മാർഗ്ഗത്തിന്റെ ഉപജ്ഞാതാവെന്നും മുള്ളതിനു വേണ്ട തെളിവുകളും ഇപ്പോൾ ലഭിച്ചിട്ടുണ്ടു്. 'ക്രിയാ ക്രമകരീ' എന്ന ലീലാവതീ വ്യാഖ്യാനത്തിൽ വൃത്തപരിധ്യാനയനത്തെപ്പറ്റി ചർച്ച ചെയ്യുന്ന സന്ദർഭത്തിൽ (ലീലാവതീ, II. 40), വ്യാഖ്യാതാവു പറയുന്നു—"യദ്യേവമതിസൂക്ഷ്മഃ പരിധിഃ കഥമാനേയം ശകുത ഇതി ചേദുച്യതേ മാധവഃ.....ഇത്യാദിനാ ലഘുതരപരിധ്യാനയനോപായഃ അനുനൈവോക്തം യഥാ—

'വ്യാസേ വാരിധിനിഹതേ
ആപ്തേ വ്യാസസാഗരാഭിഹതേ
ത്രിശരാദിവിഷമസംഖ്യാക്ഷേത്രഗുണം
സ്വം പൃഥക് ക്രമാത് കുരുത്.

* * * *

ഹരണാപ്തിസൂക്ഷ്മഃ സ്യാത്" ഇതി."

ഇതിൽ നിന്നും കിട്ടുന്ന പരിധ്യാനയനശ്രേണി ഇതാണു്—

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n+2}+\cdots$$

മാധവന്റെ കാലം (1350—1410)[1] കഴിഞ്ഞു മുന്നൂറോളം വർഷങ്ങൾ കഴിഞ്ഞു മാത്രമാണു് സ്കോട്ട്ലണ്ടുകാരനായ ജേംസ് ഗ്രെഗറിയും (1638–1675) ജർമ്മൻ ഗണിതജ്ഞനായ ഗോത്ഫ്രീഡ് വില്യം ലൈബ്നിറ്റ്സും (1646–1716) സൂക്ഷ്മപരിധ്യാനയനത്തിനുള്ള ഈ ശ്രേണി കണ്ടുപിടിച്ചതെന്ന വസ്തുത കേരളീയർക്കു് ഏറ്റവും അഭിമാനകരമായ ഒരു കാര്യമാണു്.[2]

ഗോളവിദ് മാധവൻ

പില്ക്കാലത്തെ ഗ്രന്ഥകാരന്മാർ മാധവനെ ആദരപൂർവ്വം സ്മരിക്കുകയും അദ്ദേഹത്തിനേർതായി അനേകം ഗണിതന്യായങ്ങളും സ്വകൃതികളിൽ ഉദ്ധരിക്കുകയും ചെയ്യുന്നു. ഈ ഉദ്ധരണങ്ങളിൽ പലതിലും മാധവന്റെ പേരോടു കൂടി 'ഗോളവിദ്' എന്നൊരു ഉപനാമം കൂടി ചേർത്തു കാണുന്നുണ്ടു്. സ്ഫുടനിർണ്ണയത്തിന്റെ ആദ്യശ്ലോകത്തിൽ അച്യുതപ്പിഷാരടി "വന്ദേ ഗോളവിദശ്ച മാധവമുഖാൻ" എന്നു മാധവനെ സ്മരിക്കുന്നു. ഈ മാധവൻ വേണ്വാരോഹമെന്ന കൃതിയുടെ കർത്താവു കൂടിയാണെന്നു് ഈ ശ്ലോകത്തിന്റെ വ്യാഖ്യാനം എടുത്തു പറയുന്നുമുണ്ടു്—"മാധവോ വേണ്വാരോഹാദിനാം കർത്താ."

കേരളീയനായ നീലകണ്ഠസോമയാജി തന്റെ പരമഗുരുവായ പരമേശ്വരാചാര്യരെപ്പറ്റി പ്രസ്താവിക്കുമ്പോൾ മറ്റൊരു "പരമേശ്വരസ്യ....മാധവാദിഭ്യോ ഗോളവിദ്ഭ്യോ ഗണിതഗോളയുക്തിരപി ബാലൂ ഏവ സമുക് ഗ്രഹീതാ"[3] എന്നു കുറിച്ചുന്നു. മറ്റൊരിടത്തു മാധവനെ സ്മരിക്കുമ്പോൾ "തച്ച സംഗമഗ്രാമജേന ഗോളജ്ഞവിദാ മാധവേന പ്രദർശിതം"[4] എന്നും വേറൊരിടത്ത് "അത ഏവ ഗോളവിദാ മാധവേന വിക്ഷേപവതാം സ്ഫുടാപകമാനയനേ ഗണിതവിശേഷഃ പ്രദർശിതഃ[5] എന്നും പ്രസ്താവിച്ചുന്നു. ഇതിൽ നിന്നു മാധവൻ വിശിഷ്ടു ഗോളഗണിതത്തിൽ പ്രാമാണികനായ ഒരാചാര്യനായിരുന്നുവെന്നും പില്ക്കാലത്തെ ഗ്രന്ഥകാരന്മാർ "ഗോളവിദ്" എന്ന ബിരുദം കൂട്ടിച്ചേർത്ത് അദ്ദേഹത്തെ പരാമർശിച്ചിരുന്നുവെന്നും വിശ്വസിയ്ക്കുണം.

വേണ്വാരോഹം

മാധവന്റെ കൃതിയായി ഇതുവരെ അറിയപ്പെട്ടിരുന്ന സമ്പൂർണ്ണഗ്രന്ഥം വേണ്വാരോഹം മാത്രമാണു്. തിഥിയും നക്ഷത്രവും പിശകില്ലാതെ ഗണിയ്ക്കുന്നതിനു വേണ്ടി ചന്ദ്രസ്ഫുടം സൂക്ഷ്മമായറിവാൻ ഒരു നൂതനമായ മാർഗ്ഗം പ്രതിപാദിയ്ക്കുന്ന ഗ്രന്ഥമാണു വേണ്വാരോഹം. ഈ വിശിഷ്ടകൃതി തൃക്കണ്ടിയൂർ അച്യുതപ്പിഷാരടിയുടെ ദോഷാവ്യുഖ്യാനത്തോടു കൂടി തൃപ്പണിത്തുറ സംസ്കൃതകോളേജിൽ നിന്നും പ്രസിദ്ധപ്പെടുത്തി വരുന്നുണ്ടു്.[6] ആകാശത്തിൽ നിരന്തരം ചരിച്ചു കൊണ്ടിരിയ്ക്കുന്ന ജ്യോതിസ്സുകളിൽ വച്ചു് ഏറ്റവും ശീഘ്രമായി സഞ്ചരിയ്ക്കുന്നതു് ചന്ദ്രനാണു്. ഇതിനു് ഒരു ദിവസത്തിൽ 13 ഡിഗ്രിയോളം നീക്കമുണ്ടു്. ഒരേ ദിവസത്തിൽ തന്നെ ഗതിയ്ക്കു ഗണുമായ ഭേദം വരുന്നതും ഈ ഗ്രഹത്തിനാണു്. ഇക്കാരണങ്ങൾ കൊണ്ടു മറ്റു ഗ്രഹങ്ങളുടെ സ്ഫുടമെന്ന പോലെ ചന്ദ്രസ്ഫുടം ഓരോ ദിവസത്തിനും സൂര്യോദയത്തിനു മാത്രം കണക്കാക്കിയാൽ മറ്റു സമയങ്ങൾക്കുള്ള സ്ഫുടം ഉദയസ്ഫുടം കൊണ്ടും ദിനഗതികൊണ്ടും തൈരാശികം ചെയ്തു വരുത്തിയാൽ സൂക്ഷ്മമായിരിയ്കയില്ല. സാധാരണ ചെയ്തു വരാറുള്ളതുപോലെ ഉദയാസ്തസ്ഫുടങ്ങളും ഉദയാസ്തഗതികളും കൊണ്ടു ഗണിച്ചാലും വളരെ സൂക്ഷ്മതവരികയില്ല. ഇതു പരിഹരിയ്ക്കുന്നതിനു് ഒരു ദിവസത്തിൽ ൪പത്തു കാലങ്ങൾക്കുള്ള ചന്ദ്രസ്ഫുടങ്ങൾ സൂക്ഷ്മമായി വരുത്തി അവയിൽ നിന്നും ഇഷ്ടസമയത്തേയ്ക്കുള്ള സ്ഫുടം തൈരാശികം ചെയ്തു ഗണിയ്ക്കുന്നതിനുള്ള ഒരു സുഗമമായ മാർഗ്ഗമാണു വേ

1 ഇക്കാര്യം താഴെ സമർത്ഥിയ്ക്കുന്നുണ്ടു്.

2 "വ്യാസേ വാരിധിനിഹതേ" എന്നിത്യാദിപദ്യങ്ങൾക്കു ഗണിതശാസ്ത്രചരിത്രത്തിലുള്ള പ്രാധാന്യം ഇതിനു മുമ്പു തന്നെ ചൂണ്ടിക്കാണിയ്ക്കപ്പെട്ടിട്ടുണ്ടു്. ('വൃത്തപരിധിയിലൂടെ': പി. കെ. കോര, മംഗളോദയം, 20 (1119–20) 187–90, രാമവർമ്മമഹത്തമനുരാൻ, ടി. 253–54; കെ. കഞ്ചുണ്ണിരാജാ, ടി. 256). 1500-ൽ രചിച്ച തന്ത്രസംഗ്രഹത്തിൽ ഈ ശ്ലോകങ്ങൾ കാണുന്നതുകൊണ്ടു് അതിനു മുമ്പും, പുതുമനസോമയാജിയുടെ കരണപദ്ധതിയിൽ ഈ മാർഗ്ഗം പ്രസ്താവിച്ചിട്ടുള്ളതുകൊണ്ടു് അതിന്റെ രചനയ്ക്കു മുമ്പും ഈ ശ്ലോകങ്ങൾ കേരളത്തിലറിയപ്പെട്ടിരുന്നുവെന്നാണു് ഈ ലേഖനങ്ങളിൽ കാണിച്ചിട്ടുള്ളതു്. കരണപദ്ധതിയുടെ കാലം ഈ ലേഖനങ്ങളിൽ സൂചിപ്പിച്ചിട്ടുള്ളതു പോലെ 1430 അല്ലെന്നും, 1530-നു ശേഷമാണെന്നുമുള്ളതിനു് 5–2–1956-ലെ മാതൃഭൂമി ആഴ്ചപതിപ്പിലെ "പുതുമനച്ചോമാതിരിയുടെ കാലം" എന്ന ലേഖനം നോക്കുക. ഇക്കണക്കിനു 1400-നു മുമ്പു തന്നെ ഈ മാർഗ്ഗം കേരളത്തിലറിഞ്ഞിരുന്നുവെന്നതിനു വ്യക്തമായ തെളിവു കിട്ടിയാൽ ചരിത്രപരമായി അതിനുള്ള പ്രാധാന്യം വ്യക്തമാണല്ലോ. ഉത്തരേന്ത്യയിലെ ഗണിതജ്ഞന്മാരും തന്നെ ഈ ശ്രേണികളെപ്പറ്റി പറയുന്നില്ലെന്നുള്ളതും ഇവിടെ പ്രസ്താവ്യമാണു്. ഈ ശ്ലോകത്തെ തുടർന്നു ഈ മാർഗ്ഗത്തിനു ചില പ

3 ആര്യടേിയഭാഷ്യം, ഗോളപാദം, തിരുവിതാംകൂർ സർവ്വകലാശാല ഹസ്തലിഖിതഗ്രന്ഥശാലയിലെ ഗ്രന്ഥം, നംബർ T. 298, പേജു് 258.

4 ടി. പേജു് 132.

5 ടി. പേജു് 186.

6 വേണ്വാരോഹത്തിൽ ആകെ 74 ശ്ലോകങ്ങളുണ്ടുള്ളതു്. ശ്ലോകൈരേകോനഷഷ്ട്യുത്തരശതാധ്യായക്രിയാക്രമം' എന്നു് അവസാനശ്ലോകത്തിൽ കാണുന്നതിനെ ആടിസ്ഥാനപ്പെടുത്തി ഈ കൃതിയിലെ ശ്ലോകസംഖ്യ 59 ആണെന്നു് ഉള്ളൂർ പറയുന്നതു് ശരിയല്ല (കേ. സാ. ച. II. 99). ഈ 59 ശ്ലോകങ്ങൾ ഇതിലെ ക്രിയ

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ണ്ഡാരോഹത്തിലെ ക്രിയ— സാധാരണയായി ഉപയോഗിച്ചു വരുന്ന 'ശീർഷ്രേയാദി' വരെയുള്ള വാക്യങ്ങൾക്ക് പകരം വിലിപ്പാടിവാക്യങ്ങളുപയോഗിച്ചാണ് ഇതിലെ ക്രിയ ചെയ്യുന്നത്. ഏതെങ്കിലും ഒരു ദിവസത്തേയ്ക്കു ക്രിയ ചെയ്യേണ്ട ഫലങ്ങൾ വരുത്തിക്കഴിഞ്ഞാൽ ആ ഫലങ്ങളുപയോഗിച്ചുതന്നെ പ്രയാസംകൂടാതെ ആ പര്യയം (248 ദിവസം) തീരുന്നതു വരെ ഏതു നിശ്ചിതത്തേയ്ക്കുള്ള ചന്ദ്രസ്ഫുടം സൂക്ഷ്മമായി വിവാൻ സാധിയ്ക്കുമെന്നതു മാത്രമല്ല, ഒരു പര്യയമവസാനിച്ചാൽ ഒരു ക്ലപ്തധ്രുവം കൂടി അടുത്ത പര്യയത്തേയ്ക്കുള്ള ഫലങ്ങൾ കണക്കാക്കാവുന്നതുമാണ്.[7]

മാധവന്റെ അവിജ്ഞാത കൃതികൾ

വേണ്ടാരോഹത്തിന്റെ അവസാനഭാഗത്തു കാണുന്ന

'ശ്രീലം രാജ്ഞഃ ശ്രിയേ' കൃത്വാ
പ്രാക് പുനരേനം നിർമ്മിതം
വിലിപ്പടികാടികം വാക്യജാതം
തേനേയമിരിതാ. (ശ്ലോ. 73)

എന്ന പദ്യത്തിൽ നിന്ന് അങ്ങിനെയൊരു കൂട്ടം വാക്യങ്ങൾ കൂടി മാധവൻ രചിച്ചിരിയ്ക്കുണമെന്ന് അനുമാനിയ്ക്കപ്പെട്ടിട്ടുണ്ട്. എന്നാൽ പ്രസ്തുതവാക്യങ്ങളെന്താണെന്നോ, അവയുടെ ഉപയോഗമെന്താണെന്നുവെന്നോ ഉള്ള കാര്യത്തെപ്പറ്റി പരിനിശ്ചിതമായ അറിവില്ലായിരുന്നുവെന്നു വേണം പറയുവാൻ. എന്റെ ഗ്രന്ഥപരിശോധനകൾക്കിടയിൽ ഈ വാക്യങ്ങളുടെ നാലു താളിയോലഗ്രന്ഥങ്ങൾ കണ്ടെത്തുവാനിട വരികയും ഗ്രന്ഥകർത്തൃനിർദ്ദേശവും മറ്റും മുന്നിലും ഇല്ലായിരുന്നവെങ്കിലും വേണ്ടാരോഹത്തിൽ മാധവൻ തന്നെ സൂചിച്ചിട്ടുള്ള ഇതിലെ ആദ്യവാക്യം കൊണ്ട് ഇവയുടെ കർത്തൃത്വം നിസ്സംശയപ്പെടുത്തുവാൻ സാധിയ്ക്കുകയും ചെയ്തു. സൂക്ഷ്മചന്ദ്രഗണനത്തിനും, വിശേഷിച്ചു വേണ്ടാരോഹക്രിയയ്ക്കു വേണ്ടി വികലാദിയായി തയ്യാറാക്കിയിട്ടുള്ള 248 ചന്ദ്രവാക്യങ്ങളാണിവ.[8]

സ്ഫുടനിർണ്ണയത്തിന്റെ വ്യാഖ്യാതാവ് 'മാധവോ വേണ്ടാരോഹാദിനാം കർത്താ' എന്നു പറയുന്നതിൽ നിന്നും മാധവൻ പല കൃതികളും രചിച്ചിട്ടുണ്ടെന്നും ധ്വനിയ്ക്കുന്നുണ്ട്. പക്ഷേ വേണ്ടാരോഹവും 'ശ്രീലം രാജ്ഞഃ ശ്രിയ'ത്യാദി ചന്ദ്രവാക്യങ്ങളുമല്ലാതെ മറ്റു ഗ്രന്ഥങ്ങളേതാണെന്നു നിർണ്ണയിച്ചു പറവാൻ സാധുമല്ലാതെയാണിരിയ്ക്കുന്നത്.

ജ്ഞാതകർത്തൃകമായ ഒരു കേരളീയ ജ്യോതിഷവ്യാഖ്യാനത്തിൽ

'ലംബാഹ്വേ ലംബകർണ്ണഃ സുശീൽ
ത്രിജ്യാപര പരാഹതേ'
ഇത്യുക്തത്വാത് ഗോമവിദാ
ലഗ്നപ്രകരണേ പി ച

എന്നൊരു പദ്യം കാണുന്നുണ്ട്.[9] ഇവിടെ 'ഗോളവിദാ' എന്ന പദം ആ ബിരുദം കൊണ്ടു പ്രസിദ്ധനായ ഒരു വ്യക്തിയെ കുറിക്കുന്നതു പോലെ തോന്നുന്നു. 'ഗോളവിദ്' എന്ന് ഈ വ്യാഖ്യാതാവു സൂചിപ്പിയ്ക്കുന്ന ലഗ്നപ്രകരണകർത്താവ് നമ്മുടെ മാധവാചാര്യനാകുവാൻ ന്യായമുണ്ട്. ലഗ്നാനയനത്തെപ്പറ്റിയും മറ്റും സമഗ്രമായി പ്രതിപാദിയ്ക്കുന്നതും എട്ടധ്യായങ്ങളടങ്ങിയതുമായ 'ലഗ്നപ്രകരണം' എന്നൊരു കൃതി അറിയപ്പെട്ടിട്ടുണ്ട്.[10] 'ലംബാപ്പേ' എന്നിതുദാഹരണശ്ലോകം ഈ ലഗ്നപ്രകരണത്തിലുണ്ടു താനും (അധ്യായം V, ശ്ലോകം 8). പക്ഷേ ഇതിന്റെ മാതൃകാഗ്രന്ഥങ്ങളിലൊന്നിലും ആരംഭത്തിലാകട്ടെ, സമാപ്തി ശ്ലോകത്തിലാകട്ടെ ഗ്രന്ഥകർത്താവിന്റെ പേർ കാണാത്തതു കൊണ്ട് ഇതു മാധവകൃതമാണെന്നു തീർച്ചയായി പറവാൻ സാധുമല്ല.

വേണ്ടാരോഹത്തിന്റെ ഒരു ഗ്രന്ഥത്തിൽ അതു കഴിഞ്ഞു 'ഗോളവാദം' എന്നൊരു കൃതി കൂടി ഉണ്ടായിരുന്നതായറിയാം. ഈ ഗ്രന്ഥം ഒരു നോക്കു കാണുവാൻ ചെയ്ത ശ്രമം വിഫലമായെന്നു മാത്രമല്ല, പ്രസ്തുത ഗ്രന്ഥം തീരെ നശിച്ചുപോയതായിട്ടാണറിവാനിടയായതു.[11] ഒരു പക്ഷേ ഈ 'ഗോളവാദ'വും വേണ്ടാരോഹകർത്താവിന്റെ കൃതിതന്നെയാണെന്നും, ഈ കൃതി മൂലമാണ് അദ്ദേഹത്തിന് 'ഗോളവിദ്' എന്ന ബിരുദം സിദ്ധിച്ചതു എന്നും വന്നു കൂടായ്കയില്ല.

മാധവകർതൃകമായി ധാരാളം ശ്ലോകങ്ങൾ പില്ലാലത്തെ ഗണിത കൃതികളിൽ ഉദ്ധരിച്ചിട്ടുണ്ടെന്നു ഇതിനു മുൻപു സൂചിപ്പിച്ചിട്ടുണ്ടല്ലോ. ഈ ഉദ്ധരണങ്ങളിൽ പാടിഗണിതം, ക്ഷേത്രഗണിതം, ജ്യാഗണിതം, ഗോളഗണിതം എന്നിതുദാഹരിച്ചു വിഷയങ്ങളെല്ലാം അന്തർഭവിക്കുന്നുണ്ട്. 'തത്രാഹ മാധവഃ', 'മാധവേന പ്രദർശിതഃ', 'സംഗമഗ്രാമജോ മാധവോഽ പ്യേവമാഹ', 'സംഗമഗ്രാമജ മാധവനിർമ്മിതം പദ്യം ച ശ്രുതം', 'മാധവോക്ത പ്രകാരേണ', 'മാധവോക്ത ന്യായേന' എന്നും മറ്റുമുള്ള അവതാരികകളില്ലാതെ ഈ ഉദ്ധരണങ്ങളും ഏതേതു കൃതികളിൽ നിന്നുമെടുത്തവയാണെന്നുള്ള വിവരം ഈ ഗ്രന്ഥകാരന്മാർ കുറിച്ചിട്ടില്ല. ഇവയെല്ലാം ഗണിതവിധികളുംകൊള്ളുന്ന 'മുക്തകങ്ങൾ' മാത്രമാണെന്നു പറഞ്ഞു തള്ളുവാൻ നിവൃത്തിയില്ല. ഉദ്ധരണങ്ങളുടെ വൈവിധ്യവും ബാഹുല്യവും നോക്കുമ്പോൾ മാധവൻ ബൃഹദ്ഗ്രന്ഥകർത്താവെന്നു തന്നെ വിചാരിക്കേണ്ടിയിരിയ്ക്കുന്നു. മാധവന്റെ കൃതികളുടെ താളിയോലഗ്രന്ഥങ്ങൾ കിട്ടാത്ത സ്ഥിതിയ്ക്ക് പില്ലാലത്തെ ഗണിതഗ്രന്ഥങ്ങളിൽ മാധവനേരായി കൊടുത്തിട്ടുള്ള ഉദ്ധരണങ്ങളെല്ലാം സമാഹരിച്ചെടുത്തു ക്രമപ്പെടുത്തി നിഷ്പ്രയാസമായി പഠിച്ചാൽ മാത്രമേ മാധവൻ കേരളീയഗണിതത്തിനു ചെയ്തിട്ടുള്ള സംഭാവനകളുടെ ഒരു സാമാന്യ രൂപമെങ്കിലും ലഭിയ്ക്കുകയുള്ളു.

ദേശവും കാലവും

വേണ്ടാരോഹത്തിലെ 13–ാംശ്ലോകത്തിൽ ഗ്രന്ഥകർത്താവു തന്റെ നാമവും ഗൃഹനാമവും കൊടുത്തിട്ടുണ്ട്:—

ബകളാധിഷ്ഠിതത്വേന വിഹാരോ
യോ വിശേഷ്യതേ
ഗൃഹനാമനി സോഽയം
സ്യാന്നിജനാമനി മാധവഃ

അച്യുതപ്പിഷാരടി ഈ ശ്ലോകത്തിനിപ്രകാരം അർത്ഥം പറയുന്നു:—"യാതൊരു വിഹാരം ബകളാധിഷ്ഠിതത്വം കൊണ്ടു വിശേഷിയ്ക്കപ്പെടുന്നതു, ഇവൻ ഗൃഹനാമത്തിങ്കൽ അവനാകുന്നതു. ബകളം= 'ഇരഞ്ഞി'; വിഹാരം=പള്ളി. 'ഇരഞ്ഞിനിന്ന പള്ളി' എന്നു ഇല്ലപ്പേർ. തന്റെ നാമത്തിങ്കൽ മാധവൻ." ഈ ഏഴാചാര്യനെ സംഗമഗ്രാമജമാധവനെന്നും പറയാറുണ്ടെന്നു മുൻപു കണ്ടുവല്ലോ. സംഗമേശ്വരന്റെ മഹാക്ഷേത്രം കൊണ്ടു പ്രസിദ്ധമായ സംഗമഗ്രാമം ഇരിങ്ങാലക്കുടയാണു. ഈ ഗ്രാമത്തിൽ ഇരിഞ്ഞാലപ്പള്ളി എന്ന പേരിൽ ഒരില്ലം ഇന്നുമുണ്ട്. ഈ ഇല്ലങ്ങളിലൊന്നിലെ അംഗമായിരിയ്ക്കാം മാധവൻ. 'ഇരഞ്ഞി നിന്ന പള്ളി' എന്ന ഇല്ലപ്പേരു പില്ലാലത്ത് 'ഇരിഞ്ഞാറപ്പള്ളി'യെന്നോ, 'ഇരിഞ്ഞിനവള്ളി' യെന്നോ രൂപാന്തരപ്പെട്ടിരിയ്ക്കുണം.

മാധവന്റെ കാലം ഏതാണ്ടു സൂക്ഷ്മമായി തന്നെ നിർണ്ണയിയ്ക്കുവാൻ സാധിയ്ക്കും. ദ്യുഗണിതപരമേശ്വരന്റെ പ്രധാനഗുരുനായിരുന്നുവെങ്കിലും മാധവനും പരമേശ്വരന്റെ ഒരു ഗുരുവായിരുന്നുവെന്നു കേട്ടെല്ലൂർ നീലകണ്ഠസോമയാജി തന്റെ ആര്യഭടീയഭാഷ്യത്തിൽ പറയുന്നു—"പരമേശ്വരസ്യ ഗുരു പരമേശ്വരാത്മജ നാരായണ—മാധവാദിഭ്യോ ഗോളവിദ്യോഗണിത ഗോള യുക്തീരപി ബാല്യ ഏവ സമുപഗതവാൻ" ഇത്യാദി. അൻപത്തിയഞ്ചു വർഷത്തെ ഗോള നിരീക്ഷണത്തിനു ശേഷം ക്രി. പി. 1430–ൽ ദ്യുഗണിതം രചിച്ച പരമേശ്വരന്റെ ബാല്യഗുരുവായ മാധവൻ 14–ാം ശതകത്തിന്റെ ഉത്തരാർദ്ധത്തിൽ ജീവിച്ചിരുന്നുവെന്നു കരുതുന്നതിൽ തെറ്റില്ല.

വേണ്ടാരോഹം ഏതു കാലത്തു രചിച്ചുവെന്നതിനു ആ ഗ്രന്ഥത്തിൽ ഉപയോഗിച്ചിട്ടുള്ള കലിദിനവർഷം മുതലായവയിതായ തെളിവാണു. വേണ്ടാരോഹക്രിയയിലെ സൂര്യമധ്യമാനയനത്തിനു ഉല്ലാല കലിദിനത്തിൽനിന്നും 'ദിനനായകശാസ്യ'(15,02,008) എന്ന ദിനവർഷവും (ക്രി. പി. 1010) 'ആദിക്രിയം' (5180) എന്ന ചന്ദ്രമന്ദകേന്ദ്രവർഷവും (390 വർഷം) കിഴിക്കാൻ ഗ്രന്ഥകർത്താവു നിർദ്ദേശിക്കുന്നു. കൂടുതൽ പര്യായങ്ങളുണ്ടെങ്കിൽ ഈ

(ശേഷം 50–ാം പേജിൽ)

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7 വേണ്ടാരോഹക്രിയയുടെ യുക്തിയും ക്രിയാചെലവും ഇതിനുള്ള വഴിയും തുടർന്നുള്ള സംസ്കൃതകോളേജിലെ പ്രസിദ്ധീകരണങ്ങളിൽ വിശദീകരിച്ചിട്ടുണ്ട്.

8 ഈ പ്രസിദ്ധീകരണത്തിൽ ഈ വാക്യങ്ങൾ തന്നെ ബന്ധമായി കൊടുത്തിട്ടുണ്ട്.

9 മദിരാശി ഓറിയന്റൽ മാനുസ്ക്രിപ്റ്റ്സ് ലൈബ്രറി, Mal. D. 339–ാം നമ്പർ ഗ്രന്ഥം, 'പരഹിതതന്ത്രം', പേജ് 66.

10 മദിരാശി ഓറിയന്റൽ മാനുസ്ക്രിപ്റ്റ്സ് ലൈബ്രറി, Mal. D. 339, pp. 32–42; തിരു. സ്കൃ. ഗ്രന്ഥശാല 414–B, L. 1360–C.

11 കൊച്ചി എളങ്കുന്നപ്പുഴ നടുവിൽ കോവിലകത്താണ് ഗ്രന്ഥമുണ്ടായിരുന്നതു. പക്ഷേ ചില ദിവസങ്ങൾക്കു മുമ്പ് പ്രസ്തുത കോവിലകത്തു ചെന്ന പരിശോധിച്ചപ്പോൾ ആ വീടെയുണ്ടായിരുന്ന ഗ്രന്ഥങ്ങളെല്ലാം ഒന്നോടെ ചിതലെടുത്തു പോയതായിട്ടാണ് കണ്ടതു. കൂടുതൽ വിവരങ്ങൾക്കു മാതൃഭൂമി ആഴ്ചപ്പതിപ്പ്, 17–6–1956, പേജ് 4 നോക്കുക.

Scanned by TapScanner

സംഗമഗ്രാമമാധവൻ

(18-ാം പേജിൽ നിന്നു തുടർച്ച)

വസ്തുള്ള ഗതി പ്രത്യേകം കണ
ക്കാക്കി കൂട്ടിക്കൊടുവാനും നിർദ്ദേ
ശിക്കുന്നു. അതു കൊണ്ട് ഇതാണു
ഗ്രന്ഥരചനാകാലമെന്നു വിശ്വ
സിക്കാം. ഇത് (1010+390) =

1400 ക്രീ. പി.യ്ക്കു ശരിയാകും.
1430-ൽ ഒഴുണിതം രചിച്ച പര
മേശ്വരന്റെ ഗുരുവായ മാധവൻ
1400-ൽ വേണ്വാരോഹം രചി
ച്ചുവെന്നത് സർവ്വഥാ സാധുവ
മാണ്. അതു കൊണ്ട് കേരള
ജ്യോതിഷത്തിനു മരുമാനസ്തംഭ
മായി പരിലസിച്ച സംഗമഗ്രാമ
മാധവൻ ജീവിച്ചിരുന്നത് ക്രി.പി.
1350-നും 1410-നും ഇടയ്ക്കാണെ
ന്നു നൂയമായി സങ്കല്പിക്കാം.

Paper 8

K. V. Sarma (1957)

Oru jyōtiṣhagranthavari (in Malayalam)

Mathrubhumi Weekly (Malayalam) dated 19 May 1957.

K. V. Sarma

For a brief biography of K. V. Sarma, see 79.

Source: A scanned copy of the article was obtained from the library attached to the Ramavarma Appan Thampuran Smarakam, Ayyanthol, Thrissur, Kerala, curtsey the authorities of the Smarakam, on 7 January 2025.

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Malayalam article

Oru jyōtiṣagranthavari

K. V. Sarma

Mathrubhumi 19.5.1957

pages 18-19

ഒരു ജ്യോതിഷഗ്രന്ഥവരി

കെ. വി. ശർമ്മ

മാതൃഭൂമി 19.5.1957
പേജ് 18-19

ഒരു ജ്യോതിഷഗ്രന്ഥവരി

കെ. വി. ശർമ്മ, എം. എ., ബി. എസ് സി.

സംസ്കൃതവകുപ്പ്, മദിരാശി സർവ്വകലാശാല

തിലകളത്തൂർ ഭട്ടതിരിയിൽ തുടങ്ങി അച്യുതപ്പിഷാരടി, തുപ്പാണിക്കര പുതുവാടം, നാവായിക്കുളത്ത് ആഴ്വാതി, പുലിമുഖത്തു പോറ്റി, രാമനാശാൻ, കൃഷ്ണനാശാൻ എന്നിങ്ങനെ ഒരു ജ്യോതിഷഗ്രന്ഥപരമ്പര ഭാഷാജാതകപദ്ധതി മുതലായ കൃതികളുടെ കർത്താവായ നെടുമ്പയിൽ കൃഷ്ണനാശാൻ (1756–1812) തന്റെ ആറന്മുളവിലാസം ഛംസപ്പാട്ടിൽ വിവരിച്ചിട്ടുള്ളത് സുവിദിതമാണല്ലോ.[1] ഇതുപോലെ മറ്റു ചില കേരളീയജ്യോതിഷ്കുകളുടെ കാലത്തേയും കൃതികളേയും പറ്റി വിവരങ്ങൾ നൽകുന്ന ഒരു ഗ്രന്ഥവരി ബറോഡയിലെ ഓറിയന്റൽ ഇൻസ്റ്റിറ്റ്യൂട്ടിലെ ഒരംഗമായ ശ്രീ എം. രാഘവൻനമ്പ്യാർ അടുത്ത കാലത്തു കണ്ടെടുത്തിട്ടുണ്ട്. പ്രസ്തുത ഇൻസ്റ്റിറ്റ്യൂട്ടിൽ സൂക്ഷിച്ചിട്ടുള്ളതും സൂര്യസിദ്ധാന്തത്തിന്റെ ഒരു മലയാളവ്യാഖ്യാനം എഴുതിയിട്ടുള്ളതുമായ ഒരു പഴയ താളിയോല ഗ്രന്ഥത്തിലെ (നംബർ 9886) ഒരോലയിലാണ് ഈ രേഖ കാണുന്നത്. ഈ ഗ്രന്ഥവരി ആരാണെഴുതിയതെന്നോ ഏതു കാലത്താണെഴുതിയതെന്നോ നിശ്ചയമില്ല. ഇതിൽ സ്മരിച്ചിട്ടുള്ള ഗ്രന്ഥകാരന്മാരിൽ ഏറ്റവും അവ്വാചീനൻ മേല്പത്തൂർ നാരായണഭട്ടതിരിയാണ്. ഈ ഗ്രന്ഥം പകർത്തിയ ലേഖകനോ ഇതുപയോഗിച്ച ഒരു ഗണിതജ്ഞനോ തനിയ്ക്കറിവുള്ള വിവരങ്ങളെ ഇവിടെ രേഖപ്പെടുത്തിയിരിയ്ക്കുന്നു. 'ഇതി ശ്രുതം മയാ' എന്നവസാനത്തു കുറിച്ചിട്ടുണ്ട്. ഈ ഗ്രന്ഥവരി പ്രസാധനം ചെയ്യുകയും ഇതിൽ നിന്നു ലഭിയ്ക്കുന്ന നൂതന വിവരങ്ങളെ ചൂണ്ടിക്കാണിയ്ക്കുകയുമാണ് ഈ ലേഖനം കൊണ്ടുദ്ദേശിയ്ക്കുന്നതും. ആദ്യമായി ഇതിൽ പ്രതിപാദിച്ചിട്ടുള്ള ഗ്രന്ഥകാരന്മാർക്കു ക്രമനംബർ കൊടുത്തു ഗ്രന്ഥവരി താഴെ ചേർക്കുന്നു.[2]

"1. ഗോവിന്ദൻ തലക്കുളത്തൂർ ഭട്ടതിരി. 'രക്ഷേൽ ഗോവിന്ദമക്കഃ' എന്ന് അദ്ദേഹത്തിന്റെ ജനനകലി. കേരളേശ്വരസമീപത്ത് ഇല്ലമാകുന്നു.

2. പുതുമനച്ചോമാതിരി. കരണപദ്ധതി കർത്താ.

3. പരമേശ്വരൻ വടശ്ശേരി നമ്പൂരി. നീളായാസ്സൈമതീരേബ്ധേസ്തീരസ്ഥഃ പരമേശ്വരഃ. അയമേവ ശകാബ്ദസംസ്കാരകർത്താ.

4. അസ്യ തനയോ ദാമോദരഃ.

5. അസ്യ ശിഷ്യോ നീലകണ്ഠസോമയാജീ. ഇദ്ദേഹം തന്ത്രസംഗ്രഹം, ആര്യഭടീയഭാഷ്യം മുതലായ ഗ്രന്ഥങ്ങൾക്കു കർത്താവാകുന്നു. 'ലക്ഷ്മീശനിഹിതധ്യാനൈഃ' ഇത്യസ്യ കലിനാ കാലനിർണ്ണയഃ.

6. പൂർവ്വോക്തദാമോദരസ്യ ശിഷ്യഃ ജ്യേഷ്ഠദേവഃ. ഇദ്ദേഹം പറങ്കൊട്ട നമ്പൂരിയാകുന്നു. യുക്തിഭാഷാഗ്രന്ഥത്തെ ഉണ്ടാക്കിയതും ഇദ്ദേഹം തന്നെ.

7. ജ്യേഷ്ഠദേവന്റെ ശിഷ്യൻ തൃക്കണ്ടിയൂർ അച്യുതപിഷാരടി. ഇദ്ദേഹം സ്തംഭനിർണ്ണയം, ഗോളദീപിക മുതലായ ഗ്രന്ഥ(ങ്ങൾക്ക) കർത്താവാകുന്നു.

8. അച്യുതപിഷാരടിയുടെ ശിഷ്യൻ മേൽപത്തൂർ നാരായണഭട്ടതിരി. ഇദ്ദേഹം നാരായണീയം, സപ്തസ്വരം മുതലായ ഗ്രന്ഥങ്ങൾക്കു കർത്താ. 'ആയുരാരോഗ്യസൌഖ്യ'മിത്യാദി കലിനാകാലനിർണ്ണയഃ.

9. മാധവൻ ഇലഞ്ഞിപ്പുള്ളി എമ്പ്രാൻ. വെണ്പാരൊഹം, മുഹൂർത്തമാധവീയം, പ്രശ്നമാധവീയം മുതലായ ഗ്രന്ഥങ്ങൾക്കും, അഗണിതപഞ്ചാംഗതിനും കർത്താവാകുന്നു.

10. ബ്രഹ്മസൂര്യവസിഷ്ഠരോമശപൌലിശാഃ പഞ്ചസിദ്ധാന്തകർത്താരഃ.

11. മയോപി സൂര്യസിദ്ധാന്തസ്യ പ്രഷ്ടൃതാൽ കർത്താ.

12. ഗർഗ്ഗഃ ഏതേഷാം സമാനകക്ഷ്യതാൽ സോപി പൂജ്യതമഃ.

13 ശ്രീപതിഃ സിദ്ധാന്തശേഖരശ്രീപതി പദ്ധത്യാദീനാം കർത്താ.

14. ലല്ലഃ ശിഷ്യധീവൃദ്ധിദമിതി പ്രസിദ്ധസ്യ തന്ത്രസ്യ കർത്താ.

15. മുഞ്ജാലകോ മാനസകർത്താ. മാനസമേവ ലഘുമാനസം.

16. ഗോവിന്ദോ മുഹൂർത്തരത്നാദീനാം കർത്താ.

17. പരമേശ്വരഃ ദൃഗ്ഗണിതകർത്താ ഇതി ശ്രുതം മയാ."

മറ്റു വിധത്തിൽ അറിവുള്ള പല സംഗതികൾക്കും ഈ ഗ്രന്ഥവരിയിൽ നിന്നു സ്ഥിരീകരണം ലഭിയ്ക്കുന്നതു കൂടാതെ ഇതിനു മുമ്പറിവില്ലാത്ത ചില വിവരങ്ങളും ഇതിൽ നിന്നു കിട്ടുന്നുണ്ടെന്നുള്ളതു ശ്രദ്ധേയമാണ്.

1. 'തലക്കുളത്തൂർ ഗോവിന്ദഭട്ടതിരി'യുടെ ഇല്ലം കേരളേശ്വരത്താണെന്നുള്ള വസ്തുത ഈ ഗ്രന്ഥവരിയിൽനിന്നു കിട്ടുന്നു. 'രക്ഷേൽ ഗോവിന്ദമക്കഃ' എന്ന കലിദിനം (കൊല്ലം 412, എ. ഡി. 1237) ഭട്ടതിരി തന്റെ ദശാധ്യായി എഴുതിയവസാനിപ്പിച്ച ദിവസമാണെന്നു[3] ഒരഭിപ്രായമുണ്ട് (കൊളത്തേരി ശങ്കരമേനോൻ, ഭാഷാജാതകപദ്ധതി, അവതാരിക, vii-viii). ഇതു ഭട്ടതിരിയുടെ ജന്മദിനം തന്നെയാണെന്ന പക്ഷമാണ് ഈ ഗ്രന്ഥവരിയിൽ കാണുന്നത്.

3. 'പരമേശ്വരൻ നമ്പൂതിരി. അയാൾ ശകാബ്ദസംസ്കാരകർത്താ' എന്നതു ശരിയല്ല തന്നെ. പ്രാചീനമായ പരമേശ്വരണിതത്തിന്റെ ഒരു പ്രധാനാംശവും 'പരമേശ്വരഗ്രഹാഭാവോനാത്' എന്ന ശ്ലോകത്തിലെന്നപോലെ വിഷ്കംഭത്തുമാണ് ശകാബ്ദസംസ്കാരം.

6. ജ്യേഷ്ഠദേവൻ: തൃക്കണ്ടിയൂർ അച്യുതപ്പിഷാരടിയുടെ ഉപരാഗക്രിയാക്രമത്തിന്റെ അന്ത്യശ്ലോകത്തിൽ:

പ്രോക്തഃ പ്രവയസോ ധ്യാനാത്
ജ്യേഷ്ഠദേവസ്യ സദ്ഗുരോഃ
വിച്യുതാശേഷദോഷേണേ
തൃച്യുതേന ക്രിയാക്രമഃ

എന്നും, പ്രസ്തുത കൃതിയുടെ ഒരു ഭാഷാവ്യാഖ്യാനത്തിൽ—"വയോധികനായി തന്റെ ജ്യേഷ്ഠദേവനെന്ന പ്രസിദ്ധനായിരിയ്ക്കുന്ന ഗുരുവിന്റെ ധ്യാനം കൊണ്ട് മനസ്സിന്റെ മാലിന്യാദികളായിരിയ്ക്കുന്ന ദോഷങ്ങളോട് വേർപെട്ടിരിയ്ക്കുന്ന അച്യുതനാൽ ഇങ്ങനെ ഗ്രഹണത്തിന്റെ ക്രിയാക്രമം ചൊല്ലപ്പെട്ടു" എന്നും കാണുന്നതിൽ നിന്നു പിഷാരടിയുടെ ഗുരുവായി ജ്യേഷ്ഠദേവനെന്നൊരു പണ്ഡിതനുണ്ടായിരുന്നുവെന്നു മാത്രമേ അറിവുണ്ടായിരുന്നുള്ളൂ. പ്രസ്തുതജ്യേഷ്ഠദേവൻ (ഇതൊരു ബിരുദപ്പേരോ കേരളീയനാമത്തിന്റെ സംസ്കൃതീകരണമോ ആവോ?) പറങ്കൊട് (പറങ്ങോട്?) ഇല്ലത്തെ അംഗമാണെന്നും യുക്തിഭാഷാകർത്താവാണെന്നും[3] മാത്രമല്ല, അച്യുതപിഷാരടിയുടെ ഗുരുവിനു പരമേശ്വരനോടുള്ള ശിഷ്യപ്രശിഷ്യബന്ധവും—പരമേശ്വരൻ, ദാമോദരൻ, ജ്യേഷ്ഠദേവൻ, പിഷാരടി എന്ന ക്രമത്തിനു—ഈ ഗ്രന്ഥവരി അറിയിയ്ക്കുന്നു. ജ്യേഷ്ഠദേവന്റെ കാലവും ഇതിൽ നിന്നു നിർണ്ണയിയ്ക്കാം. യുക്തിഭാഷ എ. ഡി. 1500–ൽ നീലകണ്ഠസോമയാജി രചിച്ച തന്ത്രസംഗ്രഹത്തെ ആശ്രയിച്ചെഴുതിയതായതു കൊണ്ട്[4] അതിനു ശേഷമാകണം യുക്തിഭാഷയുടെ രചന. നീലകണ്ഠസോമയാജിയും (എ. ഡി. 1443–1550)[5] ദാമോദരന്റെ ശിഷ്യൻ തന്നെയാകയാൽ നീലകണ്ഠനെ ഉപജീവിച്ചെഴുതിയ ജ്യേഷ്ഠദേവൻ നീലകണ്ഠന്റെ കനിഷ്ഠനായ ഒരു സതീർത്ഥ്യനെന്നു വേണം വിചാരിയ്ക്കുവാൻ. വയോധികനായിരുന്ന കാലത്താണ് ജ്യേഷ്ഠദേവൻ അച്യുതപ്പിഷാരടിയുടെ (എ. ഡി. 1550–1621)

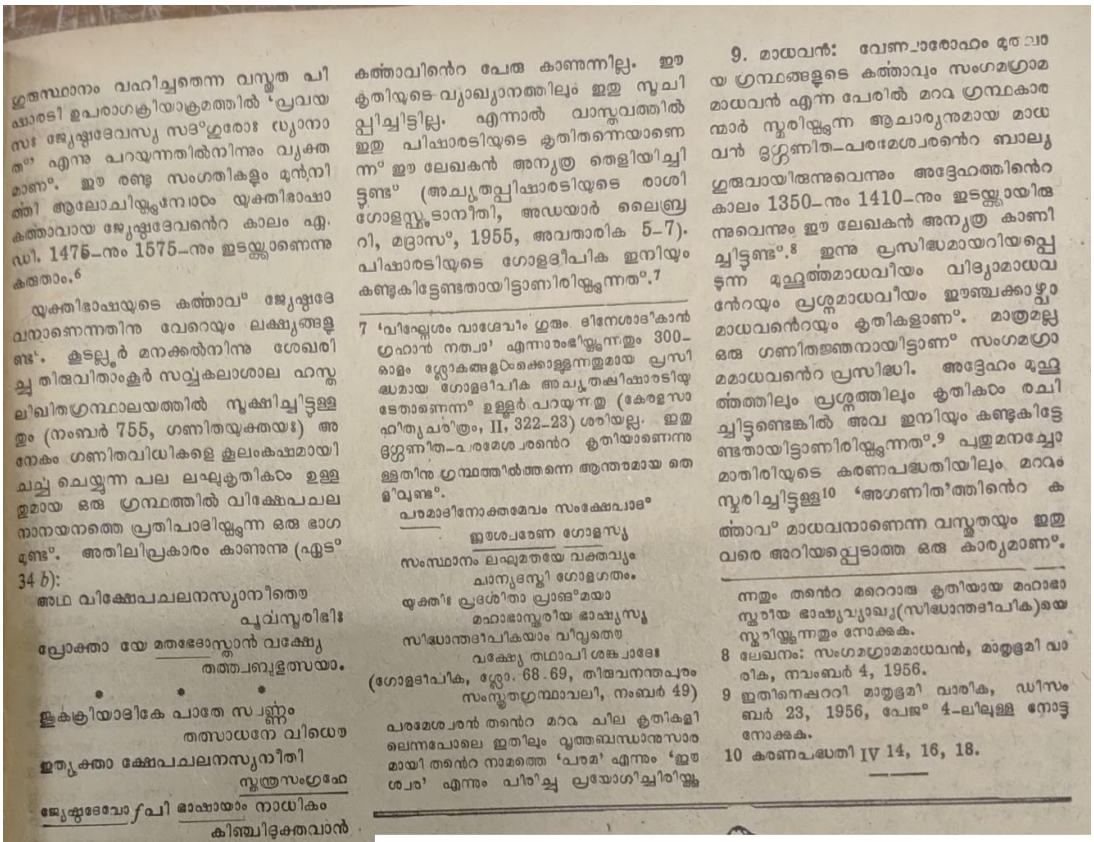
1 ഉള്ളൂർ, കേരളസാഹിത്യചരിത്രം, II, 321. കൃഷ്ണനാശാന്റെ ബഹുമുഖമായ സാഹിത്യസേവനത്തേയും ശിഷ്യപരമ്പരയേയും പറ്റിയുള്ള വിവരങ്ങൾക്ക്, 'ഭാഷാജാതകപദ്ധതി,' തിരുവനന്തപുരം, 1101, പ്രസ: കൊളത്തേരി ശങ്കരമേനോൻ, അവതാരിക; ഉള്ളൂർ, കേരളസാഹിത്യചരിത്രം, II. 321–22, III. 519–527, മുതലായവ നോക്കുക.

2 പ്രസാധനത്തിനായി ഈ ഗ്രന്ഥവരി സമയം പകർത്തി അയച്ചു തന്നതിനു ശ്രീ നമ്പ്യാരോടു ലേഖകനുള്ള കൃതജ്ഞത ഇവിടെ രേഖപ്പെടുത്തട്ടെ.

3 യുക്തിഭാഷാകർത്താവു് പറങ്കൊട്ടില്ലത്തെ അംഗമാണെന്നതു പ്രസ്തുതമാണ്.

4 യുക്തിഭാഷ ആരംഭിയ്ക്കുന്നതു തന്നെ—'അഥ തന്ത്രസംഗ്രഹത്തെ അനുസരിച്ചു നിന്നു തന്ത്രഗണിതത്തിൽ ഉപയോഗമുള്ള ഗണിതങ്ങളെ മുഴുവനും ചൊല്ലുവാൻ ഇടയുണ്ടാകുന്നതാണ്...' എന്ന അവതാരികയോടു കടപ്പാടുണ്ട്. യുക്തിഭാഷയിൽ അവിടവിടെ സോമയാജിയുടെ തന്ത്രസംഗ്രഹത്തേയും സിദ്ധാന്തശേഖരത്തേയും സൂചിയ്ക്കുന്നുണ്ടെന്നതും ശ്രദ്ധേയം.

5 സോമയാജിയുടെ കാലനിർണ്ണയത്തിനു തെളിവുകൾ 'സിദ്ധാന്തദർപ്പണ'ത്തിന്റെ അവതാരികയിൽ (അഡയാർ ലൈബ്രറി, 1955) ഈ ലേഖകൻ കാണിച്ചിട്ടുണ്ട്.



ഇതുസ്ഥാനം വഹിച്ചതെന്ന വസ്തുത പിഷാരടി ഉപരാഗക്രിയാക്രമത്തിൽ 'പ്രവയസഃ ജ്യേഷ്ഠദേവസ്യ സദ°ഘ്രോഃ ധ്യാനാത്' എന്നു പറയുന്നതിൽനിന്നും വ്യക്തമാണ°. ഈ രണ്ടു സംഗതികളും മുൻനിർത്തി ആലോചിയ്ക്കുമ്പോൾ യുക്തിഭാഷാകർത്താവായ ജ്യേഷ്ഠദേവന്റെ കാലം എ.ഡി. 1475–നും 1575–നും ഇടയ്ക്കാണെന്നു കരുതാം.[6]

യുക്തിഭാഷയുടെ കർത്താവ° ജ്യേഷ്ഠദേവനാണെന്നതിനു വേറെയും ലക്ഷ്യങ്ങളുണ്ട്. കൂടല്ലൂർ മനക്കൽനിന്നു ശേഖരിച്ചു തിരുവിതാംകൂർ സർവ്വകലാശാല ഹസ്തലിഖിതഗ്രന്ഥാലയത്തിൽ സൂക്ഷിച്ചിട്ടുള്ളതും (നംബർ 755, ഗണിതയുക്തയഃ) അനേകം ഗണിതവിഡികളെ ക്രമീകരിച്ചു ചെയ്യുന്ന പല ലഘുകൃതികളും ഉള്ളതുമായ ഒരു ഗ്രന്ഥത്തിൽ വിക്ഷേപചലനാനയനത്തെ പ്രതിപാദിയ്ക്കുന്ന ഒരു ഭാഗമുണ്ട°. അതിലിപ്രകാരം കാണുന്നു (ഫുട° 34 *b*):

അഥ വിക്ഷേപചലനസ്യാനീതൌ
പൂർവ്വസൂരിഭിഃ
പ്രോക്താ യേ മതഭേദാസ്താൻ വക്ഷ്യേ
തത്ത്വബുഭുത്സയാ.

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ക്രിയാടികേ പാതേ സ ച്ഛന്നം
തത്സാധനേ വിധൌ
ഇത്യുക്താ ക്ഷേപചലനസ്യാനീതി
സൂത്രസംഗ്രഹേ
ജ്യേഷ്ഠദേവോ ഽപി ഭാഷായാം നാധികം
കിഞ്ചിദുക്തവാൻ

• • •

തന്ത്രസംഗ്രഹത്തെ ഉടനു പറയുന്ന ഭാഷ തന്ത്രസംഗ്രഹത്തിന്റെ ഭാഷയായ യുക്തിഭാഷതന്നെയാണെന്നു പറയുന്നതിൽ അത്യുക്തിയൊന്നുമില്ല. ഇവിടെ ഉപയോഗിച്ചുള്ള 'അപി'ശബ്ദം ഇതിനുപോദ°ബലകവുമാണ°. ഈ ഭാഷയുടെ കർത്താവ° ജ്യേഷ്ഠദേവനുമാണ° എന്നതു ഈ ഉദ്ധരണത്തിൽനിന്നു മനസ്സിലാക്കേണ്ടതു°.

7. തൃക്കണ്ടിയൂർ അച്യുതപ്പിഷാരടി: സ്ഫുടനിർണ്ണയം, ഗോളദീപിക. സ്ഫുടനിർണ്ണയത്തിന്റെ കിട്ടിയേടത്തോളം മാതൃകകളിലൊന്നുംതന്നെ (തിരുവനന്തപുരം ഹസ്തലിഖിതഗ്രന്ഥശാല L. 848-B; C. 946-B; C. 809-C; C. 1945-B; മദിരാശി ഹസ്തലിഖിതഗ്രന്ഥശാല R. 3799-B മുതലായവ) കർത്താവിന്റെ പേരു കാണുന്നില്ല. ഈ കൃതിയുടെ വ്യാഖ്യാനത്തിലും ഇതു സൂചിപ്പിച്ചിട്ടില്ല. എന്നാൽ വാസ്തവത്തിൽ ഇതു പിഷാരടിയുടെ കൃതിതന്നെയാണെന്ന° ഈ ലേഖകൻ അനുമാനിച്ചിട്ടുണ്ട° (അച്യുതപ്പിഷാരടിയുടെ രാശിഗോളസ്ഫുടാനീതി, അഡയാർ ലൈബ്രറി, മദ്രാസ°, 1955, അവതാരിക 5–7). പിഷാരടിയുടെ ഗോളദീപിക ഇനിയും കണ്ടുകിട്ടേണ്ടതായിട്ടാണിരിയ്ക്കുന്നതു°.[7]

9. മാധവൻ: വേണ്ണാറോഹം മുതലായ ഗ്രന്ഥങ്ങളുടെ കർത്താവും സംഗമഗ്രാമമാധവൻ എന്ന പേരിൽ മറ്റു ഗ്രന്ഥകാരന്മാർ സ്മരിയ്ക്കുന്ന ആചാര്യനുമായ മാധവൻ ദുഗ്ഗണിത–പരമേശ്വരന്റെ ബാല്യഗുരുവായിരുന്നുവെന്നും അദ്ദേഹത്തിന്റെ കാലം 1350–നും 1410–നും ഇടയ്ക്കായിരുന്നുവെന്നും ഈ ലേഖകൻ അനുമാനിച്ചിട്ടുണ്ട°.[8] ഇന്നു പ്രസിദ്ധമായിരിയ്ക്കുന്ന മുഹൂർത്തമാധവീയം വിദ്യുന്മാധവീയം ശ്രീഭൂതമാധവീയം ഈഞ്ചക്കാഴ്ച്ചമാധവന്റെയും കൃതികളാണ°. മാത്രമല്ല ഒരു ഗണിതജ്ഞനായിട്ടാണ° സംഗമഗ്രാമമാധവന്റെ പ്രസിദ്ധി. അദ്ദേഹം മുഹൂർത്തത്തിലും പ്രശ്നത്തിലും കൃതികൾ രചിച്ചിട്ടുണ്ടെങ്കിൽ അവ ഇനിയും കണ്ടുകിട്ടേണ്ടതായിട്ടാണിരിയ്ക്കുന്നതു°.[9] പുരുമനയ്ക്കോമാതിരിയുടെ കരണപദ്ധതിയിലും മറ്റും സ്മരിച്ചിട്ടുള്ള[10] 'അഗണിത'ത്തിന്റെ കർത്താവ° മാധവനാണെന്ന വസ്തുതയും ഇതുവരെ അറിയപ്പെടാത്ത ഒരു കാര്യമാണ°.

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7 'വിഭ്രല്ലശം വാശ്വേവം ഗ്രഹം ദിനേശാദ്കാൻ ഗ്രഹാൻ നത്വാ' എന്നാരംഭിയ്ക്കുന്നതും 300–ഓളം ശ്ലോകങ്ങളുൾക്കൊള്ളുന്നതുമായ പ്രസിദ്ധമായ ഗോളദീപിക അച്യുതപ്പിഷാരടിയുടേതാണെന്ന° ഉള്ളൂർ പറയുന്നു (കേരളസാഹിത്യചരിത്രം, II, 322–23) ശരിയല്ല. ഇതു ദുഗ്ഗണിത–പരമേശ്വരന്റെ കൃതിയാണെന്നുള്ളതിനു ഗ്രന്ഥത്തിൽത്തന്നെ ആന്തരമായ തെളിവുണ്ട°.

പരമാദിനോക്തമേവം സംക്ഷേപാദ°
ഇദശപരേണ ഗോളസ്യ
സംസ്ഥാനം ലഘുമതയേ വക്തവ്യം
ചാന്യദസ്മൽ ഗോളഗതം.
യുക്തിഃ പ്രദർശിതാ പ്രാജ്ഞ°മയാ
മഹാഭാസ്കരീയ ഭാഷ്യസ്യ
സിദ്ധാന്തദീപികയാം വിവൃതൌ
വക്ഷ്യേ തഥാപി ശങ്കാദേഃ
(ഗോളദീപിക, ശ്ലോ. 68–69, തിരുവനന്തപുരം സംസ്കൃതഗ്രന്ഥാവലി, നംബർ 49)

പരമേശ്വരൻ തന്റെ മറ്റു ചില കൃതികളിലെന്നപോലെ ഇതിലും സ്വന്തഭാസ്കരസമീരമായി തന്റെ നാമത്തെ 'പരമ' എന്നും 'ഇദശപര' എന്നും പിരിച്ചു പ്രയോഗിച്ചിരിയ്ക്കുന്നതും തന്റെ മറ്റൊരു കൃതിയായ മഹാഭാസ്കരീയ ഭാഷ്യവ്യാഖ്യ(സിദ്ധാന്തദീപിക)യെ സ്മരിയ്ക്കുന്നതും നോക്കുക.

8 ലേഖനം: സംഗമഗ്രാമമാധവൻ, മാതൃഭൂമി വാരിക, നവംബർ 4, 1956.

9 ഇതിനെപ്പറ്റി മാതൃഭൂമി വാരിക, ഡിസംബർ 23, 1956, പേജ° 4–ലിലുള്ള നോട്ടു നോക്കുക.

10 കരണപദ്ധതി IV 14, 16, 18.

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6 ഉള്ളൂരിന്റെ സംസ്കൃതകോളേജ° ഗ്രന്ഥശാലയിലുള്ള ഒരു യുക്തിഭാഷാഗ്രന്ഥത്തിന്റെ അവസാനത്ത°:

ആലേഖി യുക്തിഭാഷാ വിപ്രേണ
ബ്രഹ്മദത്തസംജ്ഞേന
'യേ ഗോള പഥസ്ഥാഃ സ്യുഃ'
കലിരഹിതാഃ ശോധയന്തു സന്തഃ

എന്നൊരു പദ്യം കാണുന്നതിൽനിന്നു യുക്തിഭാഷയുടെ കർത്താവു ബ്രഹ്മദത്തനാണെന്നും രചനാകാലം 'യേ ഗോള പഥസ്ഥാഃ സ്യുഃ' എന്ന കലിദിനത്തിനു തുല്യമായ 1750 എ.ഡി.യാണെന്നും ഊഹിച്ചിരുന്നതു (യുക്തിഭാഷാ, മംഗളോദയം, 1123, അവതാരിക, പേജ° 5) ശരിയല്ലെന്നും ബ്രഹ്മദത്തൻ പ്രസ്തുത ഗ്രന്ഥത്തിന്റെ ലേഖകൻ മാത്രമാണെന്നും ഈ കലിദിനം ഗ്രന്ഥം പകർത്തിത്തീർന്ന കാലമായാണ° കരുതുന്നതെന്നും വിചാരിയ്ക്കേണ്ടിയിരിയ്ക്കുന്നു. 'ആലേഖി' എന്ന മാതൃമാനുള്ള ബ്രഹ്മദത്തൻ പ്രയോഗവും

Paper 9

K. V. Sarma (1958)

Date of Mādhava - A little known Indian Astronomer

Quarterly Journal of The Mythic Society, Bangalore, Vol. XLIX, no. 3, 1958. Available in *Facets of Indian Astronomy (A Collection of Articles of Prof. K. V. Sarma)*, edited by Prof Siniruddha Dash, Rashtriya Sanskrit University, Tirupati, 2009, pp. 270–273.

Notes

The Mythic Society is an institution based in Bangalore devoted to Indic studies and established in 1909 by F. H. Richards the then Collector of the Bangalore Contonment District.

K. V. Sarma

For a brief biographical sketch of K. V. Sarma, see p. 79.

Source: Internet Archive

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DATE OF MĀDHAVA- A LITTLE KNOWN INDIAN ASTRONOMER*

Āryabhaṭa's school of Indian astronomy (499 A.D.) had much following in the South of India, especially in Kerala where the system has been assiduously studied and developed and followed to this day. Significant improvements and corrections came to be made in a number of works in this school written by a long line of Kerala scholars. One of the more important of these last mentioned writers is Mādhava of Saṅgama grāma and author of a work called *Veṇvāroha,* who, however, does not find mention in the printed records of Indian mathematics and astronomy. In fact, even in his own home province his contributions have yet to be fully evaluated.

Mādhava is frequently quoted as an authority by Kerala astronomers.[1] Two of his significant contributions to mathematics known from these quotations are his enunciation of formulae for the accurate determination of the circumference of a circle and the value of π by the method of indeterminate series, a method which was re-discovered in Europe nearly three hundred years later.[2] by James Gregory (1638-75) and Gothfreid Wilhelm Leibnitz (1646-1716), and the formula for the Sine of the sum of two angles, Sine (A+B) = Sine A, Cos B+ Cos A. Sine B. Thus in his commentary *Kriyākramakarī* on the *Līlāvatī*, section of circles (II.40), Nārāyaṇa discussing the various views on the determination of the circumference of a circle says :

यद्येवमतिसूक्ष्मः परिधिः कथमानेतुं शक्यत इति चेद् अत्राह माधवः - - - - । इतोऽपि लघुतरपरिध्यानयनोपायः अमुनैवोक्तम् । यथा-

व्यासे वारिधिनिहते रूपहते व्याससागराभिहते ।
त्रिशरादि विषमसंख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात् ।।
- -
लब्धः परिधिः सूक्ष्मो बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात् । इति ।

Quarterly Journal of Mythic Society, Bangalore, Vol. XLIX. no. 3. 1958.

This would give the value-

$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots\ldots + 1/n\text{-}2 - 1/n + 1/n{+}2$ which is the same basic series attributed to Gregory and Leibnitz[3].

Quite often when Mādhava is mentioned, his name is prefixed with a title *Golavid* (expert on spherics). Thus Acyuta Piśāraṭi in the opening verse of his *Sphuṭanirṇaya* paying homage to his predecessors in the field, says वन्दे गोलविदश्च माधवमुखान् , and the commentary on the work explains, माधवो वेण्वारोहादीनां कर्ता Nīlakaṇṭha Somayāji says in his commentary on *Āryabhaṭīya* - अत एव गोलविदा माधवेन विक्षेपवतां स्फुटापक्रमानयने गणितविशेषः प्रदर्शितः (Mss. trav. Uni. Mss. Library , T. 298, p. 186); तच्च सङ्गमग्रामजेन गोलतत्त्वविदां माधवेन प्रदर्शितम् (*ibid.* p. 138.). It would seem from such references that Mādhava was especially proficient in spherical astronomy. A Manuscript of Mādhava's *Veṇvāroha*[4] contains in continuation of that work, another work called *Golavāda* and it is possible that the *Golavāda* too is a work of Mādhava and that it was this work that got him the title *Golavid.*

The only work of Mādhava hitherto known as *Veṇvāroha*[5], an astronomical manual which enunciates an ingenious method for the accurate calculation of the longitude of the moon. Herein the author deals elaborately with the preparation of a table using which the longitudes of the moon could be read out at short intervals of one and a half *nāḍikā*-s each, the same table being valid for a cycle of 248 days.

Towards the end of *Veṇvāroha*, the author mentions another work of his, a set of astronomical *Vākyas* (short formulae expressing specific numbers in the *Kaṭapāyādi* notation) commencing with the *vākya* 'शीलं राज्ञः श्रिये' *cf.*

'शीलं राज्ञः श्रि[illegible] कृत्वा प्राक् पुनर्येन निर्मितम् ।
विलिप्तिकादिकं वाक्यजातं तेनेयमीरिता ।। (verse no. 73)

This work, which I could identify and trace to four Manuscripts,[6] is a set of 248 *vākyas*, giving the daily motion of the moon correct to the second, for accurate calculations of the moon and especially in the preparation of his *Veṇvāroha* table.

No other full-fledged work of Mādhava is known so far. However, in later works we meet with numerous quotations from Mādhava comprising all aspects of Mathematics and astronomy and it is quite probable that he has written other works from which these quotations are taken. The statement of the commentator of the *Sphuṭanirṇaya*, previously referred to, माधवो वेण्वारोहादीनां कर्ता, also supports this view. An anonymous commentary on a fragmentary astronomical work[7] has the verse-

'लम्बाप्ते लम्बकर्णः स्यात् त्रिज्यापरपराहते '।
इत्युक्तत्वाद् गोलविदा लग्नप्रकरणेऽपि च ।।

The expression *Golavidā* here would seem to refer to a specific author, Mādhava being the only author from Kerala thus styled might be the author of this work. A work *Lagnaprakaraṇa* in eight chapters is known[8] and the above is the eighth verse in Ch. V, thereof.

That Mādhava belonged to Saṅgamagrāma, a village. The name Saṅgama, is testified by all authors quoting him by prefixing his name with the expression *Saṅgamagrāmajaḥ*. Saṅgamagrāma in Kerala is a place called Iriññālakkuṭa in Cochin, famous for the temple of Saṅgameśvara (Bharata). In verse 13 of his *Veṇvāroha*, Mādhava gives also his family-title.

बकुळाधिष्ठितत्वेन विहारो यो विशेष्यते ।
गृहनामनि सोऽयं स्यान्निजनामनि माधवः ।।

The Malayalam commentary on this work gives the local form of the word *Iraññi ninna paḷḷi*; cf. "*Bakulam = Iraññi , vihāram = paḷḷi. Iraññi ninna paḷḷi enna illa-p-per.*" This house exists even now in that place, though the present descendants have failed to preserve their scholarly tradition.

It is possible to fix the date of Mādhava from the epoch he gives in the *Veṇvāroha*. For the calculation of the mean Sun, it gives the epoch of दीननम्रानुशास्य (15,02,008) *Kali* days (i.e. 1010 A.D.) and आदिकूर्म (5180) cycles of the moon, corresponding to 390 years, the two together giving the date 1400 A.D. as the date of the work. This date of the author is also supported by other evidences. Thus Nīlakaṇṭha

Somayāji who lived and studied at the house of Paramaeśvara, says about the latter: परमेश्वरस्तु... माधवादिभ्यो गोलविद्भ्यो गणितगोलयुक्तीरपि बाल्य एव सम्यग् गृहीत्वा[9], etc. Parameśvara wrote his *Dṛggaṇita* in 1430 A.D. (*Śaka* 1353) which date is mentioned at the end of that work 'एवं दृग्गणितं शाके त्रीषु-विश्व-मिते कृतम्' It is quite possible that Mādhava wrote his work thirty years before that of his disciple. We may, therefore, consider him to have lived between 1350 and 1410 A.D.

NOTES AND REFERENCES:

1. *Cf.* Nīlakaṇṭha Somayāji, in his several works; Nārāyaṇa in his Commentary on the Līlāvatī; and Śaṅkara in his Commentary on *Tantrasaṅgraha*, etc.
2. On Mādhava's date 1350-1410, see further.
3. It may also be mentioned here that Mādhava enunciates further corrections to this basic series, an investigation into which is beyond the scope of this article, and should form a separate study.
4. Belonging to the Elamkunnappuzha Naṭakkal Kovilakam, Ernakulam.
5. This work with the Malayalam commentary of Acyuta Piṣāraṭi is being issued serially in the *Ravi Varma Granthāvali,* Vol. II-III (1955-56), Sanskrit College, Trippunithura, by the author, and published later as no.7, *Sri Ravivarma Skt. Ser.*, Sanskrit College, Trippunithura,1956.
6. This has been given astronomy an appendix to the above edition.
7. Mss. Madras Mal. D. 339, p. 66.
8. For Mss. see Kerala Uni. Mss. Library, 414-B, L. 1360-C.
9. *Āryabhaṭīya Bhāṣya* Mss. Kerala Uni. Mss. T.298, p. 258.

Paper 10

K. V. Sarma (1973)

Mādhava, the Author of *Sphuṭacandrāpti*

Extract from
Saṅgamagrāma Mādhava's *Sphuṭacandrāpti (Computation of True Moon)* critically edited with an English translation by K. V. Sarma, published by Vishveshvaranand Institute, Hoshiarpur, Panjab in 1973, pp. 12–14.

K. V. Sarma

For a brief biographical sketch of K. V. Sarma, see page 79.

Source: Internet Archive

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Mādhava, the Author

Among the several astute astronomers of Kerala of the middle ages, Mādhava of Saṅgamagrāma holds a position of eminence. Till recently, he was unknown to the scholarly world, especially outside his native land. He hailed from the village of Saṅgamagrāma, the modern Irinjalakkuda, near Cochin, in Central Kerala, the name of his house being *Iraññi ninna palḷi* to be identified with one of the two still existing houses in the village, named Iriññanavaḷḷi and Iriññārappaḷḷi. This information the author gives in his *Veṇvāroha* and is corroborated by his commentator Acyuta.[1] Mādhava was the teacher of Parameśvara, (A.D. 1360-1455), the promulgator of the *Dṛggaṇita* school of Kerala astronomy and is frequently quoted in the mediaeval astronomical literature of Kerala with the appellation of *Golavid* ('Adept in Spherics'). Thus Nīlakaṇṭha Somayāji (1444-1545 A.D.), while referring to Parameśvara in his *Āryabhaṭīya-Bhāṣya* says : *Parameśvaras tu ... Mādhavādibhyo 'Golavidbhyo' Gaṇita-gola-yuktīr api bālya eva samyag gṛhītvā ...*.[2] Acyuta Piṣāraṭi uses the same appellation for Mādhava in the introductory verse to his *Sphuṭanirṇaya : vande 'golavidaś' ca Mādhavamukhān* etc.

Works of Mādhava

The *Veṇvāroha*[3] depicting a facile method for the computation of the Moon and the Moon-sentences,[4] commencing with *śīlam*

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1. *Cf.* बकुलाधिष्ठितत्वेन विहारो यो विशेष्यते ।
गृहनामनि सोऽयं स्यान्निजनामनि माधवः ॥ (verse 13)

Com. in Malayalam . यातोरु विहारं बकुळाधिष्ठितत्वं कोण्टु विशेषिक्कप्पेटुन्नतु, इवन् गृहनामत्तिङ्कल् श्रवनाकुन्नतु । बकुळम् = इरञ्ञि । विहारम् = पळ्ळि । 'इरञ्ञि निन्न पळ्ळि' एन्नु इल्लप्पेरु । तन्टे नामत्तिङ्कल् माधवन् ।

On this, see K.V. Sarma, Introduction to his edn. of *Veṇvāroha*., *op. cit.*, pp. 6-7.

2. Edn., *Trivandrum Sanskrit Series*, No. 185, p. 154. For other similar statements, see *ib.*, p. 75 ; *tac ca Saṅgamagrāmajena 'Golatattvavidā' Mādhavena pradarśitam*; p. 108 : *ata eva 'Golavidā' Mādhavena kṣepavatām sphuṭāpakramānayane gaṇitaviśeṣaḥ pradarśitaḥ*

3. Cr. ed. by K. V. Sarma, (Tripunithura, 1956), with the Malayalam commentary of Acyuta.

4. Ed. as Appendix I, below.

rājñaḥ śriye were the only works of Mādhava known till recently. It has been shown lately[1] that he is the author also of *Lagnaprakaraṇa* in six chapters, *Agaṇita*, an extensive work on the computation of planets, using novel methodologies and two short tracts entitled *Madhyamānayanaprakāra* and *Mahājyānayanaprakāra*. Still another work possibly composed by him is the *Golavāda*, which seems to have helped to stabilise his appellation as '*Golavid*'. *Sphuṭacandrāpti*, edited here, is the latest addition to his works.[2]

Besides the said full-fledged works, a number of stray verses of Mādhava are quoted by later astronomers like Nīlakaṇṭha Somayāji, Nārāyaṇa, commentator of the *Līlāvatī*, Śaṅkara, commentator of *Tantrasaṅgraha* etc. One of his significant contributions to mathematics, known from these quotations, is his enunciation of formulae for the accurate determination of the circumference of a circle and the value of π by the method of indeterminate series, a method which was re-discovered in Europe nearly three hundred years later by James Gregory (1638-75 A. D.) and Gottfried Wilhelm Leibnitz (1646-1716 A. D.).[3] His *jīve paraspara-nyāya* contains the enunciation, probably for the first time in India, of the formula for the sine of the sum of two angles, sine (A+B)=sin A. Cos B + Cos A. Sine B. The study and interpretation, in terms of modern mathematics, of the enunciations of Mādhava in his stray verses and in his full-fledged works is bound to yield valuable results in the history of Hindu astronomy.

Date of Mādhava

Certain directions given in the *Sphuṭacandrāpti* give a general indication of its date of composition. Thus, for the calculation of the Mean Sun we are asked to subtract from the current Kaliday the

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1. See K.V. Sarma, 'Date of Mādhava, a little-known Indian astronomer', *Quarterly Jl. of the Mythic Society*, (Bangalore), 49 (1958-59) 183-86 ; Intro. to the editon of *Veṇvāroha*, pp. 8-9 ; *History of the Kerala school of Hindu astronomy*, (Hoshiarpur, 1972), pp. 51-52, 151.

2. In my *History of the Kerala school of Hindu astronomy, op. cit.*, pp. 32-33, 117, I had mentioned this text as an anonymous work related to the *Veṇvāroha* of Mādhava. However, a closer examination of the work with a view to editing it established its being an earlier work of Mādhava himself.

3. On this, see *History of the Kerala school of Hindu astronomy, op. cit.*, pp. 20-26.

Khaṇḍa 15,02,008 and 5180 anomalistic cycles of the Moon. If there be further days, the number of such days has to be multiplied by the mean daily motion and added (verses 20-21). This would show that the work was composed about this time. This date would be 15,02,008 days (Kali 4112, A.D. 1010) *plus* 5180 anomalistic cycles of the Moon (390 years), *i.e.*, about 1400 A.D.

Mādhava's recently identified work, *Agaṇita*, also gives a clue to his date. Indicating the *śodhyābda-s* ('deductive years') for the computation of the planets, the author states :

1320
शकाब्दात् 'नरलोको'नाद् राघवैर्धीत्सुना कुजः ।
1318
'दिव्यलोको'ननीलाग्रैम् तत्त्वज्ञैर्भाजिते बुधः ॥

1340
'नवलोको'नसारङ्गैर्गजैराप्ते गुरुर्भवेत् ।
1158
'हेमपुण्यो'नशाकाब्दात् सारवैर्गोवरैर्भृगुः ।
1301
'यज्ञलोको'ननीतांशैर्धर्मैराप्ते शनिर्भवेत् ।
1276
'तीर्थप्रियो'नशाकाब्दाद् गन्धज्ञैर्माजिते तमः ॥

The 'deductive years' for the different planets Mars etc. are Śaka 1320, 1318, 1340, 1158, 1301 and 1276, corresponding to A. D. 1398, 1396, 1418, 1236, 1379 and 1354. In consonance with the principle of *śodhyābda-s*, these figures represent the largest number of years possible to be cut off for the different planets at the time when the work was written. The date of composition of the work would thus, be just ahead of the largest 'deductive year' mentioned, which in the present case, is A.D. 1418.

A clue to the date of birth of Mādhava is provided by that of his younger contemporary and pupil, Parameśvara who was born *c.* 1360.

From the above considerations Mādhava could be supposed to have lived between A. D. 1340 and 1425.

Paper 11

P. P. Divakaran (2018)

Who is Mādhava?

Extract from
P. P. Divakaran's *Mathematics in India* published jointly by Hindustan Book Agency and Springer in 2018, pp. 282–290.

P. P. Divakaran

P. P. Divakaran retired as professor of mathematical physics from the Tata Institute of Fundamental Research, Mumbai in 1996. Subsequently, he has spent time at several institutions as adjunct or visiting professor. His book, *Mathematics of India: Concepts, methods and Connections*, Hindustan Book Agency/Springer, 2018, grew out of research done at the Inter-University Centre for Astronomy and Astrophysics, Pune, as the holder of a Senior Award of the Homi Bhabha Fellowships Council.

Source: Private copy.

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9.4 Who was Madhava? . . . and Narayana?

The defining mathematical event of the five centuries separating Madhava from Shankaranarayana took place far from the geographical limits of Kerala and

that of course was the life and work of Bhaskara II. To keep the chronology in perspective, it is useful to remember that at the time of his birth (1114) the Namputiris were already on the ascendant in Kerala, the Perumals having taken a mortal blow to their empire just a few years before. Suryadeva was born towards the end of the 12th century and there is no Bhaskara II (but plenty of Bhaskara I) in his books, perhaps an indication that they were well separated geographically. Considering how widely Bhaskara's writings came to be disseminated, the greater surprise is that the first commentary on a work of his from (possibly) his natal region or nearabouts (Gaṅgādhara on *Līlāvatī*) is as late as 1420 ([Co], [SiSi-P]); indeed, Parameshvara's, also on *Līlāvatī*, may well be earlier.

Political and military events in the northern Deccan during the period 1200–1350 probably had much to do with the absence of any material related to Bhaskara II datable to this time. As noted in Chapter 8.1, towards the end of the 12th century Muhammad Ghori had devastated all of India north of the Vindhya with a ferocity unmatched since the invasion of the Hunas. Ghori was followed by a succession of sultans belonging to different Afghan-Turkic clans based in Delhi. Two clans, the Khiljis and the Tughlaqs after them, became ambitious and led attacks into peninsular India, plundering and killing on a massive scale. Eventually, the Tughlaqs withdrew to Delhi, leaving behind a fragmented polity in the Deccan, with several Islamic chieftains exercising authority over their individual domains. About the same time (the 1330s), the assertively Hindu Vijayanagar kingdom was founded somewhat further south and quickly grew into a rich and powerful empire and a bastion of Hindu orthodoxy and of Hindu scholarship. Barely 20 years after its founding, what can rightly be called a Vedic workshop was set up, with many pandits working under the direction of the great and famous Sayana; it is to them that we owe the survival of the Vedic corpus as we know it today.

If the pattern of cultural and religious persecution of earlier invasions is anything to go by, traditional scholarship in any field would have found it difficult to sustain itself during the heyday of the Delhi sultans. In the Deccan, the establishment of the Vijayanagar kingdom changed that. The profession of scholarship had by then become a monopoly of Hindu learned men; the Jains had abdicated from their long-held attachment to mathematics, Mahavira being the last of them, and there were hardly any Buddhists left in India, leave alone Buddhist scholars. Practitioners of the astronomical sciences, now part of an exclusively Hindu knowledge system, would have found as warm a welcome from the Vijayanagar dispensation as Sayana and his Vedic experts did. It is reasonable to think that that is what actually happened; that a revival of astronomical-mathematical activity, a coming into the open of what had perhaps remained a private endeavour till then, was made possible under the Vijayanagar umbrella. It was long dead by then in the north but a footnote may be added: astronomy did make a return to the north Indian plains but only towards the end of the 16th century, after the Moghuls had brought a measure of stability and tolerance to the region. The Daivajña family, one of

whom served emperor Jahangir while living in Benares, produced some fine astronomers, especially strong on the work of Bhaskara II. The Daivajña were a community of Brahmins domiciled on the west coast all the way down to the Tulu country (but not Kerala) and among the professions they practised was astrology. It is an intriguing thought that the intimate link with astrology may have been a factor in seeing astronomy over the hard times it went through during this unsettled period; the grandson of Bhaskara II describes himself in the plaque he erected in his ancestor's honour as a *Daivajña*. Even – especially – the powerful liked to know what the stars had in store for them.

In any event, our default choice of provenance for mathematicians who lived between 1300 and 1600, and about whom there is no independent information, is either Kerala or the territory under the control of Vijayanagar – the empire lasted till 1565 – subject of course to revision if and when future evidence comes to light. There is no viable alternative and nothing that we know now contradicts it.

The Vijayanagar empire extended to the west coast, including the Tulu country – the base of the spiritual preceptor of the two founding brothers, Vidyaranya, was on the eastern edge of that region – but not any part of Kerala. The cultural separation between the Brahmins north and south of the river Netravati had become more pronounced with the flow of time but exchanges did take place, more into Kerala than the other way. It is to this diffusion that the relatively small community of the Emprantiris of Malabar can trace its origin. As in other instances of social mobility between castes of roughly equal status, these fine distinctions tended to disappear over time though there are still Brahmin families whose caste apellation remains Emprantiri, as there are other similar subcaste survivals.

This is the background against which we have to assess the reference to Madhava as an Emprantiri. The information appears reliable, at least as reliable as Jyeshthadeva being a disciple of Damodara and the author of *Yuktibhāṣā*, because it comes from the same source, the manuscript giving the partial lineage of the Nila mathematicians (section 2 above): "Mādhavan Ilaññippaḷḷi Emprān" ([Sa-HKSHA]). Ilaññippaḷḷi (or Ilanjippalli) is a village or house name whose identification we will come to in a moment. For the present, let us note that if Madhava or his immediate ancestors came from the Tulu country to settle in the Zamorin's kingdom, the question of the proximate cause of the Nila rejuvenation gets a credible answer: the arrival in its midst of someone who, apart from his own genius, brought with him fresh ideas and fresh knowledge, such as the teachings of Bhaskara II. To remind ourselves again, Parameshvara's commentary on *Līlāvatī*, an unexpected intruder among his dozen or so books on his favourite Bhaskara I, marked the first documented appearance of Bhaskara II on the Kerala scene, maybe the first ever commentary anywhere on any work of his (not counting Narayana's independent monographs on subjects inspired by him). Subsequently, Bhaskara II became an inseparable part of Nila mathematical education. Not only does Shankara devote an

exhaustive *vyākhyā* to *Līlāvatī*, Nilakantha has numerous citations from it and the verse with the list of powers of 10 in *Yuktibhāṣā* is lifted bodily from it.

The fact nevertheless is that, aside from the caste name, there is no other positive evidence for Madhava's Tulu roots. There is, instead, an absence of anything that ties his ancestry to a local background (unlike all his followers), no reference to a favourite temple, for example. No source has anything to say about his teacher(s), and that is very striking in the Nila context where we know the lines of succession in full detail. Nilakantha, otherwise so generous with useful historical and personal asides, is silent on Madhava's intellectual descent: his own is traced to Damodara and Ravi to Parameshvara to Madhava but there it stops. (He even takes the trouble to tell us that Shankaranarayana, 600 years in the past, was a pupil of Govinda). On the other hand, scattered among his rich tributes to Madhava's mathematics in the *Āryabhaṭīyabhāṣya* are a few personal anecdotes which, if we are willing to clutch at straws, may actually point to a connection away from the Nila. In the commentary on *Gaṇita* 12 (the rule for the sine table) he says that he was pleased to have found a new method for the addition formula for sines until he discovered a verse of Madhava's containing the same method; general trigonometric identities are a theme which came to Kerala with Madhava. He also speaks of a meeting he had, while travelling, with an astrologer (the word used is *Daivajña*, not normally employed in Kerala in that sense) born in the Mushika country from whom he learned a stanza composed by Madhava about operations with zero; Mushika is far from the Nila, in the extreme north of Kerala, and shared the Netravati river as the common frontier with the Tulu country. These are at best faintly suggestive of possible northern links, specifically with the mathematics of Bhaskara II (trigonometric identities and operations with zero), that did not exist before Madhava. In the end, however, the only directly personal piece of information we get from Nilakantha is that Madhava was born in a village whose name in its Sanskrit version is Sangamagrama. It is as though Nilakantha was as much in the dark about the pedigree of his great-grand-guru as we are. That is hard to account for if Madhava was from the same milieu as the rest of them with their roots deep in the already ancient Brahmin villages on the Nila.

An appeal to the names of his village and house resolves the mystery partially. Of the two such names we have come across so far, one in Malayalam and the other in Sanskrit, Sangamagrama is not a possible Sanskritisation of Ilanjippalli, obviously not phonetically and also not semantically. A *paḷḷi* originally was a Buddhist retreat house (a *vihāra* in Sanskrit). The word often survived in the names of houses and their compounds after Buddhism's disappearance from Kerala and also lent itself as a suffix to the names of villages – there are thousands of them in southern India – where, it is generally presumed, such institutions once stood. But *ilaññi* is a flowering tree, not the Sanskrit *saṃgama*, a confluence of streams. Here help is at hand from Madhava himself: in his short tract on the moon's positions, *Veṇvāroha*, he says that he, Madhava, was born in a house named (*gṛhanāma*) *bakulādhiṣṭhita . . . vihāra*, which is identical

with Ilanjippalli if we take *paḷḷi* in its meaning of a house, not the whole village (*bakula* is the Sanskrit name of *ilaññi*), i.e., both names refer to his house and Sangamagrama is the village where it was situated.

But where was it? The general view among scholars has always been that Sangamagrama is the town of Irinjalakkuda, some 70 kilometers south of the Nila. It is difficult to see why: there is neither phonetic nor semantic equivalence in the two names; if anything, the similarity, phonetic, is with Ilanjippalli, and even that is not compelling (Malayalam generally does not confuse *r* and *l*). In any case, we have just seen that Madhava's own words contradict the identification unless Ilanjippalli is both a house and its village (that happens occasionally and, astonishingly enough, it happens with the Malayalam equivalent of Sangamagrama, see below) and we discount the significance of the name Sangamagrama altogether. Local historians periodically turn up Brahmin houses in the extended neighbourhood of Irinjalakkuda with names approximating phonetically (but not in meaning) to Ilanjippalli but that does not solve the problem as long as Sangamagrama remains unidentified. House names do change (for example Nilankantha's), village names hardly ever; it is common sense that locating the village must take priority over pinpointing the house.

Tirunavaya itself, right on the Nila, is sometimes called Trimurtisangamam in the temple legends on account of the presence of all three of the main Hindu gods on the two banks, but not Sangamagrama in any record. If we accept, on the evidence of the "Empran", that Madhava's family had its origin in the Tulu country or contiguous parts of the Deccan plateau, another possibility opens up. The area is dotted with villages with *saṃgama* or its Dravidian equivalent *kūḍal* or *kūḍala* in their names (occasionally both as in the most celebrated such place, Kudala-sangama, where Basava, the poet-saint of Karnataka, lived in the 12th century, a contemporary of Bhaskara II) and with a temple consecrated to Saṃgameśvara, the lord of the confluence. Somewhat like Aryabhata the *aśmakīya*, it is entirely reasonable (and a common occurrence in Kerala) for a family to be known by its original domicile. Nilakantha says (in several places) that Madhava was actually born in Sangamagrama, *saṃgamagrāma-ja*. (He also says that Aryabhata was born in the republic of Asmaka, *aśmaka-janapada-jāta*; perhaps *jāta* is only meant as a turn of phrase). Whether Madhava was born in, or in a family from, a village whose name or its Sanskritisation is Sangamagrama, the possibilty that that village was in Karnataka cannot be dismissed.

But, as it happens, there is a much better candidate for Sangamagrama in the Nila heartland itself. This is a village, now a small town, on the river, barely 10 kilometers upstream from Tirunavaya as the crow flies. It is called Kudallur (Kūḍalūr) and its exact literal Sanskrit translation is Sangamagrama: *kūṭal* in Malayalam (as in all Dravidian languages) means a merging or confluence and *ūr* means village. The name comes from its situation at the confluence of the Nila with its main tributary, the river Kunti. If we take Nilakantha at his word, that Madhava was born in Sangamagrama, not merely of it, it would mean

that his family was settled on the Nila before his birth but it really does not make much difference as long as it is accepted that they were recent arrivals who continued in their scholarly vocation. In accordance with custom, Madhava would probably have begun his education within the family. The curriculum would have included, in addition to the standard lessons in Vedic knowledge, language, grammar, etc., the branch(es) of knowledge the family considered its own; if they came from Karnataka, it is not a stretch to suppose that it included the mathematics of Bhaskara II *via* the line of Narayana (see below). Madhava's family tutor, very likely his father, was probably not a highly distinguished scholar, or we would have known about him. On the same ground, if there were astronomers and mathematicians living in the Nila basin at that time, possibly Parameshvara's other teachers, Madhava was not apprenticed to any of them. We should perhaps take Nilakantha's silence as an acknowledgement that Madhava's genius was his own.

To summarise, the most plausible scenario in my view is that the forefathers of Madhava ("Madhavan Empran") migrated from the Tulu coast or thereabouts to settle in Kudallur a generation or two before his birth, in a house whose present identity is unknown. My telling of the story has been convoluted but it is a complicated story, with very few fixed points, and those I have respected. Besides, as in the case of Aryabhata, the magnificence of his mathematical achievements permits and justifies, I think, a certain latitude in our imagined reconstruction of his life.

There are two postscripts to the story of Madhava.

For several centuries, no one seems to know how many, there has existed a Brahmin household known as Kudallur Mana, not in Kudallur village itself nor on the river, but a few kilometers from both. It is a splendid wooded estate now, dominated by a large traditionally built house with many signs still of its antiquity, and it used to be still more extensive in the past. The family history has it that they were originally from Kudallur village and that the main branch was obliged to relocate to the present Kudallur Mana four or five hundred years ago (another example of the persistence through space and time of family origins and family names) for reasons which are not relevant here. What is relevant is that, until about a century back, it was a renowned centre of learning, a *gurukulam* where successive generations of brilliant scholars shared their knowledge and their wisdom with young students and fellow scholars alike.[11] Their speciality was the *vedāṅga*, the six limbs of the Veda, one of which is astral science (*jyotiṣa*). During the time about which there is information, however, their fame rested more on their deep knowledge of the limbs that deal with aspects of language and grammar and of metaphysics. They were champions of the linguistic philosophy of Bhartrhari and literally worshipped

[11]Most of the information given here comes from personal enquiries and contacts; some comes from a fine essay on the Kudallur masters in Alamkod Leelakrishnan's Malayalam book with the (translated) title *Along the Banks of the Nila* (2003), published by D. C. Books, Kottayam.

Patanjali: there is still an antique stone idol of the great grammarian in the house. (The very few images of the conventional gods are recent).

But those who would seek the mathematical spirit of Madhava in this sanctuary of learning or in its earlier setting in Sangamagrama village itself (of the house, nothing remains) need not give up all hope. They may not have been astronomers but the Kudallur Namputiris ("Embrantiri[s] . . . succeeded in later times in securing, or being thought fit to assume, the name of Nambutiri" - Logan in the *Malabar Manual*) were rationalists who believed in and taught the system of metaphysics known as *lokāyata*, the most extremely materialistic and this-worldly of Indian philosophical schools. One of its credos was the contingency of all knowledge: knowledge is not for all time and must be renewed and revalidated constantly through the exercise of our senses and our mind. This is the epistemic current that runs through Nilakantha's last books and it has been suggested that, for all his expertise in ritual and his standing in the Brahmin hierarchy, he can easily be taken for a *lokāyata* sympathiser in his scientific philosophy, at least in his later life. (It has also been said of Aryabhata). The Kudallur masters, while emphatically rejecting any other-worldly influence on the sensible universe, maintained that there was no beginning or end to time and so no need for a supernatural act of creation or annihilation. One would dearly like to know how well they knew their Aryabhata. The house still has a large room called the library, its wooden racks now empty, the manuscript collection dispersed. According to the family, a part of it went to the collection of the old Sanskrit College in Trivandrum which – it is a fact – has several palm leaf manuscripts, including mathematical ones of the Nila school, bearing the Malayalam equivalent of the inscription 'ex libris Kudallur Mana'. Among them is a copy of the *Āryabhaṭīya* dated 1552; Jyeshthadeva was still alive.

It would be a fitting conclusion to the Madhava story if Kudallur was really his home, and to know that he had as brilliant a following in his progeny – in a different field, but that was not unusual – as in his mathematical descendents.

The other remark concerns a more substantive issue and, historically, is more in the nature of a preface than a postscript: who was the conduit for Madhava's familiarity with the works of Bhaskara II? The natural candidate for this role is Narayana (or possibly a school or the family that he belonged to). As we have seen in passing and will discuss in detail below, one interest of Madhava and the Nila mathematicians was the geometry of cyclic figures – to which Bhaskara and Narayana, building on Brahmagupta, made seminal contributions – and its adjunct, trigonometric identities. (The number theory of the quadratic Diophantine equation was not taken up in Kerala, for whatever reason). Narayana was not just the most prominent figure of the period between Bhaskara and Madhava; the broad range of his work and its superlative quality entitle it to a place alongside the very best of Indian mathematics of any period. No personal details are known about him except his father's name, an apparently trivial bit of information but useful all the same (see section 2 of this chapter). He wrote at least two books, *Gaṇitakaumudī* (1356) on general mathematics and a monograph on algebra and number theory, *Bījagaṇitāvataṃsa*

(date unknown). The latter is clearly modelled on Bhaskara's *Bījagaṇita* and the geometry of *Gaṇitakaumudī* (Chapter 8.4) is just as clearly inspired by the geometry in *Līlāvatī*. Narayana's backward linkage to Bhaskara is obvious enough.

So is the forward linkage to the Nila school. To the points already made in Chapter 8.5 (the *Yuktibhāṣā* proofs of his theorems on cyclic quadrilaterals), we can in fact add a new element of an entirely different sort. The last few chapters of *Gaṇitakaumudī* are devoted to a thorough treatment of combinatorial problems. This is somewhat unexpected. Though Varahamihira (in his other avatar as an astrologer-diviner) and Mahavira have some combinatorial results, the subject was not part of mainstream mathematics and had remained more or less the preserve of specialists in theories of prosody, music and dance. In particular, Narayana goes over the recursion relation satisfied by the binomial coefficients ${}^nC_m$ or $N_{n,m}$ used in answering Pingala's question 5 (Chapter 5.5) and relates these coefficients to the so-called sums-of-sums, numbers recursively defined as $S_k(n) = \sum_{i=1}^{n} S_{k-1}(i)$ (with suitable initial values) of which the lowest nontrivial one, $S_2(n)$, was known to Aryabhata (Chapter 7.1). To anticipate what will be described in detail later, these numbers (for suitable k and n) occur as the exact coefficients in the sine and cosine series before the infinitesimal limit is taken and, in the limit, give rise to the familiar reciprocals of factorials. For the power series, the general $S_k(n)$ is not required, only its value in the limit of large n and that can be obtained by methods already employed in connection with the π series; nevertheless, *Yuktibhāṣā* (as well as Shankara's *Kriyākramakarī*) gives the exact formula. Neither author has an attribution or gives a *yukti*, Jyeshthadeva without explanation and Shankara with a dismissive remark about the "comfort of the not so intelligent". There are more ways than one of deriving the exact expression, all within the demonstrated capability of the Nila mathematicians, but the question remains: where did they get it from? If from Madhava, why is the proof not given? In the case of *Yuktibhāṣā*, the silence is truly puzzling; it is one of only two instances in the whole book in which proof is withheld. The fact is that the only place where the value of $S_k(n)$ is given for general k and n, before *Yuktibhāṣā*, is Narayana's book. It may be surprising that there is no mention of Narayana's name, but no more surprising than that he is not acknowledged in connection with the third diagonal either, where it is absolutely certain that he was the originator of the theorems of which the proofs are found in *Yuktibhāṣā*; indeed, Narayana's name cannot be found in any of the writings of the Nila authors. I tend to the view that they learned about his results not from his books, but from Madhava's mouth, and only selectively (though it still does not explain why no proof is given; we will return to this point later).

From the date (1356) of *Gaṇitakaumudī* we know that Narayana was a generation or two senior to Madhava, perfectly timed to have had a direct impact on him or on his immediate ancestor(s) if the geography allowed it. So, where did he live? Almost certainly not anywhere in the Kerala of his time (as has sometimes been suggested), in view of the fact that Kerala does not know him

by name whether in written records or in orally transmitted folk history. Besides, Pandita was not a common honorific (as distinct from the common noun *paṇḍita* for learned men) among Kerala Brahmins: very occasionally bestowed on great scholars of grammar and philosophy but not on a mathematician or an astronomer. If, on the other hand, he was from Karnataka, it would position him ideally to be both the bearer of the heritage of Bhaskara and its transmitter – as well as of his own mathematics – to Kerala *via* Madhava's family. The idea is not so far fetched as it may seem. Aside from Bhaskara (and, through him, Brahmagupta), the mathematician with whom he has most in common is Mahavira (who himself had a strong commonality of interests with Bhaskara and whose Karnataka connection is not in doubt): Mahavira was one of the very very few mathematicians with an interest in the combinatorics of prosody. In geometry the connection is even stronger, as seen for example in their shared predilection for numerical (integral) geometry; a reading of Sarasvati Amma's book ([SA]), with its systematic ordering of the contributions of individual geometers to each topic, brings this out very clearly. What would be far fetched is to locate him somewhere in northern India, as has also been suggested, for no better reason than that many of his manuscripts were found there. That, as we should know, can be grossly misleading; by the same token, Aryabhata (and lots of others) will qualify overwhelmingly as a son of Kerala (which also, astonishingly, has been suggested in the past). As always in India, manuscripts and the knowledge they contained travelled with their owners, as Bhaskara went to Benares with the Daivajñas and as Narayana might well have. The crucial historical circumstance is that 14th century north India was, after 200 years of the Delhi sultans, an intellectual desert; in contrast, Karnataka was witnessing a reemergence of order during Narayana's very lifetime. With the founding of the Vijayanagar empire came also a cultural and intellectual resurgence driven by people like Vidyaranya, its guide and mentor and a spiritual figure of great authority with his base in an old centre of Hindu scholarship in the hills to the east of the Tulu coast. There is a folk belief that Sayana was Vidyaranya's brother; his Vedic workshop must have been humming with activity during the most productive years of Narayana's life.

As often, a disclaimer is due. We really do not know who Narayana was. Making him an inhabitant of the Vijayanagar kingdom, even if on its periphery, and assigning him the role of the bridge between Bhaskara II and Madhava solve several puzzles but it is nevertheless a speculative and provisional solution, to be confirmed or rejected by future discoveries and analyses.[12]

[12] I have recently learned that there was a well-known Trivikrama Pandita, a religious philosopher, who lived in the 13th-14th century in the extreme north of the present state of Kerala, right on its border with coastal Karnataka. (He even had a son named Narayana Pandita – not, alas, the Narayana of our interest though the two were rough contemporaries). Pandita is one of the common family names of the Daivajña community, to this day. Another small step on the largely untrodden path from Bhaskara II to Madhava?

Part III

DOCUMENTS AND PAPERS

Paper 12

John Warren (1825)

Excerpts from "*Kāla Saṃkalita*"

Excerpts from:
John Warren, *Kala Sankalita: A Collection of Memoirs on the Various Modes According to which the Nations of the Southern Parts of India Divide Time*, College Press (1825), pp. 92–94, 331–332.

Note

The Mādhava-Gregory series for π and several other series which converge much faster than the Mādhava-Leibniz series, the invention and the rationale for all of which have been attributed to Mādhava, appeared in print for the first time in *Kāla Saṃkalita* authored by John Warren and published in 1825.

John Warren

John Warren (1769– 1830) was an army captain and later Lieutenant Colonel with Her Majesty's 33rd Regiment of Foot, East India Company in India, surveyor and amateur astronomer. While working as a surveyor in the Great Trigonometrical Survey he re-discovered what became the Kolar Gold Fields and in later life he documented Indian astronomy and time-keeping in his book Kala Sankalita. He was in charge of the Madras observatory during 1805–1811. (Wikipedia)

Source: Internet Archive

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Section I.

Description of the *Sanku* or Gnomon.

The Sanku is a strait Rod, Pole or Pillar of Stone, such as we invariably see placed in front of every Pagoda in India, placed perpendicularly on an horizontal plane. The Hindus trace a Meridional line by describing concentric circles from the point on which the center of the Pillar

Its construction.

is to rest on the ground, precisely in the same manner as Europeans do.

Divisions.

Whatever be its height, the Sanku is divided into 12 angulas, or digits, and each angula is subdivided into 60 vinculas. It thus serves as a scale for measuring the Ch'hya or Chaya, the length of the Meridional shadow; and a Rod is accordingly made of the same dimensions and divisions for that purpose.

In marking alternately the points where the top of the shadow cuts any of the concentric circles, they chuse the time of 5, 6 and 7 dandas (or Indian hours of the *murta* account 60 to a day) before and after noon: This being done the arcs are bissected; the Meridian line is traced, and the four Dikas, or cardinal points; with the *Asta Dikas*, the four intermediate divisions are easily determined.

Dimensions of the Equatorial circle, and parallels of Latitude.

Before entering into the resolution of the Problems which depend on the length of the Meridian shadow, it is proper to enquire how the Hindus compute the dimensions of the Equatorial circle, and thence those of the parallels of Latitude of any given place.

Ratio of the diameter to the circumference,

Of their manner of resolving geometrically the ratio of the diameter to the circumference of a circle, I never saw any Indian demonstration: the common opinion, however is, that they approximate it in the manner of the ancients, by exhaustion; that is, by means of inscribed and circumscribed Polygons. However, a Native Astronomer who was a perfect stranger to European

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Geometry, gave me the well known series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$ &c. to *unit* (*) being the ratio of the area of the Circle to the square of its diameter, or that of an Arc of 45° to Radius unit,—and $4\times(1-\frac{1}{3}+\frac{1}{5}$ &c.) equal to the circumference, the diameter being 1. This person reduced the five first terms of the series before me, which he called *Bagah-Anoobanda*, or *Bagah Apovacha;* to shew that he understood its use. This proves at least, that the Hindus are not ignorant of the doctrine of series; but I could not understand whether he pretended to make out his ratio of the Earth's diameter 1600 to Equatorial Circle 5059 (that which he used in all his computations) by means of these expressions.

Be this as it may, it is certain that according to their Trigonometrical Tables, the Radius, or Sine of 90° being equal to 57° 18′ (†), the diameter would be to the circumference as 1 : 3,14136, &c. (‡) so that dividing the diameter of the Earth into 1600 yojanas, it would give the Equatorial Circle 5026,176 yojanas. But it is somewhat singular to observe, that they should have preferred for constant use another ratio much less accurate, by their own account.

Practical Rule for finding the dimensions of the Equatorial Circle in yojanas.

Dividing the diameter, as before stated, into 1600 parts, and multiplying the square of that number by 10, the root of the product $\sqrt[2]{10\times1600}=5059,6$ yojanas gives the dimensions of the Equatorial Circle. Or taking the ratio as $1 : \sqrt{10}$, otherwise 1 : 3 . 1619, &c. they have the same 5059,04 yojanas.—In all calculations of the Hindus that I have seen, they content themselves with using 5059yo, which is somewhat nearer to their Tabular ratio: but in the following calculations I have used the mean or 5059,3, which difference, however, is of little importance, considering the means that are used for determining the Palabah, the principal Element.

Quantity used 5059,3 yojanas.

Sometimes when the Almanac makers pretend to be very accurate, they divide the diameter into 20,000 parts, and then using the above formula $\sqrt[2]{10\times20,000}$ they have 62832y for the

(*) I owe the following Note to Mr. Hyne's favour. "The Hindus never invented this series; it was communicated with many others, by Europeans, to some learned Natives in modern times. Mr. Whish sent a list of the "various methods of demonstrating the ratio of the diameter and circumference of a Circle employed by the Hindus "to the Literary Society, being impressed with the notion that they were the inventors. I requested him to make "further inquiries, and his reply was, that he had reasons to believe them entirely modern and derived from Europeans, observing that not one of those who used the Rules could demonstrate them. Indeed the pretensions of the "Hindus to such a knowledge of Geometry, is too ridiculous to deserve refutation." I join in substance in Mr. Hyne's opinion, but do not admit that the circumstance that none of the Sastras mentioned by Mr. Whish, who used these series could demonstrate them, would alone be conclusive. It cannot certainly be denied, that the inventors of the system of Hindu Astronomy possessed a knowledge of Geometry which their successors have not entirely preserved, and if we bring the question home to ourselves we are compelled to acknowledge, that thousands (even among the well informed) use La Place's formulæ without understanding the principles of their construction.

(†) The European Arc is =57° 14′ 24″,8.

(‡) Do. as 1 : 3 14159 &c.

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dimensions of the Equatorial Circle: but all they gain is, that they exhibit the same ratio into minuter parts, without any nearer approach to truth.

FRAGMENT II.

On certain infinite Series collected in different parts of India, by various Gentlemen, from Native Astronomers.—Communicated by George Hyne, Esq. of the H. C.'s Medical Service.

I have stated in a Note at the foot of page 93 of the Key to the Siddhanta Chandra Mana, (article Hindu Gnomonics) that in Mr. Hyne's opinion the Hindus never invented the Series referring to the Quadrature of the Circle which were found in their possession in various parts of India; and that Mr. Whish, from whom he had obtained some of those which were communicated to the Madras Literary Society, after having first expressed a belief that they were indigenous, had subsequently reasons for thinking them entirely modern, and derived from the Europeans; observing that not one of the *Jyautish Sastras* who used these Rules, were capable of demonstrating them.

Since the time that I wrote the Note referred to, Mr. Hyne has done me the favour to communicate to me an account of the Series which had come to his knowledge; and I now lay the same before the reader in that Gentleman's own language, being well persuaded that it cannot fail to interest much all the votaries of science.

"My Dear Sir,

I have much pleasure in communicating the Series, to which I alluded in a former note to you, regarding the quadrature of the circle; and which some have supposed to have been invented by the Hindoos.

Let d be the diameter of a circle, and c its circumference: then the value of c may be obtained by any of the following formulæ.

(1) $c = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} \ldots = 4d\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \pm \frac{1}{2n-1}\right)$.

(2) $c = 12\frac{1}{2}d\left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} \ldots \pm \frac{1}{(2n-1).3^{n-1}}\right)$.

(3) $c = 2d + \frac{4d}{2^2-1} - \frac{4d}{4^2-1} + \frac{4d}{6^2-1} \ldots \quad 4d\left(\frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \ldots \frac{1}{2n-3.2n-1}\right)$.

(4) $c = \frac{6d}{3^2-1} + \frac{8d}{6^2-1} + \frac{8d}{10^2-1} \ldots = 4d\left(\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \ldots \frac{2}{4n-3\,4n-1}\right)$.

(5) $c = 4d - \frac{8d}{4^2-1} - \frac{8d}{8^2-1} - \frac{8d}{12^2-1} \ldots = 4d\left(1 - \frac{2}{3.5} - \frac{2}{7.9} \ldots \frac{2}{4n-5.4n-3}\right)$.

(6) $c = 3d + \frac{4d}{3^3-3} - \frac{4d}{5^3-5} + \ldots = 4d\left(\frac{3}{4} + \frac{1}{2.3.4} - \frac{1}{4.5.6} + \ldots \frac{1}{2n-3.2n-1.2n}\right)$.

(7) $c = \frac{16d}{1^5+4.1} - \frac{16d}{3^5+4.3} + \frac{16d}{5^5+4.5} \ldots = 4d\left(\frac{4}{5.1} - \frac{4}{5.1+10.5^2} + \frac{4}{5.1+10.5^2+10.17^2} \ldots\right)$

$= \frac{16d}{5}\left(\frac{1}{2} - \frac{1}{\frac{1}{2}+2^2+1^2} + \ldots \frac{1}{\frac{1}{2}+2^2+1^2+4^2+1^2 \ldots + 4n-1^2+1^2}\right)$.

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All these series, are very easily derived from that, which expresses the arc of a circle in terms of the radius and the tangent.

Let z be an arc of a circle, of which t is the tangent; r being radius. Then, by the theory of functions, or, by the differential calculus, $z = r\left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t}{9} \cdots\right)$. If $r = 1$ and $z = 45°$, then $t = 1$; and $c = 8z = 4d\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots\right)$, which is the first series. If $z = 30°$, then $t = \frac{1}{3\frac{1}{2}}$, and $c = 12z = 6d\left(\frac{1}{3\frac{1}{2}} - \frac{1}{3 \cdot 3\frac{1}{2}} + \frac{1}{5 \cdot 3\frac{5}{2}} \cdots\right) = 12\frac{1}{2}d\left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} \cdots\right)$, which is the second series. If the difference of each pair of terms of the first series be taken successively, then $c = 4d\left(\overline{1 - \frac{1}{3}} + \overline{\frac{1}{5} - \frac{1}{7}} + \overline{\frac{1}{9} - \frac{1}{11}} \cdots\right) = 4d \times \left(\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} \cdots\right)$, which is the fourth series: and, if we begin after the first term, then $c = 4d\left(1 - \overline{\frac{1}{3} - \frac{1}{5}} - \overline{\frac{1}{7} - \frac{1}{9}} \cdots\right) = 4d\left(1 - \frac{2}{3.5} - \frac{2}{7.9} - \frac{2}{11.13} \cdots\right)$, which is the fifth series. If the two last series, which are equal to each other, be added together, and each term of the sum be divided by two, then $c = \frac{4d}{2}\left(1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \frac{2}{7.9} + \cdots\right) = 4d \times \left(\frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \cdots\right)$, which is the third series. If the terms of the following series $\frac{1}{4}\ \frac{1}{2.4}\ \frac{2}{4.6}\ \frac{3}{6.8}\ \frac{4}{8.10} \cdots$ be added and subtracted to and from those of the first, thus:

$$\begin{array}{l} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots \\ -\frac{1}{4} + \frac{1}{2.4} - \frac{2}{4.6} + \frac{3}{6.8} - \frac{4}{8.10} \cdots \\ \quad + \frac{1}{4} - \frac{1}{2.4} + \frac{2}{4.6} - \frac{3}{6.8} \cdots \\ \hline \frac{1}{4} + \frac{1}{2.3.4} - \frac{1}{4.5.6} + \frac{1}{6.7.8} - \frac{1}{8.9.10} \cdots \end{array}$$

; then

(6) $c = 4d\left(\frac{1}{4} + \frac{1}{2.3.4} - \frac{1}{4.5.6} + \frac{1}{6.7.8} - \cdots\right)$.

(7) If the terms of the series $\frac{1}{5}, \frac{2}{17}, \frac{3}{37}, \frac{4}{65}, \frac{5}{101} \cdots \frac{n}{4n^2+1}$ be added and subtracted to and from different terms of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} -$ thus:

$$\begin{array}{l} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots \mp \frac{1}{2n-1} \\ -\frac{1}{5} + \frac{2}{17} - \frac{3}{37} + \frac{4}{65} - \frac{5}{101} \cdots \pm \frac{n}{4n^2+1} \\ \quad + \frac{1}{5} - \frac{2}{17} + \frac{3}{37} - \frac{4}{65} \cdots \pm \frac{n-1}{4\overline{n-1}^2+1} \\ \hline \frac{4}{5} - \frac{4}{255} + \frac{4}{3145} - \frac{4}{16885} + \frac{4}{95 + 4.9} \cdots \mp \frac{4}{\overline{2n-1}^5 + 4\overline{2n-1}} \end{array}$$

then $c = 4d\ \frac{4}{(1^5 + 4.1} - \frac{4}{3^5 + 4.3} + \frac{4}{5^5 + 4.5} - \frac{4}{7^5 + 4.7} + \frac{4}{9^5 + 4.9} \cdots \mp \frac{4}{\overline{2n-1}^5 + 4\overline{2n-1}}\Big)$, which is the seventh formulæ.

I am, my dear Sir, most sincerely, your's,

MADRAS, 17th August 1825.

G. HYNE."

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Paper 13

C. M. Whish (1827)

On the Alphabetical Notation of the Hindus

Transactions of the Literary Society of Madras (Part 1), 1927, pp. 54–62.

Note

This is the first of the three papers published by C. M. Whish and it discusses different systems of numeration employed in the Indian texts, such as the system used in the *Āryabhaṭīya* and the *kaṭapayādi* system used in south India. It was published in the very first issue of the *Tarnsactions of the Literary Society of Madras*. A French translation of this article was published in the *Journal Asiatique* in 1835. The same issue of the *Transactions* also carried a second paper of Whish on the origin and antiquity of the Hindu zodiac.

This paper may be read in conjunction with a very scholarly article on the same topic by Sreeramula Rajeswara Sarma, published in 2012, which has also been included in this collection (see p. 709).

C. M. Whish

Charles Whish studied at the East India College at Hertford, north of London. This College, founded in 1806, trained administrators for the Honourable East India Company. The College taught mathematics and natural philosophy, Arabic, Persian, various Indian languages and law. In 1810 Whish passed the College examinations 'with credit'.

The College of Fort St George was founded in 1812 in Madras, (now Chennai) India. Its purpose was to train British men, who had recently arrived in India, in the local languages. Whish arrived at the College in 1812 when he was register to the Zillah Court of south Malabar. During his career in India, he wrote a Malayalam grammar and dictionary that were published by the College of Fort St George.

He continued to higher positions until he became a Criminal Judge in Cuddapah. He died at the age of 38 in Cuddapah. C.M. Whish was a collector of palm-leaf manuscripts in Sanskrit and other languages. After his premature death in 1833, Whish's brother, J. L. Whish, who was also employed in the service of East India Company deposited these manuscripts in the Royal Asiatic Society of Great Britain and Ireland in July 1836. A catalogue of these manuscripts list 195 items. Though the manuscripts collected by Whish are not distinguished by great age, there are many rare and valuable ones among them.

"It is extremely difficult to stand up and argue for something which the rest

of the world thinks, in Hyne's words, is 'too ridiculous to deserve attention'. It requires great courage to argue for something that you know people will ridicule you for saying. *ldots* Whish was one of the first Europeans to question the Eurocentric view of the world in general and in mathematics in particular. This was a first step in a process that still has not been completed. If it has not been completely achieved in nearly 200 years, we get some idea of how significant it was that Whish started the process. Another point worth making here is that the British treated the 'Natives' in India badly–they considered themselves the masters of the ignorant Hindus. Naturally the Hindus reacted to this attitude as one would expect, with distrust and dislike of their British masters. We make this point to emphasise how remarkable it was that Whish was able to gain the trust and help of the local Hindus. He must have been quite a remarkable man in so many different ways." (J J O'Connor and E F Robertson on C. M. Whish in the MacTutor online resource for biographies of mathematicians)

Source: Internet Archive

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No. IV.

ON THE ALPHABETICAL NOTATION OF THE HINDUS.

It has been remarked, in the introduction to a translation of the *Līlāvatī*, a Treatise on *Hindu* Arithmetic and Geometry, by *Bhāscara Achāryah*, "that if the decimal notation did not originate in India, it must at least have existed there from time immemorial, for no traces whatever are to be found of an alphabetical notation."

The fact of the *Hindus* having been the inventors of the decimal system of notation has always been too firmly supported, and is still too generally believed, to allow of its truth being at all affected by the fallacy of the argument, *that no traces whatever of an alphabetical notation are now to be found*, being proved; and it may farther be remarked, that the word *Hindasi*, which the translator of the above work calls the decimal scale of arithmetic, is not only not to be derived from any *Hindu* word, being also according to several *Arabic* lexicons, معرب, or adapted to the genius of the *Arabic* tongue, from the *Persian* word *Andāz*, but also is used by *Arabian* mathematicians to mean especially, that mode of notation which is effected by the letters of the alphabet.

Arya Bhattah, who lived confessedly when the *Caliyugam* was far advanced, is the author of a certain mathematical tract named, after him, *Aryabhatīyam*, containing four chapters, and one hundred and twenty-three Slocas of the measure called *Aryā-vrittam*. In the commentaries

the work that I have examined, it appears that this author was born hen 3600 years of *Caliyugam* were passed, in the city of *Cusuma*, hich name, with others of the same meaning with *Cusuma-pura*, namely, *Pushpapura*, &c. *the city of flowers*, is mentioned in the *Hēma-chandra* and other *Cōsahs* as being synonymous with *Pātālipuram*.

In the first chapter of the *Aryabhatīyam* are the following lines: —

" Varggākzarāni varggēvarggēvarggākzarāni kātngmauyah
Khadwinavakēswarā nava vargēvargēnavāntyavargēvā."

These, which, though small in bulk are extensive in import, embrace the following explanation: —

The consonants of the five classes of letters in the alphabet may be written in the places of *units*, *hundreds*, *tens of thousands*, *millions*, &c.; these are, *ka*, *kha*, *ga*, *gha*, *nga*; *ca*, *cha*, *ja*, *jha*, *nya*; *ta*, *tha*, *da*, *dha*, *na*; *pa*, *pha*, *ba*, *bha*, *ma*; in number *twenty-five*: the following unclassed letters, *eight* in number, namely, *ya*, *ra*, *la*, *va*, *xa*, *za*, *sa*, *ha*, are found only in the places of *tens*, *thousands*, *hundreds of thousands*, *tens of millions*, &c. The twenty-five letters from *ka* to *ma* represent the values respectively of numbers to 25, and any two of these joined together represent the sum of the two numbers which they severally express; thus ngm, nga, and ma together is 30. The unclassed letters are thus valued: *ya*, 30; *ra*, 40; *la*, 50; *va*, 60; *xa*, 70; *za*, 80; *sa*, 90; *ha*, 100. The letter *la* has the same value with the former letter *ka*. These letters, in their simple form, represent only the significant number above assigned to them; to ascertain the place of figures in the natural scale of numbers in which they are to be understood, the vowels are thus annexed to them, it being understood, that whether it be a short vowel, or its corresponding long sound, no difference occurs. Thus, *kā* or *kā*, 1; *kī* or *kī*, 100; *ku* or *kū*, 10,000; *kri* or *kri*, 1,000,000; *kli* or *kli*, 100,000,000; *kē*, 10,000,000,000; *kai*, 1,000,000,000,000; *kō*, 100,000,000,000,000; *kau*, 10,000,000,000,000,000; in the same way the other letters, thus, *ya*, 30; *yi*, 3000; *yu*, 300,000; *yri*, 90,000,000; *yli*, 3,000,000,000, &c.; and *yau*, 300,000,000,000,000,000,

or three *parārdha.* If numbers to a greater extent be wanted, a *parārdha,* or *one hundred thousand billions,* can be expressed by *anuswaram,* or *visarga.*

The letters here alluded to are those used in writing the *Sanscrita* language, and are too well known to require being quoted. The system of English orthography, by which they are here expressed, is that followed in my paper on " The European Alphabet of the *Sanscrita* Language."

The author then proceeds to state examples in astronomy of the application of the rule: thus, in the present age of the world, " the sun will revolve in its course from *Aries* to *Pisces, Kyughri* years." In this *Khyu* is 20,000; *yu* is 300,000; and *ghri* is 4,000,000; their sum, 4,320,000 years for the duration of the Kaliyugam. " The moon will, during the period, revolve about the earth *cayagiyinguxuchlri* times:" in this *ca* is 6; *ya,* 30; *gi,* 300; *yi,* 3000; *ngu,* 50,000; *xu,* 700,000; *chlri,* 57,000,000; and their sum 57,753,336, the number of lunations during the present age.

I have not been able to discover, whether the alphabetical notation thus laid down in the *Aryabhatīyam* has been in use before the time in which its author flourished, or whether it were invented by himself to enable him to enter his calculations into *Slōkas* without the use of numbers, or numerical symbols. I am not indeed aware that it has since been in use among mathematicians, never having observed it in their works. It must, however, have been understood by them; for the three books *Līlāvatī, Laghu-bhāskarīyam,* and *Mahā-bhaskarīyam,* of *Bhāscara-Achāryah* are founded upon the principles of the *Aryabhatiyam;* the *Līlāvatī,* in particular, is composed from the second chapter of the work. The application of the rule must be very tedious in long calculations, and the promiscuous connections of letters, offensive to the eye and the ear, as violating every rule of *sandhi,* established for the better sound, and the better connecting of words, in the *Sanscrita* language; so great, indeed, is the incongruity, that the most

skilful in the characters of their country would not be able to read a line *off hand* without extreme difficulty.

Beside the alphabetical notation above explained, another system has, from time immemorial, obtained among the learned in the southern countries of the Peninsula. The great scope and variety of which this system is capable, justly entitles it to the rank that it holds in the estimation of those mathematicians who are acquainted with it; few cases occur in which the subject is not comprehended in the same words that contain the numerical value required; by which it becomes an easy *memoria-technica* in the most abstruse calculations. The alphabets thus valued are the same in the order, number, and power of the letters with the *Dēvanāgari* alphabet, whether it be the *Grantham*, *Aryya*, *Tuluh*, *Canarese*, *Telingoo*, *Mahratta*, or other character current in other southern countries of *India*.

In a certain treatise on astronomy, the rules for the value of letters are thus exhibited in a couplet of the measure called *Anashtub*:

" Nanyāvacaxca xūnyāni sankhyāq katapayādayah
Misrētuvandyahalsankhyana ca cintyō halaswarah."

This must be thus explained: The letters *na* and *nya*, and all *initial vowels*, are *cyphers*; *ka*, *ta*, *pa*, *ya*, the digit 1; *kha*, *tha*, *pha*, *ra*, 2; *ga da*, *ba*, *la*, 3; *gha*, *dha*, *bha*, *va*, 4; *nga*, *na ma*, *xa*, 5; *ca ta*, *za*, 6; *cha*, *tha*, *sa*, 7; *ja da ha*, 8; *jah*, *dha*, *la*, 9; among *double letters* the latter only hath a value, for mute consonants are to be reckoned.

It will appear from this, that in the *Sanskrita* alphabet, which contains 35 consonants, the first ten from *ka* to *nya*, have the values of 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, respectively; the second *ten* from *ta* to *na*, respectively, the same as the *former ten*; the next *five*, namely, *pa*, *pha*, *ba*, *bha*, *ma*, the value of the first *five digits*; the following *nine*, from *ya* to *la*, the *nine digits* respectively; whilst the remaining *hza*, a compound letter, is valued according to the last letter in the compound, namely, *za*, *six*. These letters can be written in any place of figures, and are increased in value, in a ten-fold proportion, accordingly; nei-

ther are they affected in power by being connected with vowels, for these have no numerical value. The first place in writing is the place of *units*, in calculating.

It is impossible, I fancy, to fix the period at which this mode of numeration first became current in southern India; but it must have been at a very early period. The following is an extract from the work called *Jaimini-sūtram*, from the second part of the first chapter.

" Atha swāmsō grahānām panca mūshikamārjārāh; tatracatushpadah; mrityaū kandusthaulyamca; dūrējalakushthadih; xēzēswapadāni; mrityu vajjāyāgnikanaxca; lābhē vanijyam; atra sarisripastanya haniza samē vāhanāducchōcchakramālpatanam; jalacarakhēcarakhetāh kandūrdushtagrandhayascaripphē tātākadaū dharmē, ucchēdharmanityatākaivalyanca."

In this *panca* is 61, which being divided by the number of signs, 12, leave *one*, which is the number required. In the same way *tatra* is 26, which, being divided by 12, leaves 2. Thus the words *mrityau, dūrē xēza ajja lābhē atra samē ripphē dharmē ucchē*, express, severally, the numbers 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12; moreover, this mode of notation is used throughout the work, and is mentioned in the same work in the following Sutram: —

" Sarvatra savarnā bhāvūrāsayaxca; na grahāh;"

in which the author explains his intention to use, in his work, numerical letters for the signs of the Zodiack, &c. but not for the planets.

Now it appears, in the astrological work called *Jyautisha-phala-ratnamālā*, whose first verse is,

" Xrī vikramārkkō jagatī talēsmin
Jīyādanuprakhyayaxā narēndrah
Puvōzayaq kōti suvarnnatō mām
Sa bāndhavam saptati vatsarāni,"

that the author of the work lived 70 years under the protection of *Vikramārkah*, of whose æra the 19th century is now passing, and this

book is founded entirely upon the *Jaimini-sūtram;* from which it occurs immediately, that the system of alphabetical notation, which I have just explained, existed nearly 2000 years ago.

In the country from which this is written, the last day of every year is entered in terms of days of the *Kaliyugam*, in a numerical line, which thus becomes the *memoria-technica* for the ensuing year, for astronomical and other purposes; and this practice, too, still continues to obtain, although 995 years have passed since a different æra has been introduced into the country: thus the last day of the month *Minah*, which concludes the year 4919 of the *Caliyuga* is expressed thus, *Tanusthastabdhisēvyah*, or 1,796,706 days of the *Caliyuga;* and the present *Kali*, as the memorial line is called, is *Kamsaghnassidhasevyah*, or 1,797,071, and continues to be the *Vākyam, word*, during the present year.

The learned in Malabar teach, that the day in which *Chērumān Perumal* went to heaven (died), or, as the Mahomedans assert, departed in their ships to Mecca, is preserved in the line *Swargga sandēha prāpyah*, which is the 3528th year of the Kaliyugah; but, unfortunately for the truth of the assertion that *Chērumān Perumal* embraced the *Mahomedan* faith, that is about the two hundredth year before the flight of the prophet of that sect. The line itself explains the doubt that obscured the history of the period, with regard to the death or disappearance of that sovereign prince of Malabar. The epoch of the present æra of Malabar, of which the 995th is now commenced, is *Achāryya-vāgabhēdyā*. The word of the *Achāryah* is irrefragable, in which it is asserted, contrary to the general opinion of the origin of the current date, that the introduction of new rules of conduct, and laws and customs, into Malabar, by a native of it, the famous *Sankara-Achāryah*, fixed a new æra. In the pagoda called *Tirukōl-Kshetram*, near *Paulghaut*, in *Kēralam*, is the following couplet of the *Anushtub* measure, on the stone steps of the tank within its precincts: —

" Prasūtan tarhi saukhyan naf pratiz thāyām ahargganah
Turuzka bhityā salilē nyastayōr atradēvayōh."

" The revolving day of the *Caliyugam* in which the two idols, whicl had been cast into this water through fear of the *Turushkars* (Ma homedans), were reinstated, is expressed thus : ' *Now joy is born unt us.*' " The words *Prasūtan tarhi saukhyannah*, which means *now joy i born unto us*, are numerically 01,786,672, in which the cypher has n value. Upon calculating the number of days of the Caliyugam, th fortunate event which it records and celebrates, is found to have hap pened in the 4892d. year of that æra, or October 20, 1790. Agai astronomers aptly call the astronomical circle, in terms of its minute *Anantapuram*, " The city of endless space ;" the numerical value of th letters in which term is 21,600. An ingenious person has entered int a *Slōka* a few words, which express the year 1820 of our æra, approach ing, in letters and also in numerical symbols ; thus *nakhăhijyā*, or, i figures, 1820. Now *jyā* means the earth, which is, according to nume rical symbols, *one; ahi* is a snake, which is *eight; nakham* means th nails of the body, 20 ; so thus, by two different methods, the same dat is expressed in the same words. The superiority of the *Hindu* ove the *Arabian* method need not be mentioned.

Let us now compare the two systems we have explained, one with th other. In the *Sadratnamālā*, the proportion of the circumference to diameter of one *parardha* (1 with seventeen cyphers), is *Bhadrāmbudh sidha janma ganita xraddāsmayad bhūpagih*, or 314,159,265,358,979,324 which, according to *Aryabhatah*'s system, would present this incon gruous form, Kyaukrojhlaicnēgllijlrichsugīsibha.

Let us also compare this system with that of symbols, used b writers on astronomy in *India* from time immemorial. The followin is an extract from the second chapter of the *Sūryya Sidhāntam*, i which the twenty-four sines of the quadrant are exhibited by numerica symbols :

" Tatwāswinōnkābdhi kritā rūpa bhūmi dharārtavah
Khānkāshtau pancha sūnyēschāh bāna rūpa gunēndavah
Sūnya lōchana panchaika chidra rūpa munīndavah
Viyachchaudratidhritayō gunarandhrambarāswinah

Muni shadyamanētrāni chandrāgnikrita dasrakāh
Panchashta vishayākshīni kunjarāswinagāswinah
Randhra panchāshtakayamah vaswadryankayamastathā
Kritāshta sūryya jivalanāh nagādri sasi vahnayah
Shatpancha lōchana gunāh chandra nētrāgnivahnayah
Yamādri vahni jivalanārandhra sūnyārnnavagnayah
Rūpāgni sāgara gunāli vasutri krita vahnayah."

These five and a half Slokahs of the Anushtub measure, contain the es in terms of minutes of the circumference, thus:

	Arc.	Sine.		Arc.	Sine.
1	3° 45′	225	13	48° 45′	2585
2	7 30	449	14	52 30	2728
3	11 15	671	15	56 15	2859
4	15 0	890	16	60 0	2978
5	18 45	1105	17	63 45	3084
6	22 30	1315	18	67 30	3177
7	26 15	1520	19	71 15	3256
8	30 0	1719	20	75 0	3321
9	33 45	1910	21	78 45	3372
10	37 30	2093	22	82 30	3409
11	41 15	2267	23	86 15	3431
12	45 0	2431	24	90 0	3438

The same 24 sines in terms of *minutes* of the circumference, with the *conds* and *thirds*, are thus laid down in the *Tantrasangrahah*, in six ōkas of the same measure with the former, in numerical letters.

" Xrezthannāmavarizthānām hīmādrirvēda bhāvanāh
Tapanobhanu suktajnyo madhyamam vidhi dohanam
Dhigajyo naxanam kaztam hrinna bhogasayambika
Mriga haranaresoyam virorana jayolsakah
Mulam vixudhannalasya ganezu viralanarah
Axudhi gupta ennora xrih sankukarno nagexwarah
Tanujo garbhajomitram xri manatra sukhi sakhe
Sariratrauhimaharo vegajnyah pathi sindhurah
Chayala yogajonilo nirmalo nasti satkule
Ratrau darpanamabhrangam nagastunganakhobali
Dhiroyuva katha lolah pūjyo narijanairbhagah
Kanyagarenagavalli devo viswasthali bhriguh."

	Arc.	Sine.		Arc.	Sine.
1	3° 45′	224′ 5″ 22‴	13	48° 45′	
2	7 30		14	52 30	
3	11 15		15	56 15	
4	15 0		16	60 0	
5	18 45		17	63 45	
6	22 30		18	67 30	
7	26 15		19	71 15	
8	30 0		20	75 0	
9	33 45		21	78 45	
10	37 30		22	82 30	
11	41 15		23	86 15	
12	45 0		24	90 0	3437′ 44″ 48‴

It is unnecessary to mention which of these appear to be the most convenient mode of expressing a sum of high numbers, or to multiply instances of its application in astronomy, and on other occasions; suffice it to say, that even though the decimal scale of arithmetic have existed from time immemorial in *India*, it is still well ascertained that an alphabetical notation has been in use 1600 years ago in northern *India*, the date of whose origin cannot probably now be fixed; and that a totally different system of notation, by the letters of the same alphabet, has been in use in the south of *India*, the period of whose origin, though its existence is traced back for near two chiliads of years, is also indeterminable.

C. M. WHISH.

Paper 14

C. M. Whish (1827)

On the origin and Antiquity of the Hindu Zodiack

Transactions of the Literary Society of Madras (Part 1), 1927, pp. 63–77.

Note

During 1860–1899, this article was frequently cited in several studies of Indian astronomy, such as the popular translation of the *Sūryasiddhānta* by Ebenezer Burgess and William Whitney, the review article by John Burgess of the European studies of Indian astronomy in the 18th and 19th centuries, and the scholarly monograph of George Thibaut (in German) on Indian Astronomy, Astrology and Mathematics.

C. M. Whish

For a biographical sketch of C. M. Whish, see p. 109.

Source: Internet Archive

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No. V.

ON THE ORIGIN AND ANTIQUITY OF THE HINDU ZODIACK.

THE opinion of M. Montucla, that the Indian Zodiack was borrowed, either directly or mediately, from the Greeks, appears to me so well founded, upon the observations that I have made on perusal of a variety of Hindu works, that I deem it necessary, even although totally unacquainted with that learned author's arguments, to produce these observations before the public. By this publicity I shall rescue M. Montucla from apparent carelessness in research, and from the contempt of those who deem his opinion "a notion bordering on phrenzy;" while the opinion of the public may be directed to a channel of intelligence, from which it has been of late years turned, in consequence of men having too strictly relied upon the information of wily Brahmans and interested teachers; information dressed in all the purity of classic diction, in which the elegant Sir William Jones "knew so well to shine."

My intention is to prove, that the Indians have borrowed the dodecatemory division of the heavens, with the figures, &c., of the zodiacal constellations, immediately from the Greeks. Sir William Jones says *, "The Brahmans were always too proud to borrow their science from the Greeks, Arabs, Moguls, or any nation of Mlechch'has, as they call

* See p. 302, vol. ii. Asiatic Researches, 4to. edition.

those who are ignorant of the Vedas, and who have not studied the language of the gods.

"They have often repeated to me the fragment of an old verse, which they now use proverbially, '*na nicho yavanatparah*,' or no base creature can be lower than a Yavana; by which name they formerly meant an Ionian or Greek, and now mean a Mogul, or generally a Muselman.

"When I mentioned to different Pundits, at several times, the opinion of Montucla, they could not prevail on themselves to oppose it by serious argument; some laughed heartily; others, with a sarcastic smile, said it was a pleasant imagination; and all seemed to think it a notion bordering on frenzy."

I shall endeavour to refute these, and the few arguments of the same nature succeeding it, in the essay from which I quote.*

I shall first, then, describe the Indian sphere, as divided into a Lunar and Solar Zodiack, which, though these have already been treated upon so often, I hope to show in such a different light, as to make it subservient to the proof of a very material point — its antiquity.

The apparent path of the sun is divided into twelve portions, containing each thirty degrees. Their names are:

Mesha — The Ram.	Tula — The Balance.
Vrishabha — The Bull.	Vrischkika — The Scorpion.
Mithuna — The Pair.	Dhanah — The Bow.
Karkata — The Crab.	Makara — The Sea-Monster.
Simha — The Lion.	Khumbha — The Water-Vessel.
Kanyā — The Virgin.	Mīna — The Fish.

Their figures are thus described by Vāraha Mihira, a Hindu author

* In an essay embracing the various objects of an extensive speculation, the facts necessarily appear detached, and somewhat desultory. The chain of argument adopted by the author has, however, an obvious connection. He first endeavours to prove, by a comparison of the Greek and Sanscrit languages, that the duodenary zodiack of the Greeks was communicated, at an early period, to the Hindus; and he afterwards adduces testimony to show that, prior to this period, the Hindus were unacquainted with a duodenary division of the heavens. G. H.

of note, who flourished in an early century of the Christian æra, in the Hōra Sastram, chap. i. verse 5.:

" Matsyau ghati nri mithunam sagadam sa vīnam
Chāpī narōswa jaghanō makarō mrigasyah
Taulissasasya dahanā plaragācha kanyā
Seshāsswanāma sadrisasswacharāscha sarvē."

" Two fishes; one bearing a water-vessel: twins; a male bearing a weapon, and a female with a lute: a monster with a bow, his body that of a horse, with half that of a man: a monster, whose upper part is that of a beast, and whose nether end is that of a sea-monster: a man bearing a balance: a virgin on a raft, holding in her hand a lamp and some grain: the remainder are figured according to their particular names, and all partake of their several natures and elements." *

The commentary to this work, named Prabōdhana, remarks upon this verse, " The sign Minam contains two fishes, whose head and tail are alternately connected." (Then follow astrological remarks, here omitted.) Khumbha is a man bearing an ewer, &c.: Mithuna, twins: a youth bearing a staff, and a damsel with a Vīna: Dhanuh, a monster with the upper half of a man on a horse's body, wielding a bow: Capricorn, a monster, whose upper part is that of a beast, ending with the lower parts of the Makara: Kanyā, a virgin standing on a raft, having in her hands some grain and a lamp: the rest have the figures of those animals whose names they bear." Thus saith Yavanīswarah:

" Adyasmritō mēsha samāna rāsih
Kālasya mūrdhā gaditah purānaih
Sojāpi kasanjarakandaradri
Stīnagni dhatwā kararatna bhūmih

Vrishakritistu prathitō dwitiyah
Savaktra kanthan niyatam vidhatuh
Vanādi dhātu dwipagōkulānām
Krishīpalānāmadhivāsa bhūmih

* The reader will find a similar description of the signs by Sripeti, the author of the Retnamálà, in p. 292, vol. ii. Asiatic Researches, 4to. G. H.

Vīnāgadābhra mithunam tritiyah
Prajāpatēskandha bhujāmsa dēsah
Pranartakōtgūyakasilpaka stri
Krīdāraladyūta vihāra bhūmih

Karkī kulīrakritiramba samsthō
Pakshah pradēsō vihitaschaturthah
Kēdāravāli pulināni tasya
Dēvanarā ratnavivikta bhūmih

Simhastu sailē hrīdaya pradēsah
Prajāpātēh panjama ā hurāryāh
Tasyātavī durga guhā vanādi
Vyādhāvanī bhū bhuvana pradēsah

Pradīpikam grihya karēna kanyā
Naustha jalēshashthatikā bhruvanti
Sāchārthadhīra jatharam vidhatuh
Sanadwalā stri rati silpa bhūmih

Vīthyāstulāpānidharō manushya
Sthitassanābhī kadivastidēsah
Sulkartha vīthyā panapattanardhura
Sārdhādhivasōnnata sasya bhumih

Swabhrēshtamō vrischika vigrahastu
Prokta prabhōrmēdra guda pradēsah
Guhāvali swabhra chitatma guptih
Valmīka kīta jagaradra bhumih

Dhanivi manushyō haya paschimārdham
Tamāhurūrū bhavana pranētuh
Samasthitavyastasasasya vāji
Sutrāstrabhrāda rathaswa bhūmih

Mrigārdhamūrrō makaram
Janudwayōrdēsa muranti yathuh
Nadi vanāranya sarōdyanūpah
Swabhritya vāsō dasamah prithivyam

Skandhē turiktah purushasya kumbhō
Janghē tamēkadasamahūrāryāh
Tasyō darādhārakusasya pakshih
Trisanndhikadyūtanivēsa dēsah

Jalētumīnadwayamantya rāsih
Kālasya pādō vihitō parishthaih
Sapunya dēvadwija tirtha bhumih
Nadī samudrambudhanādhi vāsah."

These are the twelve signs.

Varaha then proceeds and says in the eighth verse:

" Kriya tāvuru jutuma kulīra lēya pārthonā juka kōrpyākhyāh
Taukshika ākokērō hridōgaschēthusi kramasah."

Thus: " Kriya, Tāvura, Juthuma, Kulīra, Leya, Parthona, Jūka, Kōrpia, thus are they called; Taukshika, Akōkēra, Hridōga, and Isthusi, in their order."

Upon which the commentary Vivaranam remarks, " These are the names without root or meaning, that is, exotic, of the twelve signs, beginning with Mēsha, the Ram," in these words, " Mēshadīnam paribhāshikāni sajnyantarāni."

We can, however, find both their meaning and their origin, thus:

Kriya	ΚΡΙΟΣ	Aries,
Tāvuru	ΤΑΥΡΟΣ	Taurus,
Juthuma	ΔΙΔΥΜΟΣ	Gemini,
Kulīra	ΚΑΡΚΙΝΟΣ	Cancer,
Lēya	ΛΕΩΝ	Leo,
Parthōna	ΠΑΡΘΕΝΟΣ	Virgo,
Jūka	ΖΥΓΟΣ	Libra,
Korpia	ΣΚΟΡΠΙΟΣ	Scorpio,
Taukshika	ΤΟΞΟΤΗΣ	Sagittarius,
Akōkēra	ΑΙΓΟΚΕΡΩΣ	Capricornus,
Hridōga	ΥΔΡΟΧΟΟΣ	Aquarius,
Isthusi	ΙΧΘΥΣ	Pisces.

These scarcely require any comment. The fourth name, Kulīra, does not, I fancy, resemble any name in the Greek tongue for a crab: but Karkin, the substantive having that meaning, is derived from ΚΑΡΚΙΝΟΣ, the word which completes the twelve Greek names of the signs; and is used in the fourth verse of the words of Yavana above

quoted. It is found also in the Hōra Sastram; thus, chapter xi. verse 9, "Karkinī lagnē tasthē jivē chandra sita," &c.; in which the noun from which karkini is derived, is Karkinī, a crab. *

The Lunar division of the firmament shall now be considered. The heavens are divided by the Hindus into twenty-seven portions, called Nakshatra, a star, or constellation. They are as follow: —

1	Aswini.	10	Maghā		Mūla
	Bharani		Pūrva Phalguni	20	Pūrvāshāda
	Kritica		Uttara Phalguni		Uttarāshāda
	Rōhinī		Hasta		Sravana
5	Mrigasirah		Chitra		Dhanishtha
	Ardra	15	Swati		Satabhisa
	Punarvasu		Visakha	25	Purvabhadrapāda
	Pushya		Anurādha		Uttarabhadrapāda
	Aslēsha		Jyeshtha		Rēvati.

A twenty-eighth constellation is occasionally placed between Uttarāshādha and Sravana, called Abhijit. It appears to be the bright star in

* The following quotation may not be deemed irrelevant, from the Hōra Sastra, chap. i. verse 4, in which it is mentioned that Kāla, or Time personified, is divided into twelve parts, and each member of this body, named Kāla Purusha, or Prajāpati, corresponds in order with the twelve zodiacal signs, thus:

"Kālāngāni varāngamānananamurōhrit krōda vasobhritō
Vasto vyajanamūrujānuyugalē janghetalonghridwayē
Mēshāswi prathamānavarska charanāschakra stitha rasayoh
Rāsi kshētra griharkshē bhāni bhavananchaikartha sa pratyayē."

"The members of Kāla are the head, the face, the breast, the heart, the region below the breast, the belly, the abdomen, the privities, the thighs, the knees, legs, and the two feet."

This puts me in mind of the members of the body we read of in the almanack. I am not aware, and have not by me a single document to assist me, whether these members refer to the signs in the order of Varāha Mihira. I have only recollection of one verse of some memorial lines to the purpose, learned in infancy:

"The noble Lion centres in the heart;
The bashful Virgin claims the belly part;"

which corresponds in part with the saying of Yavanah above quoted. — *Note by the Author.*

Lyra. It is said among Hindu astronomers, that the stars which formerly composed this constellation have disappeared. It is not improbable that the stars, which compose Lyra a double star, were, in the early ages of astronomy, distinct; but, from their proper motion, have become so united as to be now indistinct to the naked eye, which may account for the circumstance of the constellation not being discernible as described in former sasters.

The mean calculation of the extent of these twenty-seven mansions is made simply by dividing 360, the number of degrees in a circle, by 27. The true divisions, or sphutāmsa of the asterisms are different.*

I shall now proceed to the main object of my paper. I am credibly informed by a very intelligent Brahman, whose prejudices, I am convinced, will not lead him to deceive, as his information has been fully explanatory upon the subject, that, in the three Vedā, the divisions of the year and of the heavens are as I shall relate. The subject is not discussed in those works; but, being casually mentioned in divers parts, is collected, and is as follows: — The year is lunar, consisting of 360 days and nights, of which the 360 nights are called the daughters, the 360 days the sons of the sun. The names of these months are, Tapa, Tapasya, which compose the season called Sisirah; Madhu, Madhava, of the season Vasantah; Sakra, Suchi, for the season Grishma; Nabha, Nabhasya, for Varsah; Isa, Urja, for Sarat; and Saha, Sahasya, for the season Himantah. The firmament of the stars is also divided into twenty-seven parts, called the wives of the Moon.†

* The number of the stars, and the computed positions of the asterisms, are given by Sir W. Jones, in his Essay on the Antiquity of the Indian Zodiack, inserted in the second volume of the Asiatic Researches. In the ninth volume of the same work is an excellent paper by Mr. Colebrooke, "On the Indian and Arabian Divisions of the Zodiack." It contains a table of the names, symbols, places, and number of the stars, of each of the lunar constellations, with several other concomitant circumstances. From this table it appears that the asterisms do not, by any means, occupy equal divisions of the Zodiack. — G. H.

† If the author's information be correct, the following passage by Sir W. Jones, in his Essay on the Antiquity of the Indian Zodiack, will lead to the inference that different copies of the sacred books contain statements which are contradictory, and which afford evidence that they are spurious: —

Independent of this division, there is no division at all of the path of the sun into twelve signs, and no mention at all of such constellations as the Ram, the Bull, &c.. Moreover, in the Vedahs, there is no mention of the division of a month in any other way than that of the light and dark fortnights, which is subdivided into fifteen parts, or days; and no hebdomedary division is noticed. These fifteen days are called the first, the second, the third, the fourth, &c. of the light or dark half of the month. From this I am perfectly satisfied, in my own mind, that the use of the twelve signs of the heavens, and the seven days of the week, have been introduced by the Greeks, with the improvements that they certainly caused in the astronomy of the Hindus.

I have before quoted from Yavana, or Yavaniswara's ordinances: a word, then, with regard to the Yavanahs.

Varāha Mihira, the illustrious astronomer, bigoted astrologer, and rigid atheist, of an early period of our æra, mentions his authorities in the astrological work, from which I have already quoted, in the following lines, chapter vii. verse 1.: —

> " Maya yavana manindha saktipūrraih
> Diwasakarādishu vatsarāh pradishta
> Nava tithe visayāsivi bhūtā rudra
> Dasa sahita dasabhisswatungabheshu."

In these are mentioned Maya, Yavana, Manindha, Saktipurvah, (Parāsara,) as authorities upon the subject of discussion. I shall make a remark upon the two first of these.

It is acknowledged among Hindus, that the earliest astronomical rules, or opinions, which they have professed, are those of Măyă. These, they agree, were embodied in the work called the Sūrya Sidhāntah, by some disciple of his; but by whom, or in what generation of

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" The Puránics, who reduce all nature to a system of emblematical mythology, suppose a celestial nymph to preside over each of the constellations, and feign that the god Loma, or *Lunus*, having wedded twelve of them, became the father of twelve genii, or months, who are named after their several mothers." — G. H.

the discipular descent, is not known, or at least not acknowledged. Now Maya was not an Hindu, but born in a country of Mleschahs, barbarians or foreigners; an Asurah, or species of divinity, and born in Romaka.* When we consider, therefore, that he was an Asurah, an astrologer, born in the west, and named Maya, we can propose some limit to his æra, by considering that the observations of the Magi of Chaldea, among the Babylonians, allowed to be the earliest astronomers among the ancients, only extend to the year 2342 before our Saviour. Let us examine the word Magus: the g must be pronounced soft as j; thus, in the Persian, the sound is مجوس majoos, pronounced muj-joos. To adapt the word to the genius of the Sanskrita tongue, the j must be changed to y; and the word, strictly grammatical, becomes Mayah, with the Sanskrita termination. Was Assyria, pronounced Asūria, then, the birth-place of the Maya, the Asurah, the first astrologer, who propagated the science of the heavens and the influences of the heavenly bodies among the Hindus?

To Maya, mentioned above, allusion is thus made, in the commentary of the work accompanying my copy. "Maya iti kaschidasurassuryasēvayā labdha varah jyotissastra vetta Maya: this is a certain Asurah, who having adored the sun, became, in reward for his homage to that deity, skilled in astrology." I shall only further observe on this personage, that, although some may cavil at my having called the Chaldean wise men Magi, when the term is more properly applied to the followers of Zoroaster, yet, as I find myself borne out in the expression by various authors, I shall be excused, I hope, on that point.

Yavanah is the next on the list. The commentary says of him: "Yavana mlescha nāmādhipah sopi hora sastravetta yavanah: this is a mlēscha, or βαρβαρος, so named. He, too, was well skilled in the science of astrology." Yavana, or Yavaniswarah, as he is called, the second authority in astronomy among Hindus, was then a foreigner.

* Romaka is the quarter of the surface of the globe contained between two lines, passing from the north to the south pole, one through Ougein, in east longitude 75°, and Lanca; the other 90° west from those places. — *Note by the author.*

The name Yavanah, or commonly Yōna, or Yōnaka, which (for the sake of euphony) is changed to Chonaka, is the title now given to the race of people called Māpillahs, the Mahomedans of the Malabar coast. This, however, is a local application of the term. The name is more often given, by the generality of Hindus, to the Ionians or Greeks. Thus, Alexander and his army are called Yavanah. That the Yavanah mentioned by Varāha was a Greek, may be strongly confirmed by the following consideration: let us read attentively a verse extracted from a work of Varaha Mihira, the Varaha Samhitam:

" Mlēschāhi yavanasteshu samyak sāstramidam sthitam
Rishivattehi pūjyante kimpunardaivavidwijah."

By these lines are instilled into the minds of Hindus, the consideration that the Yavanas were well skilled in astronomical and astrological science. It appears, also, that their progress in the knowledge of the heavens and planetary influences were so highly esteemed among the Brahmans, that they are instructed to reverence these Mleschas after the manner of their own Rishis, and to receive their sciences as orthodox. Thus we frequently find, in the astronomical and astrological works of the Hindus, Yavanīswara, or Yavanah, speaks " thus and thus" — the Yavanas say " so and so," &c.

Then who were the Yavanahs? In the second chapter of the Hōra Sāstra, by the author just quoted, are these verses:

" Hēlissūryaschandramāssītaraśmih
Hēma vijjnyō bōdhanaschēndupūtrah
Arō vakrah krūradrik chāvanēyah
Kōnō mandassūryaputrō sitascha

Jivōngirassuragururvachasāmpatirjyōk
Sukrō bhrigurbhrigusutassita asphujiccha
Rāhustamōgurasurascha sikhītukētuh
Paryayamanyadupalabhya vadēccha lōgāt."

" The name of the Sun is Hēli; of the Moon, Chandramā and Sītarasmih; of Mercury (Hemna, or) Hema, Vit, Jnyah, Bōdhanah, and

Induputrah; of Mars, Arah, Vakra, Kruradrik, and Avanēyah; of Saturn, Konah, Mandah, Suryaputrah, and Asitah; of Jupiter, Jivah Angiras, Suraguru, Vachasāmpati, and Jyok, and in the commentary Idyah; of Venus, Sukrah, Bhriguh, Bhrigusutah, Sitah, and Asphujit, or Apsujit; of Rahuh Tamah, Agu and Asurah; of Ketuh, Sikhi, &c.

The scholar will immediately perceive that the Greek names of the planets are distinctly to be traced in Varāha's enumeration, thus:

The Sun	Heli	ΗΛΙΟΣ	Helius,
Mercury	Hēma	ΕΡΜΗΣ	Hermes,
Mars	Arah	ΑΡΗΣ	Mars,
Jupiter	Jyok	ΖΕΥΣ	Jupiter,
Saturn	Kōnah	ΚΡΟΝΟΣ	Saturnus,
Venus	Asphujit	ΑΦΡΟΔΙΤΗ	Venus.

This arrangement is far from being fanciful. Moreover Idya, a name of Jupiter, is the Idæus, a title of that god with the Latins. Angiras seems to be Anxurus, another title of the same.*

The following curious lines, in the philological work of Pānini, show that the Hindus had a knowledge of Grecian literature: —

"Sutram, Indra varuna bhava sarva rudra mrida hīmāranya yava yavana mātulācharyānāmānuk vartakam indradināmānugāmassyat hīmāranyayoamahativē yavāddōshē yavanālliphyām upāddhāya mātullābhyām vāchāryādanatwancha artham indrasya stri indrānī varunasya stri varunani bhavasya stri bhavānī sarvasya strī sarvānī rudrasya stri rudrani mridasya stri mridānī mahaddhīmam himānī mahadaranyamaranyāni dushtō yavā yavanī yavanānām

* Even Jupiter and Diespiter appear the same with Dyupatih and Divaspatih, formed on true grammatical principles, from Dyō, the atmosphere, dium, and diva, which has the same meaning, united with patih, a lord or ruler; the compound being the "Lord of the Atmosphere:" thus Varāha Mihira says,

"Sikliū bhúkha payō marut ganānam
Vasinō bhumisutādayah kramēna."

"The regents of the elements, fire, earth, æther, water, and wind, are Mars, Mercury, Jupiter, Venus, and Saturn, respectively;" in which the planet Jupiter is dyupatih and divaspatih, or the lord of the atmosphere, diespiter: Usanah and Venus, also, for that planet may have a double denomination. — *Note by the author.*

liphih yāvanānī upāddhyaayāsyā stri upaddhyayānī upaddhyāyānī matulasya stri matulānī āchāryasya strī āchāryānī."

"Indra, varuṇa, bhava, sarva rudra mrida hīma, aranya, yava, yavana, mātula, achārya, with these are conjoined na; of Indra and others it is the feminine; of hīma and aranya it expresses extension or magnitude; and by it yava is deteriorated, and the letters of the yavanas expressed, &c. Thus the feminine of Indra is Indranī, of bhava, bhavanī, &c.; from hima is derived himānī, a great dew; from aranya, a forest, is derived aranyānī, an extensive, large forest; from yava, grain, pulse, is derived yavānī, damaged or bad grain; from yavana, a certain race, is derived yavanāni, the letters of that people," &c.

I will now make a remark upon the proverbial expression mentioned by Sir William Jones, nanicho yavanāt parah, "no base creature can be lower than a Yavanna," which is well known among Hindus. This, however, refers to the Turushkaras, the Turks, or Moguls, and not to the Greeks: the term Yavana is applied to both in the present day, particularly to the Musulman, or nations of the circumcised; but formerly solely to the generous conqueror of Porus, (king Parvata or Parvataka of that age,) and rival of Chandragupta. Indeed, Vārāha Mihira uses the term Yavana more than 200 years before the empire of India had fallen a prey to the earliest Mahomedan conquests. In the Harivamsam, while relating the origin of different nations, it is specifically mentioned that the Yavanāh were formerly Kshētryahs, or of the second rank of Hindus. This fable arises, perhaps, from the knowledge of the character of this people, with which they first became particularly acquainted, namely, that of warriors.

Having endeavoured to show that the term Yavana, confessedly that of an Ionian or Greek, is to be applied, when found in works relating to astronomy, to that nation, I shall bring forward some testimonies of that particular Grecian, who introduced, no doubt, improvements into the astrology and astronomy of the Indians, since he is quoted on every subject. Though a mlescha, a barbarian or foreigner, he was received as a Rishi, on account of his extensive knowledge and the

purity of his character. Having arrived in India, in the early ages of their society, he is reported to have been the bosom friend of Garga, the sanctified Guru of Krishna. Fable apart, he was an illustrious teacher of enthusiastic scholars. His words were received as holy truths. In astrology, in astronomy, in ethics, and even in points of religious ceremonies, and the magical influence of charms, his ordinances, as proceeding from divinity, were immediately committed to their immortal language by his learned pupils, and now form a large volume, the basis of Hindu science. They are still extant, either as a collection with themselves, or scattered in the numerous commentaries of Hindu works of science; and I feel much confidence in proposing the question, "Shall not the investigator of Yavanīswara's sayings discover the golden verses of Pythagoras?" *

The author of the Hōra Sāstram also quotes the Yavanas and Yavaneswara, particularly; and also the Grecian names for the Zodiacal signs in the same form as Varāha. Of the former is the following quotation, in the chapter which treated on the age of the subject in calculating the horoscope: —

"Maya yavanah sidhasenā prabōdhamāndavya vishnugptādyaih
Ayurdāya vibhagō nrīnam samyakpuratanaih kathillah."

* In my copy of the Horā Sāstram I find these lines, having the term Yavanah underlined by me on a former perusal of the work, chapter xi. verse 1.: —

"Prāhuryavanāh swatungagaih krūraih krūra matirmahīpatih
Krāraistunavajīva sarmana pakshe kshityādhipah prajāyatē."

"The Yavanahs say," &c. The commentary (vivaranam) says, Iti yavanācharryā vadanti, "Thus say the Yavana wise men." — Chapter viii. verse 9.

"Swaih swaih pushta phalāni sarga samayē vaktirdasayah kramā
Dantyelagna dasāsubheti yavanā neschanti kēchittatha."

In this also the word Yavanāh is used in the plural number; again, chapter xii. verse 1.: —

"Nava digvasavastrikagni vedairgunitadwitrichaturvikalpajassyuh
Yavanaistrigunāhi shadchhatisākathitā vislaratotra tatsamasah."

before quoted by Sir William Jones, in his Supplement to the Essay on Hindu Chronology. My copy reads it thus:

Aslēshārdandakshinam uttaram ayanam raverdhanishthādyam
Nānamo kadāchidāsīdyēnōctano purva sastreshu
Sāmpradam ayanam savituh karkatakādyam mrigaditaschanyat
Uktābhavō vikritch pratyaksha parikshinairvyaktih."

"Certainly the southern solstice was, at some period or other, in the half of Aslesha, and the northern solstice in the beginning of Dhanishta, according to what is recorded in former sastra. At the present time the solstices are in the beginning of Karkata and Makara; the difference from what is recorded is ascertained by actual inspection of the heavens."

The longitude of the Nakshatra Dhanishta is stated to be 290°, or 110° greater than Chitra, *Spica Virginis*, and is, therefore, in the 11th of Aquarius. Its distance from the ecliptic is stated to be 36°, the number of stars four. There can be but little doubt, then, that the Nakshatra Dhanishta consists of α, β, γ, δ, Delphini, of which the extremes β and γ were, in right ascension, 307° 2′ 34″ 74, and 309° 20′ 44″ 34, in A.D. 1800. As it is difficult to say in what part of the heavens the pole at that time was, and as Dhanishta is so far removed from the equator, it would be difficult to decide in what part exactly the colure cut the equator at that period. But, as the equinoctial points had receded 38° 0′ 10″ in 1800, the approximation of the Hindu recorded colure and that of the primitive sphere of Chiron is so remarkable, that we may strongly suspect that, as the Greeks most certainly introduced their sphere among the Indians, they also acquainted them with the stars through which the colures passed at its formation. *

* It is a remarkable circumstance, that Aswini is not the first of the lunar mansions with early astronomers, but rather Kritika. Can we conceive a better reason for moving the origin of their sphere by the Hindus, than the adapting it to an innovation which met with their general consent? The commencement of the two spheres is actually identified, by the change, with each other; and, as near as we can calculate, with the primitive sphere of Chiron. — *Note by the author.*

Paper 15

Reports (1832-1833)

Reports on C. M. Whish's presentation of paper on Quadrature of Circles

- The London Literary Gazette and Journal of Belles Letters, Arts, Science &c, No. 831, December 22, 1832, p. 810.
  Source: Google Books
- The New Monthly Magazine Literary Journal, American Edition, Vol. 1, January-June 1833, p. 71.
  Source: Google Books
- The Gentleman's Magazine and Historical Chronicle December 1832, Vol. CII, Part II, p.631.
  Source: Google Books
- The Asiatic Journal and Montly Register for British and Foreign India, China, and Australia, Vol.X–New Series, January–April 1833, pp. 81–82.
  Source: Internet Archive

Note

C. M. Whish's paper on Quadrature of Circles was read at a meeting of the Madras Literary Society on 15th of December 1832. The four documents mentioned above are reports of Whish's reading of the paper in the meeting. The paper was published in print only in 1834.

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The appended report appeared in both of the following journals.

THE LONDON LITERARY GAZETTE;

AND

Journal of Belles Lettres, Arts, Sciences, &c.

This Journal is supplied Weekly, or Monthly, by the principal Booksellers and Newsmen, throughout the Kingdom; but to those who may require its immediate transmission by post, we recommend the LITERARY GAZETTE printed on stamped paper, price One Shilling.

No. 831. SATURDAY, DECEMBER 22, 1832. Price 8*d*.

THE

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NEW MONTHLY MAGAZINE

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LITERARY JOURNAL.

VOLUME I. FROM JANUARY—JUNE, 1833.

AMERICAN EDITION.

BOSTON:
PUBLISHED BY ALLEN AND TICKNOR.
CHARLES S. FRANCIS, NEW YORK,
CAREY AND HART, PHILADELPHIA.
1833.

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ROYAL ASIATIC SOCIETY.

AT the last meeting the paper read was an explanation of the Hindu formulæ for the quadrature of the circle, by C. M. Whish, Esq. of the Madras civil service. Mr. Whish first extracts several rules from various works, exhibiting the proportion of the diameter to the circumference of the circle, with a degree of accuracy which must cause Europeans to admire the means by which Hindu mathematicians have been able to extend the proportion to so great a length. In one of these works, entitled the *Tantra Sangraha*, composed in Malabar in A.D. 1608, it is stated, that if the diameter of a circle be 1, the circumference will be 3.141592653921, &c. which is an excellent approximation, being correct to the ninth place of decimals. He then goes on to shew, that a system of fluxions, peculiar to the authors from whom he quotes among Hindus, has been followed by them in establishing their quadratures of the circle; and that by the same method the sines, cosines, &c. are found with the greatest accuracy. Several different infinite series, extracted from various Brahminical works, are given in illustration; and after some notes on the dates of these works, Mr. Whish concludes by submitting a proof of the 47th proposition of Euclid, extracted from the *Yuktibhásha*, a commentary on the *Tantra Sangraha*, above mentioned.

GENTLEMAN'S MAGAZINE,

AND

HISTORICAL CHRONICLE.

FROM JULY TO DECEMBER, 1832.

VOLUME CII.

(BEING THE TWENTY-FIFTH OF A NEW SERIES.)

PART THE SECOND.

PRODESSE & DELECTARE. E PLURIBUS UNUM.

BY SYLVANUS URBAN, GENT.

London:

PRINTED BY J. B. NICHOLS AND SON, 25, PARLIAMENT STREET;
WHERE LETTERS ARE PARTICULARLY REQUESTED TO BE SENT, POST-PAID;
AND SOLD BY JOHN HARRIS,
AT THE CORNER OF ST. PAUL'S CHURCH YARD, LUDGATE STREET; BY G. G. BENNIS, 55 RUE NEUVE ST. AUGUSTIN, PARIS; AND BY PERTHES AND BESSER, HAMBURGH.

1832.

QUADRATURE OF THE CIRCLE.

At a late meeting of the Royal Society a paper was read on the Hindu formulæ for the quadrature of the circle, by C. M. Whish, Esq. of the Madras civil service. Mr. Whish first extracts several rules from various works, exhibiting the proportion of the diameter to the circumference of the circle, with a degree of accuracy which must cause Europeans to admire the means by which Hindu mathematicians have been able to extend the proportion to so great a length. In one of these works, entitled the "Tantra Sangraha," composed in Malabar in A.D. 1608, it is stated, that if the diameter of a circle be 1, the circumference will be 3.141592653921, &c. which is an excellent approximation, being correct to the ninth place of the decimals. He then goes on to show, that a system of fluxions, peculiar to the authors from whom he quotes among Hindus, has been followed by them in establishing their quadratures of the circle; and that by the same method the sines, cosines, &c. are found with the greatest accuracy. Several different infinite series, extracted from various Brahminical works, are given in illustration; and after some notes on the dates of these works, Mr. Whish concludes by submitting a proof of the 47th proposition of Euclid, extracted from the "Yuktibhásha," a commentary on the "Tantra Sangraha," above mentioned.

THE

ASIATIC JOURNAL

AND

MONTHLY REGISTER

FOR

BRITISH AND FOREIGN INDIA, CHINA,

AND

AUSTRALASIA.

VOL. X.—NEW SERIES.

JANUARY—APRIL 1833.

LONDON:
PARBURY, ALLEN, AND CO.,
LEADENHALL STREET.

1833.

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A paper by C. M. Whish, Esq., of the Madras civil service, communicated by the Madras Auxiliary Royal Asiatic Society, and entitled "On the Hindu Quadrature of the Circle," was read.

The materials for this paper are principally drawn from four *Sástras*, entitled *Tantra Sángraha*, *Yuktabhásha* (a commentary on the first), *Karana Padhatih*, and *Sadratnamalah*. Mr. Whish, in the first place, quotes several rules for finding the proportion of a circumference to a given diameter, extracted from the works of Aryabhata, who lived in the fifth century of our era; from the *Silávati*, the author of which wrote six centuries later; the *Surya Siddhanta*, and other authorities, which exhibit approximations so wonderfully correct, that European mathematicians, who seek for such proportion in the doctrine of fluxions, or the more tedious continual bisection of an arc, must admire the means by which the Hindus have been able to extend the proportion to so great a length. He then proceeds to show that a system of fluxions peculiar to these authors alone among the Hindus has been followed by them in establishing their quadratures of the circle, and that by the same method the sines,

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cosines, &c. are found with the greatest accuracy. This position is illustrated by eight different infinite series, taken from the *Sástras* above-named, the date of the first of which he proves to be A.D. 1608. Mr. Whish concludes this essay with a simple and curious proof of the forty-seventh proposition of the first book of Euclid, extracted from the *Yuktabhásha* (the second *Sástra* above-mentioned), and which, he observes, was probably the form by which Pythagoras discovered this celebrated problem.

The thanks of the Society were ordered to be returned to Mr. Whish for this communication, and the meeting adjourned to the 5th of January 1833.

Paper 16

C. M. Whish (1834)

XXXIII. On the Hindū Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four Śāstras, the Tantra Sangrahaṃ, Yucti Bhāshā, Carana Padhati, and Sadratnamāla

Transactions of the Royal Asiatic Society of Great Britain and Ireland / Volume 3 / Issue 03 / July 1834, pp 509–523.

Note

This is C. M. Whish's landmark paper which for the first time brought to the attention of the Western scholarship some of the great accomplishments of the Kerala school of astronomy and mathematics.

The references in John Warren's *Kāla Saṃkalita* to the the discovery of the series expressions for π by Indians, indicate that by 1825, the Madras Literary Society had received a communication from Whish wherein he had referred to several Indian texts which mentioned these series. Whish was however persuaded by Hyne who was a part of the elite establishment in Madras that, since the Indian astronomers as well as their śāstras seemed to be unaware of the demonstrations of these series, they must have been communicated to the Indians by Europeans in the modern times.

C. M. Whish

For a biographical sketch of C. M. Whish, see p. 109.

Source: Internet Archive

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XXXIII. *On the Hindú Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four S'ástras, the Tantra Sangraham, Yucti Bháshá, Carana Padhati, and Sadratnamála.*

By CHARLES *M.* WHISH, *Esq., of the Hon. East-India Company's Civil Service on the Madras Establishment.*

(*Communicated by the* MADRAS LITERARY SOCIETY *and* AUXILIARY ROYAL ASIATIC SOCIETY.)

Read the 15th of December 1832.

A'RYAB'HATTA, who flourished in the beginning of the thirty-seventh century of the *Cáli Yuga*,* of which four thousand nine hundred and twenty years have passed, has in his work, the *Aryab'hatiyam*, in which he mentions the period of his birth, exhibited the proportion of the diameter to the circumference of the circle as 20000 to 62832, in the following verse:

Chaturadhicam ṣatamashṭaguṇandwáshashṭistathá sahasráṇám
Ayutadwaya vishcambhasyásannó vritta pariṇáhah.†

Which is thus translated:

"The product of one hundred increased by four and multiplied by eight, added to "sixty and two thousands, is the circumference of a circle whose diameter is twice ten "thousand."

The author of the *Lilávati*, who lived six centuries after A'RYAB'HATTA, states the proportion as 7 to 22, which, he adds, is sufficiently exact for common purposes. As a more correct or precise circumference, he proposes that the diameter be multiplied by 3927, and the product divided by 1250; the quotient will be a very precise circumference. This proportion is the same with that of A'RYAB'HATTA, which is less correct than that of

* Or the sixth century of the Christian era.

† This verse is in the variety of the *Aryavrittam* measure, called *Vipula*.

Metius, *viz.* 113 to 355, though in all three the circumference is rated at more than the truth. The *slóca* in the *Lilávati* is as follows:

Vyásé bhanandagnihaté vibhacté
Chabáṇasúryaih paridhissussúcshmah
Dwávimṣatinighnévihritétha sailaih
Stúlóthavásyádvyavahárayógyah."*

A'ryab'hatta has not left behind him any record of the means by which he obtained so accurate a proportion. His work however shews so much geometrical knowledge, that there can be little doubt that he effected his approximation in the same manner with Archimedes, by a numeral calculation of the perimeters of inscribed and circumscribed polygons with the circle; for in the polygon of 768 sides to a circle whose radius is *one*, the perimeter of the inscribed polygon is 628316, and of the circumscribed 628326. The circumference of the circle will therefore be greater than 628316, but less than 628326, or nearly half their sum, namely, 628321, to which the above approximation is nearly similar.

In the *Súrya Sid'dhánta*, the radius of a circle is stated at 3438, in terms of minutes of the circumference, being about one-quarter of a minute too great; and in the following *slóca*, it appears that a very coarse rule is laid down by the author.

Yojanánám ṣatányashṭau bhúkarnó dwigunánitu
Jadvargató daṣaguṇát padam bhúparidhirbhavét.†

"Square the *yojanas* of the earth's diameter 1600, and multiply the square by 10, "and extract the root of the product: the root is the equatorial circumference of the "earth."

In such a matter, the strictest accuracy need not be required; but for a general rule, this would err considerably from the truth, being less correct than the proportion of 7 to 22; and yet we find the same in the *Cámadógdhrí*, a commentary on the *Súrya Sid'dhánta*, laid down in the following terms:

Vritta vargát daṣa hṛitat padamvyásóbhavédiha
Evam parícshitóvyásah vṛittastu viparítatah.

"The root of the square of the circumference divided by ten, is the diameter; to find the circumference of a given diameter, reverse the rule."

---

* This verse is in the *Upajátivrittam* measure.

† This verse is in the *Anushtubvrittam* measure.

In the *śástra* named *Tántra Sangraha*, we find the following lines:

Ishṭa vyásé haté nágavédavahnyabdhichéndubhih
Tithyaṣwivibudhairbhacté susúcshmah paridhirbhavét
Paridhérvyatyayáchaivamsusúcshmam vyasamánayét
Tithyaṣwivibudhairnighnát chachatancéndunétratah
Súcshmó vyásóhi védagni cṛitóbhréndu hṛitóthava
Tríṣaghna chacra liptábhyó vyásórtthéshwagnibhirhṛitéh.

Which are thus translated:

"Multiply any given diameter by 104348, and divide the product by 33215; the "quotient is a very correct circumference. To find the diameter, if the circumference "be given, reverse the above numbers. If then you multiply the astronomical circum- "ference by 33215, and divide the product by 104348, you will have the diameter *in* "*terms of minutes of the periphery; or it will be sufficiently exact if you multiply* "*the circumference by* 113, *and divide by* 355, *for a diameter.*"

The astronomer's circle, *anantapura* or "the universe," and *chachatancendunetram*, is, according to alphabetical notation and symbols respectively, 21600, that is, the terms of the minutes in the circumference of twelve signs, containing each thirty degrees. In the first proportion of 33215 to 104348, if the diameter of a circle be 1, the circumference will be 3.1415926539 21, &c., which is an excellent approximation, being correct to the ninth place of decimals, the tenth being too large; the second proportion, 113 to 355, that of Metius, is a very useful one, being as 1 to 3.1415929, &c., correct to the sixth place of decimals, the seventh being rather more than the truth.

In another work, named *Carana-Padhati*, and in one called *Sadratnamála*, are the two following verses; the first is from the former work:

Anúnanutnánananunnanutyaih
Samáhataṣchacra kálávibhactah
Chandámṣuchandrádhama cumbhipálaih
Vyásastadardhantribhamaurvicásyát.

"If the circumference of a circle in minutes be multiplied by 10000000000, and the "product be divided by 31415926536, the quotient will be the diameter of the circle "*in terms of the minutes of the circumference*, and its half will be the radius."

In the latter work is the subjoined verse:

Evanchátra parárdha vistṛitimahá vṛittasya nahocsharaih
Syádbhadrámbudhi sidha janma gaṇitasradáṣmayadbhápúgih.

" If you proceed thus (as laid down in the former verse), and measure the circum-
" ference of a great circle by 100000000000000000 parts, the circumference will be
" equal to 314159265358979324 of *such parts*."

The approximations to the true value of the circumference with a given diameter, exhibited in these three works, are so wonderfully correct, that European mathematicians, who seek for such proportion in the doctrine of fluxions, or in the more tedious continual bisection of an arc, will wonder by what means the *Hindú* has been able to extend the proportion to so great a length. Some quotations which I shall make from these three books, will shew that a system of fluxions peculiar to their authors alone among *Hindús*, has been followed by them in establishing their quadratures of the circle; and a few more verses, which I shall hereafter treat of and explain, will prove, that by the same mode also, the sines, cosines, &c. are found with the greatest accuracy.

I proceed to quote extracts from the *Tántra Sangraham*. The first, of the measure called *A'nushtubvrittam*, is from the chapter upon sines, &c.

Vyásárdham prathamannítwatatóványátguṇannayét
Sambandhanniyamanchaivam vynéyóvyása vṛittayóh.

" Having found the radius, you may construct the sines; but you must first know the
" proportion between the diameter and the circumference."

The next is of the *Gítivrittam* measure:

Vyásé varidhinihité rúpa hṛité vyásaságarábhihate
Triṣarádhi vishama sanchyá bhactamṛiṇamswam prithacramát curyát
Yatsanchyayátra haraṇé critérnivrittá hṛitistujámitayá
Tasyá úrdhwa gatáyássamasanchyá taddalamgunóntésyát.
Jadvarggó rúpayutó háró vyásábdhighátacah prágwat.
Tabhyámáptam swamṛiṇé critédhané ṣódhananchacaraṇíyám
Súcshmah paridhissasyát bahucritwóharaṇatóti súcshmascha.

" Multiply the diameter by 4, and from it subtract and add alternately the quotients
" obtained by dividing four times the diameter by the odd numbers 3, 5, 7, 9, 11, &c.,
" do thus to the extent required; and having fixed a limit, take half the even number
" next less than the last odd divisor for a multiplier, and its square *plus* one for a divisor.
" Multiply four times the diameter by the multiplier, and divide the product by the
" divisor, and add it or subtract it, according to the sign of the last quote in the series,
" from the sum of the series: thus the circumference of the given diameter will be
" obtained very correctly."

If we proceed according to the rule, we have an infinite series of the following form :

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \frac{4d}{9} - \frac{4d}{11} + \&c. \pm \frac{4d \times \frac{1}{2}p}{p^2 + 1}$$

where C=circumference, d diameter, and p the last odd divisor diminished by unity. When $d=1$ the series becomes

$$C = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \pm \&c. \frac{\frac{1}{2}p}{p^2 + 1}\right)$$

Then follows a verse of the *Gitivrittam* measure, explaining more fully the correction by which this series is brought to greater perfection.

Asmat sucshmatarónyó vilichyatécaschanápí samscárah
Anté samasanchyá dala varyassaicó gunassá éva punah
Yuga gunitó rúpayuttassamasancyá dala ható bhavéddhárah
Triṣarádi vishama sanchyá haranát paramé tadéva vácáryam.

" I now shew how the correction may be made more complete than in the former " rule: take the even figure next greater than the last odd divisor in the series $4 \times$ " $\left(1 - \frac{1}{3} + \frac{1}{5} - \&c.\right)$ that may have been fixed upon, and square its half, and increase " it by *unity ;* this is to be a multiplier : this multiplier multiply by 4, and the product " increased by unity multiply by half the original even figure ; this last product will be " a divisor : add to the result of the series the quotient of four times the diameter " multiplied by the new multiplier, and divided by the new divisor ; the sum will be a " more correct circumference."

The series by means of this correction becomes :

$$C = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c. \pm \frac{\frac{p^2}{4} + 1}{\left(p^2 + 4 + 1\right)\frac{p}{2}}\right)$$

p being here the last odd divisor increased by unity.

The author being aware how slowly the series converges, found it necessary to correct the last quote, which is done very correctly by the rule he has exhibited.

Next follows a verse of the *Anushtubvrittam* measure.

Vyásavargádravihatat padamsyát prathamam phalam
Jatastattat phaláchchápiyávadiṣchántribhirharét
Rupádyayugmasancyábhirllabdéshwéshuyathácramam
Vishamánám yutétyacté yugmayógé vṛitirbhavét.

" Multiply any given diameter squared by 12, and extract the root of the product ; " this is the first quote : divide this first by 3 for the second quote ; this second, and each " so obtained quote, divide by 3 continually, place them in order, and divide them in " succession by the odd numbers 1, 3, 5, 7, 9, 11, &c. Then add together the first, " third, fifth, &c., quotes ; and add together also the second, fourth, sixth, &c. quotes ; " then subtract the sum of the even from that of the odd, and the remainder will be the " circumference of the circle, whose diameter was given."

The series laid down in the verse is thus expressed algebraically :

$$C = \sqrt{12} \times \left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \frac{1}{11.3^5} + \&c.\right)$$

The next verse is of the *Gítivrittam* measure :

Samapancháhatayóyá rupádyayujás chaturghna múlayutáh.
Tabhisshódaṣa guṇitádvyását pṛithagá hṛitétu vishamayaté.
Sama phala yágé tyacté syátishṭa vyása sambhavah paridhih.

" Divide the diameter multiplied by 16 severally by the fifth power of the odd " numbers 1, 3, 5, 7, 9, &c., adding to each fifth power four times its root ; of the " quotes thus obtained subtract the sum of the second, fourth, sixth, &c. from that of " the first, third, fifth, and seventh, the remainder will be the circumference of the circle " whose diameter was taken."

This is an infinite series of the following form :

$$C = \frac{16\,d}{1^5 + 4.1} - \frac{16\,d}{3^5 + 4.3} + \frac{16\,d}{5^5 + 4.5} - \frac{16\,d}{7^5 + 4.7} + \frac{16\,d}{9^5 + 4.9} - \&c.$$

Or, if the diameter be *one* :

$$C = 16 \times \left(\frac{1}{1^5 + 4.1} - \frac{1}{3^5 + 4.3} + \frac{1}{5^5 + 4.5} - \frac{1}{7^5 + 4.7} + \&c.\right)$$

The next verse of the *Gítivrittam* measure is as follows :

Vyásádwaridhi nihatát pṛitagáptantryáyugvimúld ghanaih
Trighna vyásé swamṛiṇam cramasah kritwápi paridhivánéyah.

" Divide four times the given diameter by the cubes of the odd numbers 3, 5, 7, 9, " 11, 13, &c., subtracting from each its respective root ; the quotes thus obtained " alternately add to, and subtract from three times the diameter ; this is the circumfe- " rence of the circle whose diameter was given."

This is an infinite series (given in the *Carana Padhati* in nearly the same words), as follows :

$$C = 3\,d + \frac{4\,d}{3^3 - 3} - \frac{4\,d}{5^3 - 5} + \frac{4\,d}{7^3 - 7} - \frac{4\,d}{9^3 - 9} + \&c.$$

Which, if d=1, will be:

$$C=3+4\times\left(\frac{1}{2.3.4}-\frac{1}{4.5.6}+\frac{1}{6.7.8}-\frac{1}{8.9.10}+\&c.\right)$$

The author then proceeds in the same measure:

Dwyádiyujám vacritayó vyécá hárádwinighna vishcambhe
Dhanamrinamanténtyórdhwa gataujacritidwi sahitáharódwighní.

" Multiply the diameter by 4, and divide the product severally by the squares of the " numbers 2, 4, 6, 8, 10, &c., subtracting 1 from each square: the quotients alternately " add to, and subtract from, twice the diameter: rectify the sum obtained, by taking the " next odd number less than the last even figure squared, squaring it, adding 2 to the " square, doubling this sum, and with this thus obtained as a divisor, dividing four times " the diameter: this quotient add or subtract, as is required, from the sum formerly " obtained, for a very correct circumference."

The series thus obtained is:

$$C=2d+\frac{4d}{2^2-1}-\frac{4d}{4^2-1}+\frac{4d}{6^2-1}-\frac{4d}{8^2-1}+\&c.\frac{\pm}{}\frac{4d}{\overline{p-1}^2+2\times 2}$$

Where C=circumference, d=diameter, and p=the last even number squared in the series, which, if the diameter be 1, becomes,

$$C=2+4\times\left(\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\&c.\frac{\pm}{}\frac{1}{\overline{p-1}^2+2\times 2}\right)$$

This series, *viz.* $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\&c.$ can easily be proved to be equal to the arc of $90°-\frac{1}{2}$, therefore 2+4 into the series will equal the whole circumference, the diameter being *one*, as in our author's series.

Two other series are then exhibited in lines of the *Gíti* measure.

Dwyádéschaturádérva chaturadhicánánniréca vargassyuh
Háráh cunjara gunitá vishcambhaswamati calpitóbhájyah
Phalayutirádyé vrittam bhájyadalam phalavihínamányatra.

" Take the squares of the terms, diminished by 1, of the two arithmetical progressions " whose first terms are respectively 2 and 4, and ratio of progression 4, for divisors: in " the former series, divide 8 times the diameter by the divisors severally, and the sum of " the quotients is the circumference of the circle; in the latter, subtract from 4 times " the diameter the sum of the quotients of 8 times the diameter divided by the divisors " severally, and the result is the circumference in the second case."

The two series thus explained are found to be of the following forms:

$$C = \frac{8d}{2^2-1}+\frac{8d}{6^2-1}+\frac{8d}{10^2-1}+\frac{8d}{14^2-1}+\frac{8d}{18^2-1}+\&c.$$

$$C = 4d-\left(\frac{8d}{4^2-1}+\frac{8d}{8^2-1}+\frac{8d}{12^2-1}+\frac{8d}{16^2-1}+\&c.\right)$$

Which, if $d=1$, become respectively:

$$C = 8\times\left(\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\frac{1}{13.15}+\frac{1}{17.19}+\&c.\right)$$

$$C = 4-8\times\left(\frac{1}{3.5}+\frac{1}{7.9}+\frac{1}{11.13}+\frac{1}{15.17}+\frac{1}{19.21}+\&c.\right)$$

Now, let the former series within the brackets be a, and the latter b; then it can be easily proved that $a+b$, or the series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+\&c. = \frac{1}{2}$; and it has been mentioned above that the series $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\&c. =$ (when the diameter is one), the arc of $90°-\frac{1}{2}=a-b$; having therefore the values of $a-b$ and $a+b$, it can be easily proved that $a=$the arc of 45°, and $b=\frac{1}{2}-$ arc of 45°; therefore, in the first of these quadratures, $8a=$ circumference, and in the second, $4-8b=$circumference also, as taught by the author.

The author, after laying down the above series, proceeds to shew in numbers a proportion of the circumference to the diameter in the verse which has been originally quoted and translated, and then finishes his chapter by rules for finding the sines and cosines, of which mention will be made hereafter.

In the *Carana Padhati*, the sixth chapter commences thus:

Vyásácchaturghnát bahuṣah pṛithac sthát
Tripanja saptadyayugáhritáni
Vyasé chatuoghné cramaṣastwṛiṇam swam
Curyáttadásyát paridhissusúcshmah.

"Divide the given diameter multiplied by 4 severally and continually by the odd "numbers 3, 5, 7, 9, 11, &c., and the quotients thus obtained, alternately subtract "from, and add to, the diameter multiplied by 4. The result is the precise circum-"ference."

This series is the same with the first laid down by the author of the *Tantra Sangraha*, namely,

$$C = 4\times\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\&c.\right)$$

The next verse in the chapter is of the measure called *Gitivrittam.*

Vyásádwana samguṇitát prithagáptantryádyay ugvimúlaghanaih
Iriguṇita vyásé swamṛiṇam Cramaṣah cṛitwápi paridhiránéyah.

" Divide the given diameter multiplied by 4 severally and continually by the cubes of " the odd numbers 3, 5, 7, 9, 11, &c., subtracting from each cube its respective " root; the sums thus obtained alternately add to, and subtract from, 3 times the " diameter: thus you will obtain the circumference of the circle whose diameter was " given."

This series is also the same with one of the former book, namely,

$$C = 3 + 4 \times \left(\frac{1}{2.3.4} - \frac{1}{4.5.6} + \frac{1}{6.7.8} - \frac{1}{8.9.10} + \&c.\right)$$

The fourth verse of the chapter is of the measure called *Indra Vajravrittam.*

Vargairyujámvadwigunairnnirécaih
Vargícritairvarjitayugma vargaih
Vyásancha shadghnam vibhajét phalam swam
Vyásé trinighné paridhistadásyat.

" Add to three times the diameter the sum of the quotes obtained by dividing six " times the diameter by the square of twice the square minus one of the even numbers " 2, 4, 6, 8, 10, &c., subtracting from each the square of its even figure respectively. " The sum is the circumference."

This is an infinite series thus expressed algebraically:

$$C = 3d + \frac{6d}{(2.2^2-1)^2-2^2} + \frac{6d}{(2.4^2-1)^2-4^2} + \frac{6d}{(2.6^2-1)^2-6^2} + \&c.$$

Which, if the diameter = 1, becomes

$$C = 3 + 6 \times \left(\frac{1}{1.3.3.5} + \frac{1}{3.5.7.9} + \frac{1}{5.7.11.13} + \frac{1}{7.9.15.17} + \frac{1}{9.11.19.21} + \&c.\right)$$

The author proceeds with the verse originally quoted, for determining the diameter and radius in terms of minutes of the circumference, and then teaches how, by certain series, the sines, cosines, &c. are to be constructed. He next exhibits a stanza for finding an arc of the circumference of a circle by means of the sum obtained by multiplying the sine of the arc into its radius and dividing the product by its cosine. This sum, it will be readily observed, is equivalent to a tangent of that arc, for cosine is to radius as sine is to tangent; but the tangent individually does not appear in the mathematics of the *Hindús.*

The lines are as follow:

Vyásárdhénahatádabhíshṭa guṇatah Cótyaptamádyam phalam
Jyávargéna vinighnamádimaphalam tattat phalanchá harél
Crityá cóti guṇasya tatratuphaléshwécatripanchá dibhi
Bhuctéshwójayutéstyajét samayutim jívádhanussishtyaté.

"Multiply the given sine by radius, and divide the product by cosine for the first "quote; multiply this quote by the square of the sine, and divide the product by the "square of the cosine for the second quote: multiply and divide this last quote, and so "continually each obtained quote, by the square of the sine and the square of the cosine "respectively, and the quotes obtained by this means divide in succession by the odd "numbers 1, 3, 5, 7, 9, 11, &c.; then the sum of the 2d, 4th, 6th, 8th, &c. quotes "being subtracted from the sum of the 1st, 3d, 5th, 7th, 9th, &c. quotes, the remainder "will be the arc of the sine which was taken."

Note.—*If the cosine be less than the sine of the given angle, change the names of the two, and proceed as laid down in the rule.*

The infinite series thus beautifully and concisely expressed in the *Sanscrit slóca*, is the following:

$$a = \frac{r \times s}{cos.} - \frac{r \times s^3}{3\,cos.^3} + \frac{r \times s^5}{5\,cos.^5} - \frac{r \times s^7}{7\,cos.^7} + \frac{r \times s^9}{9\,cos.^9} - \text{\&c.}$$

Where a=arc; r=radius; s=sine, and cos=cosine.

In this series, the first quote equals the tangent of the arc=t; the second quote equals tangent cubed, divided by thrice radius squared; the third equals tangent to the fifth power divided by 5 into radius to the fourth power; or, algebraically, thus:

$$a = t - \frac{t^3}{3\,r^2} + \frac{t^5}{5\,r^4} - \frac{t^7}{7\,r^6} + \frac{t^9}{9\,r^8} - \text{\&c.}$$

Which is easily proved to be true by a process in fluxions, which demonstrates the fact that the fluxion of the tangent of an arc *is to* the fluxion of the arc itself *as* the square of the secant is to the square of the radius; in which case, the fluxion of the arc is proved to be equal to $\frac{r^2\,\dot{t}}{r^2+t^2}$ and if $r^2\,\dot{t}$ be divided by r^2+t^2, the quotient will be $\dot{t} - \frac{t^2\,\dot{t}}{r^2} + \frac{t^4\,\dot{t}}{r^4} - \frac{t^6\,\dot{t}}{r^6} + \frac{t^8\,\dot{t}}{r^8} - \text{\&c.} = \dot{a}$; and the fluents of each being taken, it becomes $t - \frac{t^3}{3\,r^2} + \frac{t^5}{5\,r^4} - \frac{t^7}{7\,r^6} + \frac{t^9}{9\,r^8} - \text{\&c.} = a =$ the arc itself, as is laid down in the *Carana Padhati*.

It is also seen that from this verse the matter is derived of the former series; for if (in the first series) the arc of 45° be taken, then the tangent will be equal to radius; and if radius = 1, the above series becomes $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+$ &c. $=a$; and four times this sum will be the semi-circumference, or when the diameter is 1, the whole circumference; therefore $4\times(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+$ &c.,) the circumference, as shewn in the *Tantra Sangraha*, and in the *Carana Padhati*.

I proceed now to quote some verses from the *Sadratnamála*. The first which I shall extract is from the chapter on sines, and is of the measure called *Sálini-vrittam*.

Vargádwyásasyárcanighnāt padamyat
Tatryamṣó yastécha tattannavámṣáh
Dwigna vyécaicadwi purvaujayugmah
Chinnányaicya dwyantaram vṛittanáhah.

"Square the diameter and multiply the product by 12, and extract the root of this "product; the root obtained will be the *modulus* of odd quotes, which if you divide "by 3, the quotient will be the *modulus* of even quotes. Divide each modulus "continually by 9, and the quotient thus obtained from the former, divide by double "the numbers 1, 3, 5, 7, 9, &c. minus 1 respectively, and the quotient obtained from "the latter, by double the number 2, 4, 6, 8, 10, &c. minus 1 respectively, add up the "new obtained quotes, and subtract the sum of those gotten from the even from the "sum of those gotten from the odd modulus, the remainder is the circumference of the "circle."

The next verse of the *Sarddúla-viccridita* measure is this:

Vyasárcaghnacriteh padegnibhiratónítécha tattat phaláh
Cháthaicyadyayugá hṛiteshu paridhirbhédóyugójaicyayóh.
Evanchátra parárdha vistṛiti mahávṛittasya náhácsharaíh
Syátbhadrámbudhi sidha janma gaṇitasṛadhásmayatbhúpagíh.

"Square the diameter and multiply the product by 12, and extract the root of this "product; this root divide continually by 3, and the quotients thus obtained by 1, 3, "5, 7, 9, 11, &c., and subtract the sum of the 2d, 4th, 6th, 8th of the last obtained "quotes from the sum of the 1st, 3d, 5th, 7th, 9th, &c. If you do thus, and measure "the diameter of a great circle by 100000000000000000 equal parts, the circumference "will be equal to 314159265358979324 of such parts."

The rule laid down in this verse is exactly similar to that communicated by Dr. Halley to the Royal Society of London; and is founded upon the

principles laid down in another *slóca* of the *Vasantitilaca-vrittam* measure in the same chapter, as follows:

Cotíhṛita triguṇa báhu vadhé chatasmá
Tattat phaláchcha bhujavarga hatáttu cotyáh
Crityá cṛitéshucha dharágni sarádi bhacte
Shwojaicyatastyajatuyugmayutim dhanustat.

" The radius into the sine divided by the cosine is the first quote: this multiplied by " the square of the sine, and divided by the square of the cosine, is the second quote; " this second, and those obtained continually in the same way, multiply and divide by " the square of the sine and the square of the cosine respectively: divide the quotes in " order by 1, 3, 5, 7, 9, 11, &c. respectively, and the difference of the sum of the 1st, " 3d, 5th, &c., and of the 2d, 4th, 6th, &c., will be the arc whose sine was taken."

This, of which the meaning is the same with that in the *Carana Padhati*, is expressed thus:

$$a = \frac{r \times s}{cos.} - \frac{r \times s^3}{3\,cos.^3} + \frac{r \times s^5}{5\,cos.^5} - \frac{r \times s^7}{7\,cos.^7} + \text{ &c.}$$

Now if the arc of 30° be taken, the first quote = the tangent of that arc $= \surd\frac{1}{3}$; to multiply this by the square of the sine, and divide it by the square of the cosine, it is sufficient to divide it simply by 3 (because the square of the sine of 30° is one-third of the square of the cosine of that angle): divide then each quote in succession by the odd numbers 1, 3, 5, 7, 9, 11, &c., respectively; the arc, thus obtained, multiplied by 6, will equal the semi-circumference when the radius = 1, or the whole circumference if the diameter be 1; but $6 \times \surd\frac{1}{3} = 2 \times \surd 3 = \surd 12$; therefore $\surd 12 \times \left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \frac{1}{11.3^5} + \text{ &c.}\right)$ equals the whole circumference when the diameter is *one*, according to the rule laid down in the *Sadratnamálah* in the verse " *Vyasárcaghnacriteh,*" &c., and in the *Tantra Sangraha* in the verse " *Vyasa Vargadravihatat,*" &c. before quoted. The rule also for finding the sum of any arc from the tangent is thus expressed in the same work in four *slócas* of the *Anushtub* measure.

Vrittánayana márgínábreshta jyáná dhanurnayét
Ishta jyá trijyayórghátát cótyáptam prathamam phalam
Jyá vargom gunacam critwá cotí vargancha háracam
Prathamádi phalébhyóthá néyáphalalatirmuhuh.
Ecatryádyája sanchyábhirbhactéshwétéshwanucramát
Ojánám yutitastyacte yugma yógé dhanurbhavét
Dóh cótyóralpamévéshtam calpaníyamihóditam
Muhuh critépi labd'hínánnanyathávasitirbhavet.

" According to the genius of the rules which we have laid down for the circumference " of the circle, the arc of any given sine may be found thus: take a given sine multiplied " by radius, and divided by the cosine to the given sine for the first quote; let the square " of the sine be a constant multiplier, and the square of the cosine a constant divisor, " the quotes obtained by multiplying and dividing (by these) the first quote, must be " placed in order one under the other, and divided in succession by the odd numbers 1, 3, " 5, 7, 9, 11, &c.; then subtract the sum of the even from that of the odd quotes, " and the remainder will be the arc whose sine was taken. If the sine of the given arc " be greater than its cosine, call *that* the cosine, and the cosine to it the sine."

Having thus submitted to the inspection of the curious eight different infinite series, extracted from *Bráhmanical* works for the quadrature of the circle, it will be proper to explain by what steps the *Hindú* mathematician has been led to these forms, which have only been made known to Europeans through the method of fluxions, the invention of the illustrious Newton. Let us first, however, know the age of these works; and as far as can be determined, the authors. First, then, it is a fact which I have ascertained beyond a doubt, that the invention of infinite series of these forms has originated in Malabar, and is not, even to this day, known to the eastward of the range of *Ghâts* which divides that country, called in the earliest times *Céralam*, from the countries of *Madura*, *Coimbatore*, *Mysóre*, and those in succession, to that northward of these provinces.

The author of the *Sadratnamálah* is Sancara Varma, the younger brother of the present *Rájá* of *Cadattanáda* near *Tellicherry*, a very intelligent man and acute mathematician. This work, which is a complete system of *Hindú* astronomy, is comprehended in two hundred and eleven verses of different measures, and abounds with fluxional forms and series, to be found in no work of foreign or other Indian countries.

The author of the *Carana Padhati*, whose grandson is now alive in his seventieth year, was Pathumana Sóma Yaji, a *Nambútiri Bráhmana* of *Tirusívapura* (Trichúr) in Malabar. In the first verse of the commentary on his work, he is thus mentioned:

Nutuna griha sama sutarachitayah carana padhatervidusha
Bhasham vilichati caschit balanam bhodhanarthamalpadhiyam.

" The *Carana Padhati*, a path to arithmetic, composed by Nutunagriha Sóma " Suta, I now proceed to explain in the common tongue, to adapt it to the genius of " young scholars."

Nutunagriha Sóma Súta bears the meaning in *Sanscrita* of the Malabar terms "Puthamana Sóma Yaji," the first word being the name of his house.

The last verse of his work contains its date in numerical letters in terms of days of the *Caliyuga;* a mode of dating which, in the country, is to be traced back to upwards of one thousand seven hundred years ago, as may be seen in the inscriptions taken from the pagoda of *Tiruvunnúr* near Calicut. The verse is as follows:

Iti sivapura nam agramajah Capiyajwa
Cimapi Carana padhatyahwayam tantra rupam
Vyadhita ganita metutsamyagalocyasantah
Cathitamihavidantassuntu santosha pantah.

The words *Ganita Metutsamyac,* written in numbers, amount to 1765653 in terms of days of the present age; which is found to agree with A.D. 1733, being 86 years and 271 days before the 31st of December 1819; Sóma Yaji was not, however, inventor of the system by which he formed his infinite series.

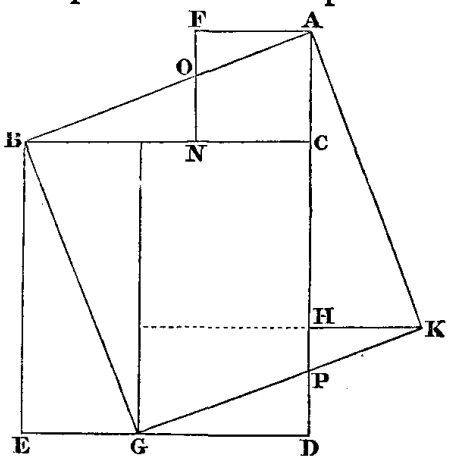
The author of the *Tantra Sangraha,* who was educated in the college of *Bráhmanas* at Trichúr (which college still exists, and which is a place of reputed sanctity, and certainly of respectable learning), has, among his works on arithmetic and astronomy, laid the foundation for a complete system of fluxions, and opened a mine of wealth in mathematics to those students who can boast of being his followers, of which they are most eminently jealous. The copy of the work, which I have obtained with some difficulty, by frequent intercourse with this interesting society, bears in itself marks of antiquity, the commentary in the vulgar tongue being written in a language which is not now generally current in Malabar, and the forms of many letters differing materially from those of the present day.

The author of the work is Talaculattúra Nambu'tiri, of *Cérala,* or Malabar; the first epithet being the name of his house, the latter the title of his rank: he flourished in the forty-sixth century of the present age, or *Caliyuga.*

The testimonies as to the author, and the period in which he lived, are the following, *viz.* The general consent of the learned in Malabar; the date which is shewn in the commencement of the work itself, namely, the year 4600 of the *Caliyuga;* the mention made of him in the first chapter of a work named *Driccaranam* by his commentator, the author of the *Yucti-Bháshá,* Cellalura Nambutiri, in which, while relating the history of the progress of astronomy, from the improvements of A'ryab'hatta up

to his own period, he thus observes: "Much difference having occurred "in astronomical calculations, in the year of the *Caliyuga* 4532 a *Bráh-* "*man* of high rank who lived on the coast of the western ocean, having "examined the heavens for twelve years, established what is laid down "in the *Tantra-Sangraha*." This is the evidence of the author of the *Yucti-Bháshá*, the commentary on the *Tantra-Sangraha*, concerning the author of the latter work: the date of the *Driccaranam* is mentioned in the latter part of the work, *viz.* the 783d of the Malabar era; and in the summary account of the periods of astronomy, it is written 4708 of the *Caliyuga*, both of which coincide with the year 1608 of the Christian era.

A farther account of the *Yucti-Bháshá*, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper: I shall therefore conclude this, by submitting a simple and curious proof of the 47th proposition of Euclid, extracted from the *Yucti-Bháshá*. In the accompanying figure, let ABC be a triangle, having the angle at C a right angle: on AC describe the square ACFN, and on BC describe the square BCDE: on ED take EG equal to AC, and on AD take AH equal to BC, draw HK perpendicular to AD and equal to AC, join BG, GK, and AK.



The mathematician will easily prove that the three spaces AFO, PGD, and BEG, parts of the squares SACFN and BCDE not included in the figure ABGK, are equal to, and identical with, the spaces KHP, OBN, and AHK, not occupied by any parts of the above two squares; *id est*, that the sum of the two squares ACFN and BCDE equals the figure ABGK; but this, from its construction, is a square, and is drawn upon the hypothenuse ABC. This is probably the form by which Pythagoras discovered the celebrated problem, which Euclid afterwards so beautifully illustrated in the 47th proposition of the first book of his 'Elements.'

Paper 17

Augustus De Morgan (1843)

Article on "Viga Ganita", in *The Penny Cyclopaedia*, Vol. XXVI, Charles Knight & Co, London, 1843, pp. 318–26.

Notes

- This is the document where the author derisively dubbed Whish's paper on quadrature of circles as a "clever hoax".
- Only that part of the nearly nine-page long article which has a relevance to the Kerala school of astronomy and mathematics has been reproduced here.
- The contributors to the Penny Cyclopaedia were not individually credited with the articles they created, although a list of their names appears in volume 27. According to this list, articles on mathematics and astronomy were contributed by Augustus De Morgan, Trinity College Cambridge, and Professor of mathematics in University College, London.

Augustus De Morgan

Augustus De Morgan (1806–1871) was a British mathematician and logician. He formulated De Morgan's laws and introduced the term mathematical induction, making its idea rigorous. (Wikipedia)

Source: Internet Archive

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dern state of science in India is generally re-
m Le Gentil downwards, to be at a low ebb.
persons, it is asserted, can even predict the time
pse; and still fewer understand the rules by
done. In this, the assertion of Le Gentil, Davis,
ey, uncontradicted by Colebrooke, we should
disposed implicitly to rely, had it not been for a
lished in our own day (*Mem. Asiatic Soc.*, vol.
r. Whish, of the Madras Civil Establishment,
Hindu Quadrature of the Circle.' If any one
ne was not known to us had simply published
on his own authority, we should have had some
that it was neither more nor less than a clever
it is published, with the Sanscrit at full length,
noirs of the Society which is best able to judge
ents.

The author begins by mentioning a work of Aryabhatta (the Aryabhatiyam, apparently unknown to Colebrooke), in which he mentions the epoch of his birth in a manner which places him at the period of Colebrooke's latest conjecture, namely, at the end of the fifth century. Here he gives the circumference of the circle at 3·1416 times its diameter. Mr. Whish then mentions three works:

1. The *Tantra Sangraha* of Talaculattura, who, by his own statement, that of his commentators, and the general consent of the learned in Malabar, wrote in A.D. 1608.

2. The *Carana Padhati*, by Pathumana Soma Yaji, whose grandson was (when the paper was written; it was read before the Society in 1832) alive in his seventieth year; written in 1733.

3. The *Sadratnamalah*, by Sancara Varma, younger brother of the rajah of Cadattanada, near Tellicherry (1832).

In the *Tantra Sangraha* it is stated that 104348 divided by 33215 is the circumference when the diameter is unity. This gives 3·1415926539, and is correct to the ninth decimal place inclusive. He adds the ratio of Metius, 355 to 113 [QUADRATURE], which, on the supposition of his having borrowed it from European books, found its way to India in less than half a century. Nothing so correct as his first statement existed in Europe before VIETA.

In the same work are (if we use the meaning of π now uniformly established) the following approximations:—

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots\dots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + - \dots \pm \frac{1}{2x+1} \mp \frac{x}{4x^2+1}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots \pm \frac{1}{2x-1} \mp \frac{x^2+1}{(4x^2+5)x}.$$

This last is very correct: three terms and the final correction give ·7853; six terms and the correction give ·785398. Then follows

$$\pi = \sqrt{12}.\left\{1 - \frac{1}{3.3} + \frac{1}{5.3^2} + \frac{1}{7.3^3} - \dots\right\}$$

$$\frac{\pi}{16} = \frac{1}{1^2+4.1} - \frac{1}{3^2+4.3} + \frac{1}{5^2+4.5} - \dots,$$

$$\pi = 3 + 4\left(\frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \dots\right)$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2^2-1} - \frac{1}{4^2-1} + \dots \pm \frac{1}{4x^2-1} \mp \frac{1}{8x^2-8x+6}$$

$$\frac{\pi}{8} = \frac{1}{2^2-1} + \frac{1}{6^2-1} + \frac{1}{10^2-1} + \dots$$

$$\frac{\pi}{8} = \frac{1}{2} - \left\{\frac{1}{4^2-1} + \frac{1}{8^2-1} + \frac{1}{12^2-1} + \dots\right\}$$

The *Carana Padhati*, besides some of the series already mentioned, gives

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{(2.2^2-1)^2-2^2} + \frac{1}{(2.4^2-1)^2-4^2} + \dots$$

$$\frac{\text{arc}}{\text{rad}} = \tan - \frac{\tan^3}{3} + \frac{\tan^5}{5} - \dots$$

the tangent is not named, being called sine ÷ cosine.

The *Sadratnamalah* gives some series, and also 3·14159265358979324 for an approximation to π. This last work, being written in our own day, requires no remark; but it is rather a staggering assertion that works written in 1608 and 1733 give series which not only were not known in Europe at those periods, but must have been discovered by something answering to the integral calculus. On this point we must suspend our opinion till these works are published, with such an account of them as may command belief in their being the genuine productions of the time at which they profess to have been written: but if we grant them existence at all, they are enough to show that there has been, in Malabar at least, knowledge enough to enable the possessors to avail themselves of the results of modern European mathematics. Mr. Whish asserts that the invention of these infinite series originated in Malabar, and that they are not 'even to this day known to the eastward of the range of Ghauts which divides that country from the countries of Madura, Coimbatore, and Mysore.' This by itself seems to us to favour the presumption that they were introduced from Europe at a comparatively recent period. We rather suspect

that Le Gentil, who was in the southern parts of the Deccan in 1763, and who used every means to ingratiate himself, that he might procure a knowledge of the native astronomical processes, in which, much against the will of the Brahmins, he succeeded by the aid of a Tamul Christian—may have been the communicator of several of these series, and that Catholic missionaries may have given the rest. All writers admit that the Hindus are apt to assign a greater antiquity to their methods and writings than really belongs to them, as indeed is evident enough from their astronomical periods. This desire on their part seems to have been met in Europe by a determination to deny their writings any antiquity whatever: it is forgotten that the mere desire to appear an exceedingly antient nation is itself a consequence of the feeling that there is already a considerable antiquity to boast of. The Chinese and Hindus, whose system has certainly lasted longer than that of any other countries with which we are acquainted, are precisely, of all the people on the face of the earth, those who most desire to have their knowledge of all kinds considered as more antient than it really can be shown to be.

Paper 18

Bapu Deva Shastri (1858)

Bhaskara's knowledge of the Differential Calculus

Journal of the Asiatic Society of Bengal, Vol. XXVII, no.3, 1858, pp. 213–216,

Note

This was the first paper ever to suggest an Indian origin to the concepts of calculus.

Bapu Deva Sastri

Bapudeva Sastri (1821–1900) was an Indian mathematician hailing from the Indian State of Maharashtra. Sastri taught Euclidean geometry at Government Sanskrit College, Varanasi during the period 1842–1889. Sastri translated the *Siddhānta Ṣiromani* (a treatise on Hindu mathematics written in the twelfth century by Bhaskaracharya) and published it in Benares in 1891. With Lancelot Wilkinson, Sastri wrote commentary on the *Sūrya Siddhānta*. (Wikipedia)

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Bháskará's knowledge of the Differential Calculus.—*By* BAPU DEVA SHASTRI, *Professor of Mathematics and Astronomy in the Government Sanskrit College, Benares.*

To the Editor of the Asiatic Society's Journal.

SIR,—It appears to be generally believed that the principle of the Differential Calculus was unknown to the ancient Hindu mathematicians. Allow me to correct this impression by the following statement regarding what Bháskaráchárya has written on the subject.

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Bháskarácháryа says that "the difference between the longitudes of a planet found at any time on a certain day and at the same time on the following day is called its rough motion during that interval of time; and that its *Tátkálika* motion is its exact motion."

The *Tátkálika* or instantaneous motion of a planet is the motion which it would have in a day, had its velocity at any given instant of time remained uniform. This is clear from the meaning of the term *Tátkálika* and it is plain enough to those who are acquainted with the principles of the Differential Calculus that this *Tátkálika* motion can be no other than the differential of the longitude of a planet. This *Tátkálika* motion is determined by Bháskarácháryа in the following manner.

"Suppose, x, x' = the mean longitudes of a planet on two successive days;

y, y' = the mean anomalies;

u, u' = the true longitudes and

a = eccentricity or the sine of the greatest equation of the orbit.

Then, $x'-x$ = the mean motion of the planet, $y'-y$ = the motion of the mean anomaly and $u'-u$ = the true motion of the planet."

Now according to Bháskarácháryа, the equation of the orbit on the first day $= \frac{a.\sin y}{\text{Rad}}$, and

that on the next day $= \frac{a \sin y'}{\text{Rad}}$;

$$\therefore \quad u = x \pm \frac{a.\sin y}{\text{Rad}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

and $u' = x' \pm \frac{a.\sin y'}{\text{Rad}}$;

$$\therefore \quad u' - u = x' - x \pm \frac{a(\sin y' - \sin y)}{\text{Rad}} \quad \ldots\ldots \quad (2).$$

Now, in order to know the instantaneous value of $u'-u$, it is necessary first to know the instantaneous value of the *Bhogya-khanḍa* or the difference between two successive sines given in *Tables of sines*. Thus, suppose the sines of the arcs 0, A, $2A$, $3A$, &c. are given in the *Tables of sines*, then

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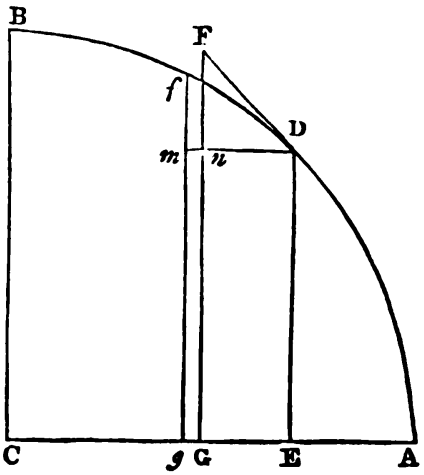
sin A—sin 0, sin $2A$—sin A, sin $3A$—sin $2A$, &c. are the *Bhogya-khaṇḍas.*

"These are not equal to each other but gradually decrease, and consequently while the increase of the arc is uniform, the increment of the sine varies"—on account of the deflection of the arc. Hence the difference between any two successive sines is not the *Tátkálika Bhogya-khaṇḍa;* but if the arc instead of being deflected be increased in the direction of the tangent then the increase which would take place in the sine is the *Tátkálika Bhogya-khaṇḍa* i. e. the instantaneous motion of the sine.



Thus, in the accompanying diagram, suppose the arc $Df = A$, then, sin Af—sin AD = $fg - DE$ = fm, the *Bhogya-khanda* of the sine DE; but this is not the *Tátkálika Bhogya-khaṇḍa* of that sine. If the arc AD instead of being deflected towards f, be increased in the direction of the tangent, so that $DF = Df = A$; then $FG - DE = Fn$, which would be the *Tátkálika Bhogya-khaṇḍa* of the sine DE i. e. the instantaneous motion of that sine."

Bháskarácháryа has determined that "the *Tátkálika Bhogya-khaṇḍa* varies as the cosine of arc, i. e. when arc = 0, its cosine equals the radius, and A = the *Tátkálika Bhogya-khaṇḍa.* And, as the arc increases, the cosine and the *Bhogya-khaṇḍa* decrease. Hence, if y be any given arc, the *Tátkálika Bhogya khaṇḍa* answering to it will be found by the following proportion.

As, R (or the cosine of an arc = 0.)

: The *Tátkálika Bhogya-khaṇḍa* (= A.)

:: Cosine y.

: *Tátkálika Bhogya-khaṇḍa* of sin y.

$$\therefore \textit{Tátkálika Bhogya-khaṇḍa} = \frac{\text{A. cos } y.}{\text{R}}."$$

The reason of the above proportion can be easily understood from the two similar triangles DCE and DFn in the above diagram.

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"After having thus determined the *Tátkálika Bhogya-khánḍa*, the instantaneous value of sin y'—sin y is found by the following proportion.

As $A : \frac{A.\cos y}{R} :: y' - y : \frac{\cos y \times (y' - y)}{R}$ (= the instantaneous value of sin y' — sin y.)

By substituting the instantaneous value of sin y'—sin y in the equation (2), the instantaneous value of $u' - u$, the true motion of the planet will be found: that is,

$$u' - u = x' - x \pm \frac{a.\cos y}{R} . \frac{y' - y}{R} \quad \ldots\ldots \quad (3)$$

This is the instantaneous motion of the planet."

This is the way in which Bháskaráchárya determined the instantaneous motion of the sun and the moon.

Equation (3) is just the differential of equation (1). As,

$$d(u) = d\left(x \pm \frac{a.\sin y}{R.}\right);$$

$$\text{or } du = dx \pm \frac{a}{R} . \frac{\cos y}{R} . dy;$$

which is similar to equation (3).

Now, the term *Tátkálika* applied by Bháskaráchárya to the velocity of a planet, and his method of determining it, correspond exactly to the differential of the longitude of a planet and the way for finding it. Hence it is plain that Bháskaráchárya was fully acquainted with the principle of the Differential Calculus. The subject, however, was only incidentally and briefly treated of by him; and his followers, not comprehending it fully, have hitherto neglected it entirely.

I have the honor to be,
Your obedient servant,
BAPU DEVA SHASTRI,

4th May, 1858.

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Paper 19

W. Spottiswoode (1860)

Note on the supposed Discovery of the Principle of the Differential Calculus by an Indian Astronomer

Journal of the Royal Asiatic Society of Great Britain & Ireland, Vol. 17, Jan. 1860, pp. 221–222.

William H. Spottiswoode

William H. Spottiswoode (1825– 1883) was an English mathematician, physicist. He was president of the Royal Society from 1878 to 1883. (Wikipedia)

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Art. VII.—*Note on the supposed Discovery of the Principle of the Differential Calculus by an Indian Astronomer.* By W. Spottiswoode, Esq. *Communicated by the Director.*

In the number of the Journal of the Asiatic Society of Bengal last received, No. III., of 1858, is a short article by Bapu Deva Shastri, Professor of Mathematics and Astronomy at the Government College of Benares, in which he has undertaken to show, that Bháskaráchárya, an astronomer who flourished at Ujjayin in the twelfth century, was fully acquainted with the principle of the Differential Calculus, one of the most important discoveries of the last century in Europe.

As this would have been a very remarkable circumstance in the history of astronomical science, it was obviously a matter of more than ordinary interest to have the accuracy of Professor Bapu Deva's statement carefully tested, and I therefore applied to our colleague, Mr. William Spottiswoode, who is well known as a mathematician, for his opinion; the answer with which he has favoured me will, I doubt not, be thought by the Society worthy of being communicated to the public through our Journal, especially as, whilst it shows that Bapu Deva's statement is not correct to its whole extent, yet it does full justice to Bháskaráchárya's penetration and science, and acknowledges that his calculations bear a very remarkable analogy to the corresponding processes in modern mathematical astronomy.

12, James Street, Buckingham Gate,
London, May 5, 1859.

My Dear Mr. Wilson,

I have read Bapu Deva Shastri's letter on Bháskaráchárya's mode of determining the instantaneous motion of a planet, with great interest, and think that we are much indebted to him for calling our attention to so important an element in the old Indian methods of calculation. It still, however, seems to me, that he has overstated the case, in saying that "Bháskaráchárya was fully acquainted with the principle of the Differential Calculus." He has undoubtedly conceived the idea of comparing the successive positions of a planet in

its path, and of regarding its motion as constant during the interval, and he may be said to have had some rudimentary notion of representing the arc of a curve by means of auxiliary straight lines. But on the other hand, in the method here given, he makes no allusion to one of the most essential features of the Differential Calculus, viz., the infinitesimal magnitude of the intervals of time and space therein employed. Nor indeed is anything specifically said about the fact that the method is an approximative one.

Nevertheless, with these reservations, it must be admitted, that the penetration shown by Bhaskara, in his analysis, is in the highest degree remarkable; that the formula which he establishes (equation 3, p. 216), and his method of establishing it, bear more than a mere resemblance—they bear a strong analogy—to the corresponding process in modern mathematical astronomy; and that the majority of scientific persons will learn with surprise the existence of such a method in the writings of so distant a period and so remote a region.

With many thanks for communicating the paper to me,

I remain, very sincerely yours,

H. H. Wilson, Esq. W. SPOTTISWOODE.

P.S. I may perhaps add, that if—

x, x' be the mean longitudes,
y, y' be the mean anomalies,
u, u' be the true anomalies

of a planet on two successive days; and a the excentricity, or sine of the greatest equation of the orbit; then $(u' - u)$, or the true motion of the planet,

$$= x' - x \; a \pm (\text{sin. } y' - \text{sin. } y).$$

And Bhaskara's method consists in showing, that the "instantaneous" value of sin. y' — sin. y (or the value which it would have if the velocity of the planet had remained uniform during the day) is $(y' - y)$ cos. y. His formula therefore becomes—

$$u' - u = x' - x \pm (y' - y) \; a \text{ cos. } y.$$

And the corresponding formula in modern analysis is—

$$du = d(x \pm a \text{ sin. } y)$$
$$= dx \pm a \text{ cos. } y \, dy.$$

W. S.

Paper 20

Charles Philip Brown (1863)

"Fraudulent Documents" in *Carnatic Chronology, the Hindu and Mohomedan Methods of Reckoning Time Explained*

Sold by Bernard Quaritch, 15, Piccadilly, London, 1863, pp. 48–49.

Charles Philip Brown

Charles Philip Brown (1798–1884) was a British official of the East India Company. He worked in what is now Andhra Pradesh, and became an important scholarly figure in Telugu language literature. In his book *Carnatic Chronology*, Brown castigated and rebuked C. M. Whish's paper on the quadrature of circle as a fraudulent document.

Source: Internet Archive

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ON SOME FRAUDULENT DOCUMENTS.

In a series of documents extending to a period of a thousand years, few have proved false, and these belong to the two last centuries.

In 1814 was printed, in Paris, "Histoire de l'Inde, ... par M. Collin de Bar, Ancient Magistrat de la cour supérieure de Pondichéry." To this are subjoined "Pièces Justificatives," No. I—XXVI. Several of these pretend to be treaties executed between French officers and the native powers, who bestow certain lands on the French. They are complete with seals, signatures, and dates. The first is between "The Company of France and Bayanor, prince de Bargaret." This is no Indian name either of a person or of a place. The others are equally absurd, and the dates will not stand enquiry.

In the Lettres Edifiantes et Curieuses, the French History of Pondichery down to 1750 is given, and in the Modern Universal History, vol. ix. p. 220, that history is recorded from French documents. In these is no mention of such treaties.

In Grose's Voyage to the East Indies, 1772, (vol. 2, page 158) it is stated that in 1754, (the commissioners sitting at Sadras, a Dutch settlement near Carangooly) the French attempted to make terms with the English, basing their claims on these documents (Le Bar. 2. 72), but the forgery was discovered and the conference was broken off. In volumes printed by French writers of more modern days I have seen no notice of these occurrences.

Bonnechose (History of France, 1856,) states that in 1747, La Bourdonnaye besieged and took Madras, which he restored for ten millions. This is also a fable.

Some papers printed in the Proceedings of the Royal Asiatic Society, by Mr. C. M. Whish, (who died in the year 1833) evince his skill in Mathematical investigation and in Sanscrit scholarship. The article *Viga Ganita*, in the Penny Cyclopædia, and pages 325, 326 of the same volume, having mentioned Mr. Whish's discoveries, with a surmise that they may be forgeries, I offer the following description of their origin.

I have reason to believe that Mr. Whish himself was the author. This belief is founded on information given me by Visvambhara Sastri, one of the tutors in the Madras College.

This learned bramhan had been intimately acquainted with Mr. Whish, and assured me in the year 1840, that the Sanscrit stanzas conveying mathematical rules were constructed under Mr. Whish's guidance in A.D. 1820, by Ayya Sastri, a professor of Sanscrit law, who, after being long employed in the College, became a Law Officer in the Court

of Sudr Udalut. Ayya Sastri was a free-thinker and derided all religions. From time to time, while framing these verses, he shewed them to my informant, who laughs at the credulity of the English in believing them ancient.

That the verses cited by Mr. Whish are apocryphal may easily be shewn. In the third volume of the Transactions, p. 509, we have a rule regarding the proportion that the diameter bears to the circumference of a circle. But this is quoted "from the Tantra Sangraham." Now this name denotes a "Summary of Magic." In the East India House, in the Madras College, and in Mr. Whish's own collection of books at the Royal Asiatic Society's Library, there are many books on the Tantras: but no one of them is named the Tantra Sangraham. Nor does this name occur in the Tantra Rajam, in the commentary on the Sarada Tilacam, or other similar treatises, which quote Tantrica authorities.

Doubtless the Tantras often mention Magic numbers, but use these in a manner as devoid of sound reason as the "Babylonic numbers" alluded to by Horace.

Were a modern professor to write mathematical rules in Greek verse, and attribute them to "Eupolis Cratinus or Aristophanes," the writers of old Comedy, the fraud would not be more daring.

In the Kala Sancalita, page 93, is a note by Mr. Hyne, who states that Mr. Whish ultimately replied that "he had reasons to believe the rules discovered to be modern: observing that not one who used the rules could demonstrate them."

Mr. Whish had a valuable collection of Sanscrit manuscripts: which was presented by his brother to the Royal Asiatic Society's Library in London. Being in the Tamil, Malayalam and Telugu character they probably will be useless in England. They should be sent to the Government Library at Madras.

Another fabrication was described to me by the same bramhan. Before the custom of burning widows was abolished, some enquiries were made in Madras regarding passages in Sanscrit law books, which were said to condemn the practice. Ayya Sastri, being required to produce some texts, readily undertook to fabricate a few laws written in Sanscrit verse, prohibiting such murders.

He took care to quote respectable authors in proof: when these passages were forwarded to Calcutta, the forgery was easily detected; but, adds my informant, an English translation was prepared and published at Madras without any mention of the detection.

The Sthala Puranas (Sanscrit legends regarding the sanctity of the idols worshipped at various pagodas), usually pretend to be extracts from the Skanda Puran, a book of vast extent, of which no complete copy has been discovered. See Wilson's learned Preface to his version of the Vishnu Puran, page xlvi. These furnish additional proof of the scorn of veracity so common among the Hindu framers of legends supporting the local superstitions.

I may adduce another forgery. In the Asiatic Journal for July 1819, (vol. viii., page 28) there are some Sanscrit lines regarding vaccination, describing the process and its effects. The writer quotes a "Sa'ctya Grandham", attributed to Dhanwantari.—Now Dhanwantari is the Indian Hippocrates: and a "Sa'ctya Grandham" "is a treatise on magic and on the worship of Demons." To attribute such a work to a physician is unreasonable.

This statement is very properly criticised in the London Christian Observer for July, 1819, page 441. The author says, "Its minuteness leads me to feel a suspicion that it is a mere forgery."

A letter written by the Abbé Dubois in the year 1805 (and printed in the Asiatic Annual Register for 1806 at the beginning of the Bengal occurrences) regarding Vaccination, is entirely silent regarding all such Sanscrit rules. Dubois was an ignorant man. The respectable chapters in his description of India are copied from Sonnerat without acknowledgement.

Paper 21

Brajendra Nath Seal (1915)

Excerpt from *The Positive Sciences of the Ancient Hindus*

Excerpt from:
Brajendranatah Seal, *The Positive Sciences of the Ancient Hindus*, Longmans Green and Co, London, 1915, pp. 75–80.

Sir Brajendra Nath Seal

Sir Brajendra Nath Seal (1864–1938) was a Bengali Indian humanist philosopher. Seal was the inaugural chair of philosophy at India's first graduate school in philosophy at the University of Calcutta. He also served as the second vice chancellor of Mysore University. (Wikipedia)

Source: Internet Archive

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Measures of Time and Space—Size of Atoms.—The Siddhánta-Śiromani gives the following measures of Time :—

30 Kshanas (क्षण) = 1 day, 30 Káshthás = 1 Kalá,
2 Ghatikás = 1 Kshana, 18 Nimeshas = 1 Káshthá,
30 Kalás = 1 Ghatiká, 30 Tatparas = 1 Nimesha,
and 100 Trutis = 1 Tatpara.

This makes a Truti of time equal to $\frac{1}{33750}$ of a second, which is nearly the measure of the Paramánu of time as given in the Vishnu Purána (*vide* Bháskara's Siddhánta-Śiromani—कालमानाध्याय).

The above measures were in use among the astronomers, but the physicists computed according to the

similar (potential) qualities of at least two component elements, it follows that a molecule of the structure EA would not manifest the energy of smell potentially contained in the Earth-atom. Hence, admitting the combination EA for a smell-less compound, the upholders of this view would suppose a molecule of the type E_2A (*i.e.* two atoms of Earth and one of Ap) to explain any bi-Bhautic compound of Earth and Ap (like the plant saps and fruit juices) which exhibits smell in addition to the peculiarities of colour and taste. *Cf.* Váchaspati's comment on Udyotakara's refutation of this view :—

अयमभिसन्धिः । पृथिव्याप्यपरमाणू तावन्नैकं द्व्यणुकमारब्धुमुहतः तयोः रूपरसस्पर्शवत्त्वेन तदारम्भसम्भवे अपि गन्धवत्त्वाभावप्रसङ्गात् । एवं पार्थिवाणु-समवेतस्य गन्धस्य एकत्वेन अनारम्भकत्वात् नापि पार्थिवपरमाणुद्वयं एकश्च पाथसीयपरमाणुः इति अणूनामारम्भकत्वे गन्धवत्त्वोपपत्तिः इति साम्प्रतं । परमाणूनां बहूनां अनारम्भकत्वात् । तथाहि त्रयः परमाणवः न कार्य्यमारभन्ते ।

Measures of Time and Space—Size of Atoms.—The Siddhánta-Śiroma*ni* gives the following measures of Time :—

30 Kshaṇas (क्षण) = 1 day, 30 Kásh*thá*s = 1 Kalá,
2 Ghatikás = 1 Kshaṇa, 18 Nimeshas = 1 Kásh*thá*,
30 Kalás = 1 Ghatiká, 30 Tatparas = 1 Nimesha,
and 100 Tru*ti*s = 1 Tatpara.

This makes a Tru*ti* of time equal to $\frac{1}{33750}$ of a second, which is nearly the measure of the Paramáṇu of time as given in the Vishṇu Puráṇa (*vide* Bháskara's Siddhánta-Śiroma*ni*—कालमानाध्याय).

The above measures were in use among the astronomers, but the physicists computed according to the

following table given both in Udayana's Kiraṇávali and Śrídhara's Nyáya-Kandali :—

30 Muhúrtas = 1 day (24 hours),
30 Kalás = 1 Muhúrta, 2 Lavas = 1 Nimesha,
30 Kásh*thás* = 1 Kalá, 2 Kshanas = 1 Lava:
18 Nimeshas = 1 Kásh*thá*,

क्षणद्वयं लवः प्रोक्तो निमेषस्तु लवद्वयम्,
अष्टादशनिमेषास्तु काष्ठा त्रिंशत्तु ताः कला।
त्रिंशत्कला मुहूर्तैः स्यात् त्रिंशद्रात्र्यहणी च ते।

(Udayana, Kiranávali.)

This makes 1 Kshana of the Nyáya-Vaiseshika equal to $\frac{2}{45}$ of a second. The Nyáya assumes that the unit of physical change (or the time occupied by any single antecedent step in a causal series before the succeeding step is ushered in) is equal to a Kshana (or $\frac{2}{45}$ of a second). The astronomers were familiar with far smaller measures of time. The astronomical Truti of time measures about the thirty-four-thousandth part of a second. This is of special value in determining the exact character of Bháskara's claim to be regarded as the precursor of Newton in the discovery of the principle of the Differential Calculus, as well as in its application to astronomical problems and computations. This claim, as I proceed to show, is absolutely established; it is indeed far stronger than Archimedes' to the conception of a rudimentary process of Integration. Bháskara, in computing the "instantaneous motion" (तात्कालिकी गति), of a planet, compares its *successive* positions, and regards its motion as constant during the interval (which of course cannot be greater than a Truti of time, though

it may be indefinitely less). This Tátkálika motion is no other than the differential of the planet's longitude, and Bápudeva Śástrí claims that both the conception of the instantaneous motion and *the method of determining it* plainly show that Bháskara was acquainted with the principle of the Differential Calculus. On the data before him, Mr. Spottiswoode remarks that Bápudeva Śástrí "overstates the case." Bháskara "makes no allusion to one of the most essential features of the Differential Calculus, viz. the infinitesimal magnitude of the intervals of time and space therein employed. Nor indeed is anything specifically said about the fact that the method is an approximate one." "With all these reservations," Mr. Spottiswoode continues, "it must be admitted that the formula he establishes and the method of establishing it bear a strong analogy to the corresponding process in modern mathematical astronomy" (viz., the determination of the differential of the planet's longitude—by no means the first step in transcendental analysis or in its application to astronomy). And Mr. Spottiswoode concludes by stating that mathematicians in Europe will be surprised to learn of the existence of such a process in the age of Bháskara (*circa* 1150 A.D.—born 1114 A.D.). Mr. Spottiswoode's second objection, that Bháskara does not specifically state that the method of the Calculus is only approximative, cannot be taken seriously. The conception of limit and the computation of errors came late in the history of the Calculi of Fluxions and Infinitesimals. For the rest, Bháskara introduces his computation expressly as a "correction" of Brahma-

gupta's rough simplification. The first objection (viz. that Bháskara makes no allusion to the infinitesimal magnitude of the intervals of space and time employed) would be more to the point if it were well founded.[1] But it is not, and Mr. Spottiswoode's error was due to the insufficiency of the data supplied to him. As a matter of fact, even Bháskara's unit, the Truti of time (or Paramánu), is exceedingly small, as the very name implies, being about one thirty-four-thousandth of a second of time. And in the passage in which Bháskara describes the process, he distinguishes between Sthúla-gati and Súkshma-gati (velocity roughly measured, and measured accurately, *i.e.* by reference to indefinitely small quantities; for Súkshma, as we have seen, has always a reference to the Anu, the indefinitely small); indeed, he expressly mentions that the Sthúla-gati takes only Sthúla-kála (finite time) into consideration, and that the determination of the Tátkálikí-gati (Súkshma-gati) must have reference to the moment (प्रतिक्षणम्), which is an indefinitely small quantity of time, being, of course, smaller than his unit, the Truti. (*Cf.* इयं किल स्थूला गतिः अथ सूक्ष्मा तात्कालिकी कथ्यते..... यदा आसन्नस्तिथ्यन्तस्तदा तात्कालिक्या गत्या तिथिसाधनं कर्त्तुं युज्यते। तथा समीपचालनच्च। अदा तु दूरतरस्तिथ्यन्तः दूरचालनं वा चन्द्रस्य तदा आद्यया स्थूलया कर्त्तुं युज्यते स्थूलकालत्वात्। यतश्चन्द्रगतिः महत्त्वात् प्रतिक्षणं समा न भवति अतस्तदर्थं अयं विशेषोऽभिहितः;—nothing can be clearer

[1] It is an error to suppose that infinitesimals are indispensable to the Calculus, but I am here speaking of the earlier Calculi, and not of the modern developments which have made the Calculus independent of infinitesimals. My claim on behalf of Bháskara is limited to the historically earlier imperfect form of the Calculus.

than this conception of "momentary" motion.—Bháskara, Siddhánta-Śiromaṇi, Gaṇitádhyáya, Gatisphutiprakaraṇa; *cf.* also प्रतिक्षणं सा न समा महत्त्वतः, *ibid.*; *cf.* also Goládhyáya, Tátkálikí-karaṇa-vásaná-prakaraṇa, where Bháskara points out that the mode of computing adopted by the Ácháryya (Brahmagupta) is a rough simplification. The idea of resolved components of motion was familiar to the astronomers (*cf.* एवं सत्याचार्य्येण लाघवार्थं इष्टघटीसम्बन्धिन्यो गतिकला अर्के प्रक्षिप्ताः। *ibid.*). I may add *en passant* that Bháskara's formula for the computation of a table of sines also implies his use of the principle of the Differential Calculus.

Paper 22

Bibhutibhushan Datta (1926)

Hindu values of π

Journal of the Asiatic Society of Bengal (New Series) Vol. XXII, 1926, pp. 25–42.

Notes

It was this paper that discussed all the series mentioned in Whish's paper for the first time after a lapse of over eighty years from the time when they were last mentioned in the 1843 article of De Morgan.

Bibhutibhushan Datta

Bibhutibhushan Datta (1888–1958) was a historian of Indian mathematics. Datta secured the master's degree in mathematics in 1914 and doctorate degree in 1920 in applied mathematics. He taught at Calcutta University where he was lecturer at University Science College, and from 1924 to 1929 he was Rhashbehari Ghosh Professor of Applied Mathematics. During the 1920s and 1930s he created a reputation as an authority on the history of Indian mathematics. He was also deeply interested in Indian philosophy and religion. In 1929 he retired from his professorship and left the university in 1933, and became a sannyasin. *History of Hindu Mathematics: A Source Book*, written by him jointly with Avadhesh Narayan Singh (1901–1954) became a standard reference in the history of Indian mathematics. In the last years of his life, as Swami Vidyaranya, he lived mainly at Pushkar (in Rajasthan).

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Hindu Values of π

By BIBHUTIBHUSAN DATTA, D.Sc.,
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EARLY VALUES—ŚULVASŪTRAS (*c.* 500 B.C.).

The problem of 'squaring the circle' was favourite with all the ancient nations of the world. In India it originated and acquired special importance in connection with the Vedic sacrifices at a very early period, before the earliest hymns of the Ṛgveda were composed (about 4500 B.C.).[1] But we do not know anything about the value which was adopted for π in those hoary ages. The earliest definite information is found in the *Śulvasūtras* (*c.* 500 B.C.). Āpastamba says[2]:

"If you wish to turn a square into a circle, stretch a cord from the centre towards one of the corners, draw it round the side and describe the circle together with the third part of the piece standing over."

Baudhāyana and Kātyāyana give the same rule in different words. This works out

$$\pi=4/\{1+\tfrac{1}{3}(\sqrt{2}-1)\}^2=3\cdot0883\ldots$$

Baudhāyana has another expression (obtained from the reverse process) of an almost equivalent value, viz.

$$\pi=4\left(\frac{7}{8}+\frac{1}{8.29}-\frac{1}{8.29.6}+\frac{1}{8.29.6.8}\right)^2,$$
$$=3\cdot0885\ldots$$

Another simpler and less accurate value has been given by all the three sūtrakāras,

$$\pi=4\left(\frac{13}{15}\right)^2=3\cdot004.$$

These values are far wide of the true one, 3·141592653... obtained in the 15th century. It will also be noted that though the last two expressions bear a striking similarity to the form obtained from the statements of the Egyptian Ahmes (*c.* 1550 B.C.),

$$\pi=4\left(1-\frac{1}{9}\right)^2=3\cdot1605\ldots,$$

the Hindu values are less satisfactory.

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[1] *Vide* Oldenberg, *Die Ṛgveda*, iii. 356.

[2] Thibaut, The Śulvasūtras, *Journ. Asiat. Soc. Beng.* (1875); Reprint, p. 26.

Period 500 A.D.—1200 A.D.

From the early centuries of the Christian era, the Hindu astronomers and mathematicians have adopted more accurate values of π. Different writers adopted different values whose order of approximation varies widely. Thus we find such expressions as $\sqrt{10}$, 22/7, 3927/1250, 21600/6876. Along with these a few writers continued to use the rough approximation 3 in those cases specially where accuracy was not of much practical consequence. We shall record here a short history of the use of each of these values without any attempt to discover the method how it was arrived at, though this aspect of the problem is not less important.

The value of $\pi = \frac{3927}{1250}$ (=3·1416).

This value is found stated for the first time in the work of Āryabhaṭa the elder (*b.* 476 A.D.). He writes[1]:

"100 plus 4, multiplied by 8, and added to 62,000: this will be the approximate value of the circumference of a circle of diameter 20,000."

That is, $\pi = 62{,}832/20{,}000$. We do not know why Āryabhaṭa did not reduce the expression to its simplest form, viz. 3927/1250, the form in which it has been adopted by the subsequent mathematicians.[2] Most probably he got it by actual measurement by a method of interpolation and retained the form in which it was first discovered. This value of π was adopted and applied by Lalla (*c.* 578 A.D.) who professed to be a disciple of Āryabhaṭa and acknowledged his authority on several occasions.[3] It was carried to Arabia by the Hindu astronomer who went to Bagdad in 772 A.D. and was adopted by Ya'qub ibn Tariq, who, along with Alfāzāri, introduced Indian Astronomy into Arabia.[4] Al-Khowarizmi (*c.*

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[1] *Āryabhaṭiya*, ed. Kern (1875), *Gaṇitapāda*, verse 10.

[2] It will be shewn later on how this unreduced form has saved Āryabhaṭa from being accused as a plagiarist.

[3] Lalla, *Śiṣyadhivṛddhida*, ed. Sudhakara Dvivedi, Benares (1886); i 1, 2; ii. 3; iv. 3, 4; xiii. 22.

There has been much difference of opinion about the time of Lalla. Sudhakara Dvivedi states the date *c.* 499 A.D. (*Gaṇaka Taraṅgiṇī*, Benares, 1892, p. 8); Sankar Balkrisna Dikshit thinks that Lalla lived about 638 A.D. (*History of Indian Astronomy*, Poona, 1896, p. 227). After a careful perusal of the above authorities, Jogeschandra Roy concludes the date of Lalla to be *c.* 578 A.D. (*Āmāder Jyotiṣī O Jyotiṣa*, Calcutta, 1903, p. 181)

[4] Al-Biruni's *India*, English trans. by Sachau (1910), Vol. I, p. 169 and Vol II, p. 67. "The diameter of the earth is 2100 *farsakh*, its circumference 6596 $\frac{9}{25}$ *farsakh*." One *farsakh*=one-half *yojana*. At other places (I. 312, 316) Al-Biruni has called them *yojanas*. Again 6596 $\frac{9}{25}$ must

825 A.D.) recorded the original form of Āryabhaṭa $\frac{\text{d}2832}{20000}$ and remarked it as being due to the Indian "astronomers."[1] In 966 A.D., the eminent scholiast Bhaṭṭotpala used it in his celebrated commentary of Varāhamihira's *Bṛhat Saṃhitā*.[2] From this writer[3] and also from Al-Biruni[4] we learn that this value of π was applied in the *Puliśa-siddhānta* known to them. It should be remarked that this was not the original *Siddhānta* of Puliśa, nor the one known to Varāhamihira or even to Brahmagupta, from either of which it differed materially. In fact, it was a recast of the same work with various changes and additions.[5] Finally we may point out that Bhāskara (*b.* 1114 A.D.) adopted the value 3927/1250 along with 22/7.[6]

Value of $\pi = \sqrt{10}$ (=3·16227 . . .)

This value of π was adopted as a close approximation (*sūkṣma*) by Brahmagupta (628 A.D.).[7] It seems, however, that he was aware of the unsatisfactoriness of this value and did, in fact, know a closer approximation though nowhere has it been expressly stated by him.[8] Still we wonder why he did not adopt Āryabhaṭa's extremely accurate value. Proceeding earlier, we find this value of π in Varāhamihira's *Pañca-siddhāntikā*,[9] in connection with the summary of the *Puliśa-siddhānta*. Hence this must have been the value adopted in the original work. But in the recasted *Puliśa-siddhānta*, as has already been stated, it was discarded in favour of Āryabhaṭa's value. In all the early *Siddhāntas*,[10] at least in their modern forms as

be a misprint for 6597 $\frac{9}{25}$ and then the value of π applied will be 3927/1250. Ya'qub got these values from his Hindu informant and they agree with Āryabhaṭa's values for the dimensions of the Earth.

[1] Cf. Rodet, Leçons de Calcul d' Āryabhaṭa, *Journal Asiatique*, seventh series xiii (1878), p. 411; Rosen, *The Algebra of Mohammed Ben Musa*, London (1831)

[2] *Bṛhat Saṃhitā* with Bhaṭṭotpala's commentary, ed. Sudhakara Dvivedi, Benares (1895), p. 53.

[3] *Ibid*, pp. 53–55.

[4] *Loc. cit.*, I. 168; II. 72.

[5] *Vide* Varāhamihira's *Pañca-siddhāntikā*, ed Thibaut and Dvivedi, Benares (1889), Introduction, p. xxvii; Sankar Balkrisna Dikshit, loc. cit., p. 163.

[6] *Līlāvatī*, ed. Sudhakara Dvivedi, Benares (1912), verse 40; *Siddhānta-śiromaṇi*, ed. Bapu Deva Sastri, *Candragrahaṇādhikāra*, verse 3 (*Vāsanābhaṣya*), p. 207.

[7] *Brāhma-sphuṭa-siddhānta*, ed. Sudhakara Dvivedi, Benares (1902), xxi. 15, p. 363.

[8] Cf. *ibid*, xxi. 16.

[9] *Loc. cit.*, iv. 1. This value seems to be common with all the Siddhāntas.

[10] *Vide Jyotiṣa-siddhānta-saṃgraha*, ed. Vindeswariprasad Dvivedi, Benares (1912): *Soma-siddhānta*, i. 50; *Brahma-siddhānta* (of the *Śākalya-saṃhitā*), i. 61; *Pitāmaha-siddhānta* (of the *Viṣnudharmottara Purāṇa*), p 3; *Vṛddha Vasiṣṭha-siddhānta*, i. 52; xii. 38.

well as in the *Sūrya-siddhānta*[1] (the old and the modern), are found application of the value $\sqrt{10}$. Thus in India $\sqrt{10}$ seems to be oldest closer approximation to the value of π and it also remained the most widely employed value up to the 11th century. It was adopted by Śrīdhara (*c.* 750),[2] Mahāvīra (*c.* 850 A.D.),[3] and Āryabhaṭa the younger (*c.* 950 A.D.).[4] Later on, however, it was abandoned.

The origin of the value $\sqrt{10}$—an explanation.

It seems very surprising that the value $\sqrt{10}$ of π continued to be popularly used in India up to the 11th century and that even by those mathematicians who were aware of closer approximations. Specially it is so in the case of Brahmagupta who was otherwise very astute and accurate, who was held in high respect and whose authority, more than that of any one else, was freely acknowledged by the celebrated Bhāskara, the greatest of Indian astronomers and mathematicians. The earliest explanation by an Indian author comes from this latter writer. Bhāskara states that $\sqrt{10}$ was adopted as the value of π because of its simple form (*sukhārtham*). At the same time he has warned his readers not to conclude that Brahmagupta and Śrīdhara, who adopted this value, were ignorant of the accurate expression of Āryabhaṭa (*vide infra*, p. 15). In the opinion of the scholiasts Nṛsiṃha (*c.* 1600 A.D.) and Raṅganatha (*c.* 1625 A.D.), $\sqrt{10}$ began to be employed for π, because 10 is the nearest whole number to the square of its actual value and from which it differs by a small quantity only[5] This seems to be the most probable explanation and it has been endorsed as such in modern times by Burgess and Whitney. These writers say[6]: "It is to be observed with regard to this latter ratio ($1 : \sqrt{10}$), that it could not possibly be the direct result of any actual process adopted for ascertaining the value of the diameter from that of the circumference or the contrary. It was probably fixed upon by the Hindus because it looked and sounded well, and was at the same time a sufficiently near approximation to the truth to be applied in cases where exactness was neither attainable by their methods, nor of much practical consequence; as in fixing the dimensions of the Earth,

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[1] *Sūrya-siddhānta*, trans. with notes by Burgess and Whitney, i. 59, p. 56.

[2] Śrīdhara, *Triśatikā*, ed. Sudhakara Dvivedi, Benares (1899), p. 34.

[3] Mahāvīra, *Gaṇita-sāra-saṃgraha*, trans. by Rangachārya, Madras (1908), vii. 60.

[4] *Mahā-siddhānta*, ed. Sudhakara Dvivedi, Benares (1910), xv, 88.

[5] $(3927/1250)^2 = 9{\cdot}86965 = 10$ nearly;
$(22/7)^2 = 9{\cdot}87755 = 10$ nearly.

[6] *Sūrya-siddhānta, loc. cit.*, p. 57.

Changing the sexagesimal notation to our decimal notation, the value of the radius will be $795\frac{409}{900}$, so that the diameter will be $1590\frac{818}{900}$. Evidently the value of π employed in either instances is 22/7. It is noteworthy that though 22/7 was commonly used for π in Greece from the time of Archimedes (*c.* 225 B.C.), it is found recorded in an Indian work of the 10th century, or at best it may have been recorded in a work, not earlier than, of the eighth century.[1] This is an important fact against the supposed probability of a Greek influence.[2]

Value of π=21600/6876 (=3·14136...): Its origin.

There is another Hindu value of π which deserves more than an ordinary notice. It is this π=21,600/6,876=3·14136... This value has been stated only by the younger Āryabhaṭa,[3] and by none else. It will be noticed that this value forms a very close approximation to Āryabhaṭa's value. It has, in fact, been derived from this latter value. For near about the same time, Bhaṭṭotpala (966 A D)[4] took $3437\frac{967}{1307}$ as the radius of a circle of circumference 21,600. The younger Āryabhaṭa simply substituted the nearest whole number for the fraction. Thus 21,600/6,876 is another expression derived from 3927/1250, with

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[1] The information as regards the knowledge and use of 22/7 as the value of π in India in the 8th century is derived mainly from the Arabic source. It has been stated by Al-Biruni that according to "the astronomical hand-book *Al-arkand*," the diameter of the Earth is 1050 *yojanas* and its circumference 3300 (I. 312, 315). This book has been stated by him to be the Arabic version of Brahmagupta's *Khaṇḍa-khādyaka* (II. 7). Again we read: "Brahmagupta uses 4800 as the number of *yojanas* of the earth's circumference in his canon *Khaṇḍa-khādyaka*, but in the amended edition he uses, instead of this, the *corrected* circumference agreeing with Puliśa [$5026\frac{14}{25}$]" (I. 312). These statements are contradictory. Two explanations are possible. Either (i) Al-Biruni has made some confusion between *Al-arkand* and *Khaṇḍa-khādyaka*, or (ii) the former work contains interpolations besides a literal translation of the latter. In support of the first supposition it may be stated that, from phonetic consideration Sachau "prefers to suppose the Sanskrit original of *Arkand*, a word like *Ārya-khaṇḍa*" (II. 339; cf. II. 304, 313). In the extant *Khaṇḍa-khādyaka* (ed. Pandit Babua Misra, Calcutta University Press, 1925, p. 26) we find the circumference of the Earth as 4800 *yojanas*. In any case, if the substance of the first statement of Al-Biruni can be traced to the Hindu source, it may be said that 22/7 was known as the value of π in India in the 8th century. The value used might have been also 3927/1295 (*vide infra*, p. 12 *sq.*).

[2] Cf. Rodet, Leçons de Calcul d' Āryabhaṭa, *loc. cit.*, p. 411.

[3] *Mahā-siddhānta*, xvi. 36.

[4] *Loc. cit.*, p. 53.

a slight change. Certainly the new form was introduced not at random. It came as a corrollary to another scientific undertaking. We shall state and explain that first. Almost all the astronomical books of the Hindus contain a table of sines. Beginning with Āryabhaṭa the elder, leaving alone Brahmagupta, it became the common Indian fashion to calculate the sines for the radius 3438. Before him the fashion seems to have been to take the radius as 120. Thus in the summary of the *Puliśa-siddhānta* as found in the *Pañca-siddhāntikā*,[1] there is a table of sines calculated for the latter value of the radius. This seems further to have been common to the other two scientific Siddhāntas as well, namely, the *Romaka-siddhānta* and the *Sūrya-siddhānta*. For, Varāhamihira had recourse to it all throughout the book wherever it was necessary to use the sines.[2] The elder Āryabhaṭa's departure from the old custom and his innovation to take 3438 for the radius were dictated by a real scientific spirit which is also marked in his other works. And this innovation has "a distinct and important advance on the method of the older siddhāntas."[3]

It was appreciated and adopted by Lalla, the younger Āryabhaṭa and Bhāskara. We also learn it from Al-Biruni, that it was followed by Balabhadra (*c.* 700 A.D.), in the recasted *Puliśa-siddhānta* and in Vijayanandin's *Karaṇa-tilaka* (*ca.* 966 A.D.).[4] In short, Āryabhaṭa's innovation was appreciated and adopted by most of the writers that followed him. Let us now enquire what suggested this improvement to Āryabhaṭa. According to the usual method of measuring arcs, the circumference of a circle contains 21600 minutes. So it is natural to take that number to represent the circumference. Having done this, the most natural and scientific value, of course to the nearest whole number, for the radius shall be 3438. To take any other value will be unscientific and artificial. With any other value we shall have not only to assume different units for the circumference and the radius but also shall be liable to adopt an inaccurate value for π. Having obtained a highly accurate expression for the ratio of the circumference to the diameter of a circle, it naturally suggested to Āryabhaṭa the elder to introduce an innovation into the table of sines which will be in keeping with it.[5] We fail to understand why

[1] *Loc. cit.*, iv. 1–11.

[2] For example see vi, 5; vii. 1, 6; viii. 12–14; ix. 13–15 and xviii. 13, 14. Also compare, Introduction, p. xxxvi.

[3] Burgess, "The sines of arcs in the Pañca-siddhāntikā," *Ind. Ant.*, 20 (1891), p. 228.

[4] Al-Biruni's *India*, I. 156, 275; II. 205. Al-Biruni says: "the *sinus totus*, according to Puliśa and Āryabhaṭa, has the relation of the diameter to the circle of 360 degrees" (I. 275).

[5] Kaye suggests that Ptolemy's value of π was used in reducing the table of sines into the new form. That must be wrong. Rodet is of same opinion as myself.

so great a mathematician as Brahmagupta failed to appreciate the scientific truth underlying this innovation and chose 3270 for the radius.[1] To keep similarity, most probably, it occurred to the younger Āryabhaṭa to choose a new form to replace the elder Āryabhaṭa's expression for π.

Dimensions of the Earth and value of π deduced therefrom: Charge of inconsistency.

There is another set of values of π which are ordinarily deduced from the dimensions of the earth as recorded in the Indian astronomical books. It will be noticed (Table I), as also pointed out and commented upon by Bhāskara,[2] that different authors have given different values. Such has also been the case with the ancient Greek investigators. But we are here more concerned with the ratio of the circumference of the earth to its diameter than with their actual values.

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[1] Brahmagupta has offered an explanation himself (*vide Brāhma-sphuṭa-siddhānta*, xxi. 16). But it is vague and at the same time unsatisfactory and unconvincing. It may be noted that the form given by him to the table of sines is not even consistent with the value of π adopted by him. For $\frac{21600}{6540}=3\cdot3027\ldots$ whereas $\sqrt{10}=3\cdot16227\ldots$

We know that Brahmagupta's form has not been adopted by any other Indian mathematicians who followed him closely in many other respects.

[2] *Siddhānta-śiromaṇi, Grahagaṇita, Madhyamādhikāra, Bhūparidhi*, sl. 1 (*Vāsanābhāṣya*). Bhāskara attempts to explain the apparent differences amongst the various investigators as regards the measurements of the dimensions of the same Earth as being due to difference in the choice of units.

Table I.

Authority.	Circumference of the Earth in *yojanas*.	Diameter of the Earth.	$\pi = \frac{\text{Circumference}}{\text{Diameter}}$.
Varāhamihira [1]	3200	$1018\frac{6}{10}$	3·14156...
Lalla [2]	3300	1050	22/7
Brahmagupta [3]	5000	1581	3·16255...
Mahā-siddhānta [4]	6625	2109 nearly	3·14129...
Puliśa-siddhānta [5] (recasted).	$5026\frac{14}{15}\left(?\ 5026\frac{14}{25}\right)$	1600	3·14183...
Bhaṭṭotpala [6]	5027	1600	3·14187...
Bhāskara [7]	4927	1581 $1581\frac{1}{24}$	3·14168... 3·1415994...

It appears from the last column that the values of π calculated from the dimensions of the earth are different from those which have been expressly stated by the respective investigators. Indeed this divergence between the values given and used appears not a little surprising. And for this the Indian authors have been charged with inconsistency by Burgess and Whitney,[8] Kaye [9] and Smith.[10]

Inconsistency more apparent than real.

The Indians have been very unjustly condemned in this matter by the foreign scholars. For on closer examination it becomes evident that the inconsistency is more apparent than real.

[1] *Pañca-siddhāntikā*, xiii. 15 *sq*. The value of the circumference was common to other Siddhāntas (*vide* iii. 14, ix. 10). I fail to trace the value for the diameter in Thibaut and Dvivedi's edition; but it has been recorded by Sankar Balkrisna Dikshit (*History*, p. 320). He is reliable in matters of Indian Astronomy.

[2] *Śiṣyadhīvṛddhida*, i. 56.

[3] *Brāhma-sphuṭa-siddhānta*, i. 36; xxi. 32. [4] *Loc. cit.*, i. 56; xvi. 35.

[5] Al-Biruni's *India*, I. 312; II. 67. $5026\frac{14}{15}$ must be a misprint for $5026\frac{14}{25}$ (*vide infra*). In the *Puliśa-siddhānta* known to Varāhamihira, the circumference of the earth is given as 3,200 *yojanas*.

[6] *Loc. cit.*, p. 55.

[7] *Siddhānta-śiromaṇi*, *Golādhyāya Bhūvanakośa*, verse 52; *Grahaṇa-vāsanā*, verse 16 (*Vāsanābhāṣya*); *Madhyamādhikāra*, *Bhūparidhi*, verse 1. In the opinion of Bhāskara $1581\frac{1}{24}$ is the most accurate value of the radius of the earth. But ordinarily the fraction has been neglected.

[8] *Sūrya-siddhānta*, loc. cit., p. 39.

[9] Kaye, *Indian Math.*, p. 32; *Journ. Asiat. Soc. Beng.*, VIII (1908), pp. 122, 3.

[10] Smith, *History of Mathematics*, Boston, Vol. II (1925), p. 308.

On the other hand the dimensions of the earth supply the much needed concrete illustrations of practical application of the abstract values adopted for π (Table II). It is to be observed that the usual method of getting at these data is to determine, first of all, the circumference of the earth by direct triangulation. In principle the plan followed in this latter operation was essentially the same as the one followed to-day, whether in India or elsewhere.. The amplitude and the length of the arc of a meridian is determined first and from these data the circumference of the earth is computed. Having thus determined the circumference, the diameter of the earth is then calculated with the help of known values of π. There is no other direct method. Hence in every case of dimensions of the earth, the value for the circumference should be accepted as more correctly representing the exact opinion of the investigator.

Table II.

Authority.	Circumference of the earth (determined).	Value of π applied.	Diameter of the earth (calculated).	Diameter (adopted).
Varāhamihira ..	3200	3927/1250	$1018\frac{6}{10\cdot14}$	$1018\frac{6}{10}$
Lalla.. ..	3300	22/7 3927/1250	1050 1050·42...	1050
Brahmagupta ..	5000	$\sqrt{10}$	1581·12	1581
Mahā-siddhānta..	6625	21600/6876 3927/1250	2109—1/24 2109—793/3927	2109[1]
Bhāskara ..	4967	3927/1250	$1581\frac{1}{24\cdot09}$	$1581\frac{1}{24}$ or 1581

In the case of Bhaṭṭotpala and the recasted *Puliśa-siddhānta* there are reasons to believe that the reverse, though unnatural, process has been followed. They adopted the value for the radius of the earth from a foregoing investigator and then deduced the circumference with help of the value of π adopted by them. Thus

$$1600\times\frac{3925}{1250}=5026\frac{14}{25}=5027 \text{ nearly}$$

or $$5027\times\frac{1250}{3925}=1600\cdot24\ldots=1600 \text{ nearly.}$$

[1] Cf. *Mahā-siddhānta*, xvi. 36 :
"A little less than 2109 is the diameter (of the Earth)."

Therefore in every case the calculated value forms a very close approximation to the adopted and recorded value. Hence it is proved that the charge of inconsistency against Indian astronomers and mathematicians is unfounded.

It should also be noted in fairness that the case of Lalla remains a little doubtful. For with him it is not easy to decide once for all whether the value 22/7 or 3927/1250 of π was applied in calculating the diameter of the earth from its circumference determined independently. With the one there is an exact coincidence between the calculated and adopted values of the diameter, and with the other a very close approximation. Remembering the degree of accuracy that was usually observed in those ages, especially as regards such matters, both may be asserted to be probable with almost equal force. But as Lalla has used the value 3927/1250 in calculating the dimensions of the planetary orbits,[1] and as he has not mentioned or applied anywhere else 22/7 as the value of π, one will be safe to conclude that Lalla has taken $\pi = 3927/1250$ in calculating the diameter of the Earth.

Dimensions of the Earth according to Āryabhaṭa.

There are various wrong notions as regards the dimensions of the earth as stated by Āryabhaṭa the elder and the value of π used by him in their calculation. Al-Biruni says: "Brahmagupta relates with regard to Āryabhaṭa, criticising him, that he fixed the circumference as 3393; that he fixed the diameter in one place as 1080, in another place as 1050."[2] He explains the second statement to be a "blunder in the text, not of the author," and further points out that the value of π employed by Āryabhaṭa was $\frac{377}{120}$. Colebrooke has stated, again on the authority of Brahmagupta and his scholiast, 3393 to be the circumference of the earth's wind (spiritus vector) whose diameter is 1080 whereas 1050 is the diameter of the earth. He further presumed the value of π taught by Āryabhaṭa to be 22/7, so that the circumference will be 3300. Here Colebrooke has quoted in his support a scholiast of Bhāskara, Ganeśa (1545 A.D.)[3] These statements are not found in the printed edition of Brahmagupta's works. There is also another consideration which throws considerable doubt on what Colebrooke presumes[4] Evidently the values of π employed in this connec-

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1 *Śiṣyadhīvṛddhida*, iv. 3, p. 28. 2 Al-Biruni's *India*, I. 168.

3 Colebrooke's *Algebra*, etc., p. xxxviii *sq.*

4 Sudhakara Dvivedi has stated in the preface that his edition is collated from three MSS., and he has also consulted Colebrooke's MS. of Prithudaka-swami's commentary.

tion are $\frac{377}{120}$ and $\frac{22}{7}$. There cannot be any good reason why one should employ two values of π in two contiguous parts of the same instance. The whole controversy will, however, be set at rest by the fact that according to the *Āryabhaṭīya*,[1] the diameter of the Earth is 1050 *yojanas* and the circumference of its wind 3375 *yojanas*! Parameśwara (*c.* 1430 A.D.), a commentator of the elder Āryabhaṭa, has quoted him as recording 3299 as the circumference of the Earth.[2] Evidently the value of π applied is what has been stated by Āryabhaṭa in his *Gaṇita*, viz., 62,832/20,000.[3] This will belie the statement of Kaye, "it is rather extraordinary that Āryabhaṭa himself never utilised this value."[4] Kaye has wrongly attributed the value 3 to Āryabhaṭa.

So-called Greek influence in Āryabhaṭa's value of π.

Being struck at the high degree of accuracy in Āryabhaṭa's value of π, many foreign scholars have suspected, more or less, the probability of certain Greek influence in it. The tenor of their minds has been put in a nutshell by Smith: "In case the value 3·1416 is due to either of the Āryabhaṭas, it may have been obtained from the Alexandrian scholars by whom it was then known and whose works may well have reached India, or it may have been found independently."[5] Besides an expression of what may be called their pious wish, a very few of these writers have thought it proper to examine the whole subject closely and dispassionately. And strangely enough even those few have suffered from ignorance or other shortcomings. It has been remarked by Rodet that "the choice of two myriads for the diameter is an argument strongly in favour of the Greek origin of Āryabhaṭa's expression for π, for the Greeks alone of all people made myriad the numerical unit of second order."[6] Here he has been followed

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[1] *Daśagītikā*, verses 5, 9.

[2] *Vide Goladīpikā*, ed. by T. Ganapati Sastri, *Trivandrum Sanskrit Series* (1916), verse 30, p. 4.

[3] Since

$$3375 \times \frac{1250}{3927} = 1074{\cdot}29\ldots = 1074 \text{ nearly},$$

so the depth of the atmosphere will be nearly 12 *yojanas*, which again agrees with the value recorded by Lalla, the younger Āryabhaṭa, Bhāskara and Parameśwara. Indeed, all Indian astronomers agree as regards the depth of the atmosphere. This is an additional proof against the presumptions of Colebrooke and Al-Biruni. Cf. Lalla's *Śiṣyadhivṛddhida*, xxi. 2 (*Grahabhramasaṃsthādhyāya*).

[4] Kaye, *Indian Math.*, p. 12; cf. also Kaye, "Āryabhaṭa." *Journ. Asiat. Soc. Beng.*, VIII (1908), p. 122.

[5] Smith, *loc. cit.*, II. 308-309.

[6] Rodet, *loc. cit.*, p. 411.

and quoted by Heath, who is almost sure of Āryabhaṭa's indebtedness to an unknown Greek writer.[1] Probably it is not known to Heath, nor to Rodet, that *ayuta*, which is the word actually used by Āryabhaṭa, occurs many times in the Vedic as well as in the Post-Vedic literatures.[2] In the Epics, it has been used times without number and has sometimes been given especial significance.[3] Thus it was in circulation in India long before the Greeks invented their myriad, or even before the Greek civilisation was born. It is not easy to understand then how one can scent foreign influence on the score of the use of such a word as *ayuta*. From other considerations, however, Rodet came to the conclusion—Heath omitted to refer him here—that Āryabhaṭa was at least ignorant of the works of Archimedes and his followers.[4] The works of these latter writers loomed so largely in the writings of the later Greek investigators that the question of the probable Greek influence in Āryabhaṭa's value of π, or in his mathematical writings, becomes highly doubtful. Before Archimedes, the Greek value of π was not satisfactory. From whom then did Āryabhaṭa receive, if so at all, his inspiration? Kaye has struck a novel line of argument. He thinks that as it has been stated by Al-Biruni that a value of π equivalent to Āryabhaṭa's value, has been employed in the *Puliśa-siddhānta*, and that as it has been stated by that distinguished traveller and also supported by some modern Indologists that that book had been transmitted from a Greek source, Āryabhaṭa must have got his expression for π also from there.[5] This surmise is scarcely admissible, for it is wholly vitiated by wrong premises. As has been already stated, the *Puliśa-siddhānta* which was known to Al-Biruni, was different from the original *Siddhānta* of Puliśa which was first transmitted, if it was so at all, from Greece.[6] Further we have it from Varāhamihira, who was anterior to Al-Biruni by five centuries, that in the *Puliśa-siddhānta* known to him, the value of π was put at $\sqrt{10}$ (*vide supra*). We have so far considered the evidence derived from the Indian sources. Looking into the Greek sources we do not find any Greek expression for π which was as accurate in value

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1 Heath, *History of Greek Mathematics*, Oxford (1921), Vol. I, p. 234.

2 *Vide* Macdonald and Keith, *Vedic Index*, Vol. I, p. 343.

3 Compare such words as *ayutādhyāpaka*, *ayutanāyin*, *ayutaśas*, etc.; even proper names with *ayuta* are found in the Epics. Cf. *St. Petersberg Sanskrit Dictionary*.

4 Rodet, *loc. cit.*, pp. 409, 411.

5 Kaye, Āryabhaṭa, *Journ. Asiat. Soc. Beng*, VIII (1908), p. 126.

6 Compare the following candid statement of Kern: "On the other hand, that Puliśa was a Greek, I do not doubt for a moment, notwithstanding that the Puliśa-siddhānta, judging from quotations and rather numerous ones, is so thoroughly Hinduised that few or no traces of its Greek origin are left" (*Bṛhat Saṃhitā*, Pref., p. 49 *sq.*).

or as alike in form as the one recorded by Āryabhaṭa. The only Greek value which comes very near to, but is not exactly equal to, Āryabhaṭa's value is that of Ptolemy. This writer expressed it in sexagesimal notation and gave $\pi = 3° 8' 30''$. When expressed as a proper fraction it works out as 377/120 which is very different from Āryabhaṭa's form 62,832/20,000. This aspect of the question is not negligible. It cannot be said that Āryabhaṭa changed the form arbitrarily and intentionally in order to suppress plagiarism, for they are not equal in value, and what is of utmost importance, Āryabhaṭa's form is not directly deducible from that of Ptolemy.[1] Moreover it has been pointed out by Whitney, a confirmed believer and ardent exponent of Greek, or probably Chaldean and Babylonian, influence in Indian Astronomy, that whatever might have transmitted from that side into India, must have come before the advent of Ptolemy.[2] We have already stated Rodet's conclusion that Āryabhaṭa was ignorant of the Alexandrian School of Mathematics. On the other hand there are others, like Burgess, who believed that Indian Astronomy and Mathematics developed independently.[3] The whole subject has been discussed at length by Sankar Balkrisna Dikshit.[4] In any case this much at least can be asserted safely that there is nothing to prove that Āryabhaṭa got inspiration from any foreign source, in establishing his value of π.

Date of Āryabhaṭa's value of π.

According to the statements made therein, the *Āryabhaṭīya* was composed in the year 499 A.D, so anything recorded in that book must bear at least that, if not an earlier date. But Kaye has made one covert attempt in the shape of creating doubt about the authenticity and date of the *Gaṇita* section of this work, wherein alone is found the rule for the value of π.[5] During recent years no one has written more copiously on the various topics of Indian mathematics

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[1] 377/120=3·141666... and 62,832/20,000=3·1416. Thus expressed in decimals, the difference between the two expressions appears negligible. But retaining the fractional form

$$\frac{377}{120} = \frac{3927{\cdot}08}{1250} = \frac{62{,}833{\cdot}33\ldots}{20{,}000}.$$

Within the order of approximation though the middle form can be replaced by the reduced value of Āryabhaṭa's form as adopted by Lalla and others, the original unreduced form of Āryabhaṭa cannot replace the last term, for the difference is too great.

[2] *Sūrya-siddhānta*, p. 330 *sq.*; cf. p 74.

[3] *Ibid.*

[4] *Loc. cit.*, p. 492 *sq.*

[5] Kaye, Āryabhaṭa, *Journ. Asiat. Soc. Beng.* (1908), p. 117; *Indian Math.*, p. 11; "The two Āryabhaṭas," *Bibl. Math.*, xiii (3) (1910), p. 203.

than Kaye. He has shown more capacity for collecting facts and marshalling them to his own end. At the same time he has excelled all the previous writers on the subject in vilifying it by faulty translations, by twisting the proper significance, by suppression of contexts and by condemning other workers and their followers, if they happen to hold opinions different from his. Hence anything falling from such a man should not be lightly brushed aside, especially as his opinions are slowly gaining ground amongst a certain section of people who are innocent of Sanskrit and who are not, nor can be expected to be, directly in touch with the fountain heads of Indian Mathematics. Kaye's surmise about the date and authenticity of the *Gaṇita* has been critically examined, and found totally false in another paper.[1] Certain points of that paper require re-statement and also further amplification as they have a direct bearing on the subject under discussion. Kaye has stated: "Another noteworthy rule given by Āryabhaṭa is the one which contains an extremely accurate value of the ratio of the circumference of a circle to the diameter, viz.

$$\pi = 3\frac{177}{1250} \ (=3{\cdot}1416) ;$$

but it is rather extraordinary that Āryabhaṭa himself never utilised this value, that it was not used by any other Indian mathematicians before the 12th century and that no Indian writer quotes Āryabhaṭa as recording this value."[2] Or again, "the Indians record an extremely accurate value at a very early date but seldom or never actually use it."[3] These statements are utterly wrong. For we have already seen how Āryabhaṭa has utilised his important discovery about the value of π in putting the table of sines in a rational and scientific form, and how he has been followed in this respect by all the *Siddhānta* writers, excepting Brahmagupta. We have also learned that Āryabhaṭa applied his value for π in calculating the dimensions of the Earth and its atmosphere, and further that it was adopted and utilised by Varāhamihira, Lalla, the younger Āryabhaṭa, Bhaṭṭotpala and Bhāskara. This last-named writer has remarked: "Thus it has been admitted by Āryabhaṭa and others that for the diameter 20,000, the circumference will be 62,832. What again has been admitted by Śrīdharāchārya, Brahmagupta and others that the square root of ten times the square of the diameter is the circumference, is gross. That because it is simple and facile,

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[1] *Vide* the author's paper, "Āryabhaṭa, the author of the *Gaṇita*," *Bull. Cal. Math. Soc*, XVIII (1927), No. 1, pp. 5-18.

[2] Kaye, *Indian Math.*, pp. 12-13.

[3] *Ibid*, p. 32.

not because they do not know (the other value)."[1] Bhāskara has nowhere quoted his authority in making such an important assertion that Brahmagupta and Śrīdhara were aware of Āryabhaṭa's value of π. We, however, learn it from Al-Biruni that Brahmagupta applied that value in his *Uttara-khaṇḍa-khādyaka*—an emended edition of his first book *Khaṇḍa-khādyaka*—in which it has been stated that the diameter of the Earth is 1600 *yojanas* and the circumference $5026\frac{14}{25}$.[2] Apparently the value of π employed is $\frac{3927}{1250}$. Thus we learn it undoubtedly that $\frac{62,832}{20,000}$ was adopted as the value of π as early as 499 A.D.

Later Hindu values—Closer approximations.

In later years the Hindus stated more accurate values of π some of which are correct up to the 8th or 10th place of decimals. In the *Tantra-samuccaya* (1426 A.D.),[3] we find applications of the values $\pi=\frac{22}{7}, \frac{355}{113}$. The second one deserves more than a passing notice. It is really the Chinese value, and forms a slightly closer approximation than the Hindu value, $\pi=\frac{62,832}{20,000}$. It was discovered in China as the "accurate" value of π in the 5th century A.D. by Tsu Ch'ung-chih. Subsequent Chinese mathematicians failed to appreciate its true significance though they adopted the other admittedly "inaccurate" value stated by Tsu, which is also the Greek value, viz. $\pi=\frac{22}{7}$. So it was gradually overlooked and was practically forgotten for several centuries until it was rediscovered and brought into prominence by Chang Yu-chin about 1300 A.D.[4] In Europe, it was discovered by Adriaen Anthoniszoon (*c.* 1600 A.D.) and his son Adriaen Metius (1571–1635 A.D.).[5] Another noteworthy

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[1] *Siddhānta-śiromaṇi, Golādhyāya, Bhubanakośa,* verse 52 (*Vāsanā-bhāṣya*).

[2] Cf. Al-Biruni's *India,* I. 156, 312; II. 67.

[3] Nārāyan, *Tantra-samuccaya,* ed. by T. Ganapati Sastri, *Trivandrum Sanskrit Series* (1919); Paṭala 2, sls. 65, 67-68; Paṭala 12, sl. 27. The author has misquoted the rule of the *Sulva-sūtra*

[4] Mikami, *The Development of Mathematics in China and Japan,* Leipzig (1913), p. 135; also compare p. 52.

[5] Anthoniszoon first obtained the approximations $\frac{333}{106}<\pi<\frac{377}{120}$ by the method of Archimedes; he then took the mean of the numerators and denominators and thus obtained the value $\pi=\frac{355}{113}$.

point is that the *Tantra-samuccaya* is not a work on mathematics but "a collection of subjects dealing with the rituals of consecration, daily worship, festivals and other kindred matters of the deities in temples which are elaborately explained in the ancient original Tantric literature."[1] So the credit of the discovery of the value $\pi=\frac{355}{113}$ in India does not belong to its author Nārāyan, who seems to be more of a compiler than an original investigator, but must go to an earlier mathematician, whose name we do not know at present. Hobson has remarked: "This latter value $\left(\frac{355}{113}\right)$ was not obtained either by the Greeks or the Hindoos, and was only rediscovered in Europe more than a thousand years later."[2] A similar remark has also been made by Mikami.[3] Evidently they are wrong so far as the Hindus are concerned.

Tantra-saṃgraha (1608 A.D.),

$$\pi=\frac{355}{113},\ \frac{104348}{33215}\ (=3{\cdot}14159265391\ldots)$$

It has been remarked by the author that the first value is accurate but the second is still more accurate. It has been pointed out by Mikami that these fractional values of π can be obtained as successive approximations from the value $\pi=3{\cdot}1415926535\ldots$ up to the 25th correct figure.[4]

Karaṇa-paddhati (1733 A.D.), $\pi=3{\cdot}1415926536$.

Sadratna-mālā (1832 A.D.), $\pi=3{\cdot}14159265358979324$.

Values in series.

The Hindus also obtained some interesting series as values of π. For these, in fact, for all the results taken from *Tantra-samgraha*, *Karaṇa-paddhati* and *Sadratna-mālā*, I am indebted to Whish's paper "*On the Hindu Quadrature of the Circle.*"[5] These books are still in MSS. and I have had no access to them.

Tantra-saṃgraha gives the series

$$\pi=4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ \ldots\ \pm\frac{p/2}{p^2+1}\right)$$

where p is the last odd divisor diminished by unity. This series converges slowly, so the author found it necessary to correct the last quotient thus

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1 *Tantra-samuccaya*, Preface.
2 Hobson, *Squaring the Circle*, Cambridge (1913), p. 24.
3 Mikami, *loc. cit.*, p. 50. 4 Mikami, *loc cit.*, p. 202.
5 *Trans. Roy. Asiat. Soc.* III (1835), p. 509.

$$\pi=4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ \ldots\ \pm\frac{q^2/4+1}{(q^2+4+1)q/2}\right)$$

where q is the last odd divisor increased by unity.

This series without the last term, in any form, is found also in the *Karaṇa-paddhati*. In Europe, the series—without any of the last terms—was given by Leibniz (1673 A.D.) and De Lagny (1682 A.D.). It is a special case of Gregory's (1671) series

$$\pi=\sqrt{12}\,.\,(1-\frac{1}{3\,.\,3}+\frac{1}{5\,.\,3^2}-\frac{1}{7\,.\,3^3}+\frac{1}{9\,.\,3^4}-\frac{1}{11\,.\,3^5}+\ \ldots\ ).$$

This series was discovered in Europe by Abraham Sharp (*c.* 1717) who used it in calculating the value of π to 72 decimal places

$$\pi=3+4\,.\left(1-\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\frac{1}{9^3-9}+\ \ldots\right).$$

This series is also given in the *Karaṇa-paddhati*

$$\pi=16\,.\left(\frac{1}{1^5+4\,.\,1}-\frac{1}{3^5+4\,.\,3}+\frac{1}{5^5+4\,.\,5}-\frac{1}{7^5+4\,.\,7}+\ \ldots\right).$$

$$\pi=2+4\,.\left(\frac{1}{2^2-1}-\frac{1}{4^2-1}+\frac{1}{6^2-1}-\frac{1}{8^2-1}+\ \ldots+\frac{1}{(p-1)^2+4}\right),$$

where p=last even number squared in the series.

$$\pi=8\left(\frac{1}{2^2-1}+\frac{1}{6^2-1}+\frac{1}{10^2-1}+\frac{1}{14^2-1}+\ \ldots\right)$$

$$\pi=4-8\left(\frac{1}{4^2-1}+\frac{1}{8^2-1}+\frac{1}{12^2-1}+\ \ldots\right).$$

It is evident that these two series when combined will give the previous series.

The *Karaṇa-paddhati* gives, in addition to those already noted, another interesting series, viz.

$$\pi=3+6\,.\left\{\frac{1}{(2.2^2-1)^2-2^2}+\frac{1}{(2.4^2-1)^2-4^2}+\frac{1}{(2.6^2-1)^2-6^2}\ldots\right\}$$

This series is equivalent to

$$\pi=3+6\,.\left(\frac{1}{1.3.3.5}+\frac{1}{3.5\ 7.9}+\frac{1}{5.7.11.13}+\frac{1}{7.9.15.17}+\ \ldots\right).$$

Paper 23

P. C. Sengupta (1932)

Infinitesimal Calculus in Indian Mathematics - Its Origin and Development

Journal of the Department of Letters (Calcutta University), Vol. 22 (1932), pp. 1–17.

Notes

After Bapu Deva Shastri's brief paper published in 1858, which has been included in this collection, this paper presented for the first time a detailed account of several facts from the history of Indian mathematics to prove that the idea of calculus had originated among the ancient Indian astronomers, Āryabhaṭa and his pupils.

Prabodh Chandra Sengupta

Prabodh Chandra Sengupta (1876–1962) was a historian of ancient Indian astronomy. He was a Professor of Mathematics at Bethune College in Calcutta and a lecturer in Indian Astronomy and Mathematics at the University of Calcutta. (Wikipedia)

Source: Internet Archive

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INFINITESIMAL CALCULUS IN INDIAN MATHEMATICS—ITS ORIGIN AND DEVELOPMENT

BY

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1. Dr. Hoppe in his article in a recent issue of the "Jahresbericht der Deutschen Mathematiker-vereinigung" has traced the history of the Infinitesimal Calculus prior to the time of Leibnitz and Newton. Unfortunately he has put on record no account of the work in this line done by the ancient Indian astronomers and mathematicians. It is proposed to present some facts in regard to this subject and to claim for the Indian mathematicians their due place in the history of the origin and development of the Infinitesimal Calculus. I have to thank Dr. Ganesh Prasad, Hardinge Professor of Pure Mathematics, Calcutta University, at whose suggestion I took up the writing of this paper.

2. The idea of the calculus originated among the ancient Indian astronomers, Āryabhaṭa and his pupils, Brahmagupta, Muñjāla and Bhāskara in their attempt to determine (*i*) the "instantaneous" or "तात्कालिक" daily motion of planets; (*ii*) the position-angle of the ecliptic at any secondary to the equator; (*iii*) the surface and volume of a sphere; (*iv*) the value of π. The necessity for determining the "instantaneous" daily motion arose in connection with the calculation of the eclipses of the sun and the moon and of the right instants of conjunctions of planets between themselves and of the conjunctions of planets with prominent stars near about the ecliptic. The calculation of the position-angle became necessary for

determining the exact point of the beginning of a solar or lunar eclipse. In the Indian method of finding the surface and volume of the sphere, we have a method of integration. In the attempt of the Indian mathematicians to find the value of π, we notice that a small arc of a circle and the chord joining its ends are taken to be equal. No attempt however is made by them to ascertain the difference between the arc and the bounding-chord.

In what follows we shall try to develop historically how the necessity spoken of above gave rise first to the idea of differentiation and the use of infinitesimal triangles, and led also to a process very akin to integration. Some of these results arrived at by Indian mathematicians have already been brought to light by European researchers. In Ball's History of Mathematics, the author says in reference to Bhāskara, "Amongst the trigonometrical formulæ there is one which is equivalent to the equation,

$$d\ (\sin\theta)=\cos\theta\ d\theta.$$"

His authority on this point is Delambre, I. 456. Some more of the results treated later on were first brought to public notice by the late MM. Bāpudev Śāstrī,[1] Spottiswoode, the then Astronomer-Royal, called it an over statement of a case. Dr. B. N. Seal[2] in his "Positive Sciences of the Ancient Hindus" has tried to meet Spottiswoode's views but has not made the point sufficiently clear whether Indian astronomers had a clear notion of the principle and methods of the Differential Calculus.

3. Differentiation.—As has been stated before the idea of differentiation arose from the attempt of the Indian astronomers to find the "instantaneous" or "तात्कालिक"

[1] Bāpudev Śāstri, Bhāskara's Knowledge of the Differential Calculus, J. A. S. B., Vol. XXVII, 1858.

[2] Dr. Seal's Positive Sciences of the Ancient Hindus, pp. 77-80.

daily motion of the planets. According to Āryabhaṭa, the equation giving the longitude, l, of the moon on any day is

$l = m - \frac{P \times R \sin (m-a)}{360}$, where m is the moon's mean longitude

from the first point of Aries, a the longitude of the apogee, and P the circumference of the moon's epicycle of apsis, R=3438′. If n and n' are the rates of the mean daily motion of the moon and her apogee, the "instantaneous" daily motion is expressed as,

$$n - \frac{(n-n')\ P \times \text{Tabular difference of sines at arc } (m-a)}{360 \times 225}$$

Here the term "sine" means the Indian "sine," that is, R sin θ where θ is the arc. The value of R is 3438′ according to Āryabhaṭa, 3270′ according to Brahmagupta, and it is different with different authors. The 'sines' are calculated at intervals of 225′ of arc. The above formula for "instantaneous" daily motion was most probably obtained in the following way:

Suppose l' to be the longitude of the moon after a fraction τ of a day, then

$$l' = m + n\tau - \frac{P}{360} \times R \sin \{m - a + (n-n')\tau\}$$

$\therefore$ Instantaneous daily motion

$$= \frac{l'-l}{\tau} = n - \frac{P}{360\tau}[R \sin \{m-a+(n-n')\tau\} - R \sin (m-a)]$$

$$= n - \frac{P(n-n')}{360 \times 225} \times \text{Tab. difference of sines at arc } (m-a).$$

This rule is ascribed by Lalla (748 A.D.) to some direct pupils of Āryabhaṭa (499 A.D.), thus—

"यच्चन्द्र-केन्द्र-गतिरेव शरद्विदस्रै- (225)
र्भक्ता फलं जनितमार्य्यभटस्य शिष्यैः ॥"[1]

But Lalla does not state what was the exact formula found by Āryabhaṭa's pupils. The rule, however, is found in the Sūryasiddhānta, as made "more accurate" (स्पष्टतर) by Varāhamihira (540 A.D.) in his Pañcasiddhāntikā—

"Multiply (the motion of the anomaly) by the difference of the sines of anomaly and divide by 225; reduce the result (to terms of the epicycle). The arc of the result is to be deducted from the mean motion in the six signs beginning with capricorn (*i.e.*, in the fourth and first quadrants of the mean anomaly) and to be added to it in the six signs beginning with cancer (*i.e.*, in the second and third quadrants of anomaly)."[2] The rule is equivalent to the expression—

$$n - \frac{P\,(n-n') \times \text{Tab. difference of sines at the arc } (m-a) \times 225}{360 \times 225 \times 7\frac{51}{60}}.$$

In Varāhamihira's table of sines, the radius is taken to be 120′, the sine of the arc 225′ is given as 7′ 51″.

In a paper on "Āryabhata's Lost Work," read in February, 1928, before a meeting of the Calcutta Mathematical Society, I proved that Varāhamihira borrowed almost the whole set of astronomical constants from this work of Āryabhaṭa and made the 'Sūryasiddhānta' up-to-date or "more accurate." As Lalla's mention of the division of $(n-n')$ by 225 is found in Varāhamihira's Sūryasiddhānta, the latter must have derived the rule quoted above also from Āryabhaṭa or his pupils.

[1] *The divisor 225 shows that $(n-n')r$ was never contemplated to be greater than 225′, although Bhāskara uses the expression,*

"अद्यतन स्वल्पन केन्द्र-दोर्ज्ययोरन्तरम्"

Grahagaṇita, Bhāskara's own commentary. Ch. VIII, stanzas 37 and 38.

[2] Pañcasiddhantikā, IX, 13.

The expression for the instantaneous daily motion of the moon is given in the same form by Brahmagupta thus—

"ग्रहमन्द-केन्द्रभुक्तिर्ज्यान्तरगुणिताऽऽद्यजीवया भक्ता ।
लब्धं स्फुटपरिधिघ्नं भगणांशहृतं कलाभिरृ ॥४१॥
मृगकर्कादावूनाधिका स्वमध्यगतिः स्फुटार्केन्दोः ।"[1]

"Multiply the planets' motion in anomaly of apsis by the tabular difference of sines (at the mean anomaly) and divide by the first sine (*i.e.*, 214); then multiply the quotient by the true periphery (of the epicycle of apsis) and divide by the degrees in the circle (*i.e.*, 360); the minutes of arc obtained applied negatively and positively in six signs of mean anomaly beginning respectively from capricorn and cancer to the mean motion will give the instantaneous daily motions of the sun and the moon."

That the tabular difference of 'sines' for any given value of the arc is proportional to the 'cosines' of the same arc is perhaps first met with in *Muñjala's Laghumānasa* (932 A.D.) as found by Dr. B. B. Dutta, D.Sc., of the Calcutta University College of Science—from a passage in this astronomical treatise—

"कोटिर्गतिघ्नीच्छेदाप्ता व्यस्तं गतिकलाः फलम् ।"

The commentary on this verse by Prasastidhara (958 A.D.) runs thus—

"भुक्तिस्फुटीकरणार्थं कोटिर्गतिघ्नी कोटिफलमुच्चभुक्त्यारहित-मध्यगत्या गुणितः स्फुटच्छेदेन विभक्तः लब्धं व्यस्तं धनर्णयोर्विपरीतं कोटौ धनभूताप्ता ऋणं धनं गतिकलाफलं लिप्तात्मकं भवति । तेन मध्यभुक्तिस्तथा संस्कृता स्फुटा भवति ।"

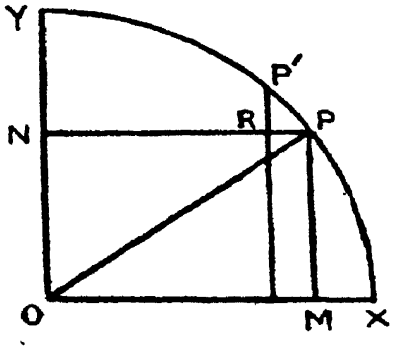
"In order to find the true daily motion, multiply the 'koṭiphala $\left[i.e., \frac{P \times R \cos(m-\alpha)}{360}\right]$ by the difference of the mean motion of the planet and of its apse, and divide the

1 Brahma Sphuṭasiddhānta, Ch. II, 41-42.

result by the rectified *chheda* or divisor: apply the result taken as minutes inversely as the koṭiphala is positive or negative, to the mean motion which thus become true."

[1] There are indeed many methods by which this relation, *i.e.*, the equation $d(\sin\theta)=\cos\theta\, d\theta$ may have been recognised. Here is the most probable way by which this was recognised.

In the figure given below let XOY be a quadrant of a circle and XP any arc, $<POX=\theta$. Measure PP′ such that the arc $=225'$.



Now $PP'=\frac{225'\times R}{3438'}$, where R=OX. Let the figure be completed as shown.

Here PM=R sin θ, PN=R cos θ,

P′R=Tabular difference of sines at the arc XP.

[1] As to Bhaskara, he knew the addition and subtraction theorems in trigonometry and actually gives the following equations :—

(1) $R\sin(\alpha\pm\beta)=\frac{R\sin\alpha\times R\cos\beta}{R}\pm\frac{R\sin\beta\times R\cos\alpha}{R}$.

(2) $R\sin(\alpha\pm225')=R\sin\alpha\times\left(1-\frac{1}{467}\right)\pm\frac{100}{1529}\times R\cos\alpha$.

(3) $R\sin(\alpha\pm60')=R\sin\alpha\left(1-\frac{1}{6569}\right)\pm\frac{10}{573}\times R\cos\alpha$.

} Jyotpatti, 21-22, 18-20, 16-17.

In the last equation Bhaskara indicates that $\sin 60'=\frac{10}{573}=\frac{60}{3438}$. Hence it was quite possible for him to recognise, that the smaller the interval at which the 'sines' are tabulated, the more strictly do the tabular differences of 'sines' become proportional to this interval and the cosine of the arc α. The earlier writers might also have arrived at the equation by an examination of the tabular differences of 'sines' as given by Āryabhaṭa and others.

The $\triangle$s PP'R and PON are similar. OP : PN = P'P : P'R

$$\therefore \text{ The Tabular difference of sines} = P'R = \frac{P'P \times PN}{OP} = \frac{R \cos \theta \times 225'}{3438}.$$

But according to Brahmagupta 225 : 3438 = 214 : 3270

$\therefore$ the tabular difference of sines

$$= \frac{R \cos \theta \times 225}{3438} \text{ or } \frac{R \cos \theta \times 214}{3270}.$$

We shall have ample proofs later on that the ancient Indian astronomers could really follow such a method.

The moon's instantaneous daily motion as found by Āryabhaṭa's pupils may now be written in the form

$$n - \frac{(n-n')P \times 225 \times R \cos (m-\alpha)}{360 \times 225 \times 3438}$$

$$= n - \frac{(n-n')P \times R \cos (m-\alpha)}{360 \times 3438}$$

$$= n - \frac{(n-n')P \times R \cos (m-\alpha)}{360 \times R}.$$

This is thus stated by Bhāskara (1150 A.D.) as follows:

"Multiply the *koṭiphala* $\left[i.e., \frac{P \times R \cos (m-\alpha)}{360} \right]$ by the rate of increase of the mean anomaly of the apsis and divide by the radius: the result taken as minutes of arc applied positively and negatively in six signs of anomaly beginning from cancer and capricorn respectively to the mean motion of the planet will give the instantaneous (तात्कालिकी) daily motion of the planet as affected by the apsis [1]."

[1] Grahagaṇita, VIII, 37, already referred to.

Another case of differentiation was worked out most probably first by Brahmagupta (628 A.D.). The geocentric longitude of a 'star planet,' say Mars, is expressed as follows :

$$l = \mathrm{M} + \mathrm{Sin}^{-1} \frac{p \sin (\odot - \mathrm{M})}{\sqrt{\{\mathrm{R} + p \cos (\odot - \mathrm{M})\}^2 + \{p \sin (\odot - \mathrm{M})\}^2}}$$

$$= \mathrm{M} + \tan^{-1} \frac{p \sin (\odot - \mathrm{M})}{\mathrm{R} + p \cos. (\odot - \mathrm{M})},$$ as expressed in the modern

way, where l is the geocentric longitude of Mars, $\odot$ denotes the mean longitude of the Sun, M the heliocentric longitude of Mars, p the radius of the epicycle of conjunction and R the radius of the concentric. The instantaneous daily motion for any value of $\odot - \mathrm{M}$ is expressed by Brahmagupta (628 A.D.) as follows :—

"शीघ्रगतिं मन्दफलस्फुटभुक्त्यूनां कुजादीनाम् ॥४२॥
शीघ्रफलभोग्यज्या संगुणितां त्वाद्यजीवया विभजेत् ।
फलगुणितं व्यासार्धं विभाजयेत् शीघ्रकर्णेन ॥४३॥
लब्धोना शीघ्रगतिः स्फुटभुक्तिर्भवति... "[1]

"Multiply the motion of the *Śīghra* (*i.e.*, of the mean sun in the case of a superior planet) lessened by the mean motion of the planet as rectified by apsis, by the tabular difference of sines used for finding the *Śighra* equation (*i.e.*, the annual parallax in the case of a superior planet) and divide the product by the first sine (*i.e.*, 214) ; multiply the result by the radius and divide by Śīghra hypotenuse (*i.e.*, the distance of the planet from the earth) ; now subtract the result taken as minutes from the motion of the Śīghra, the remainder is the instantaneous daily motion of the planet."

[1] Bṛahmasphuṭasiddhānta, II, 42½-44½.

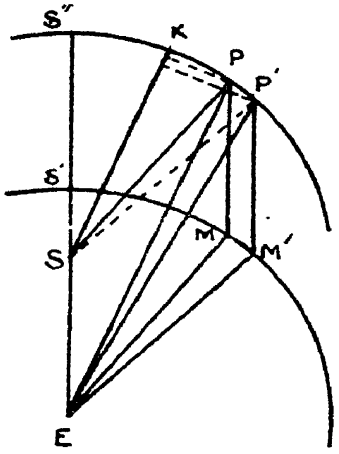
Symbolically this means that:

Inst. daily motion =

$$n' - \frac{(n'-n)\times \text{Tab. dif. of sines at the Śighra Eqn} \times R}{214\sqrt{\{R+p\cos(\odot-M)\}^2+\{p\sin(\odot-M)\}^2}}$$

where[1] n' and n are the motions of the mean sun and Mars.

[2] It is from Bhāskara that we learn the method by which the result was deduced. He makes a direct reference to Brahmagupta when he speaks of him as a **महामतिमान्** person or a genius.



Let E be the earth, S the mean sun, P the planet Mars in the circular orbit having S for the centre. Join PE, PS. With centre E and radius equal to that of the orbit of Mars describe a circle. Let ES=p and ES'=SS''=R. Produce ES cutting the circles in S' and S'' as shown in the figure. From P draw PM parallel to ESS' cutting the second circle at M. Join EM. The arc S'M=S''P and is called the *śīghra* anomaly. The angle SEP is called the rectified *śīghra* anomaly and is equivalent to our elongation. The angle PEM = ∠SPE is the *śīghra phala* or the annual parallax in the case of a superior planet. The method consists in finding the rate of increase in elongation from

[1] Here n stands for the apparent heliocentric daily motion of Mars.

[2] Grahagaṇita, VIII, 39, Bhāskara's own commentary. What is given here is but a slight modification of his method of proof.

the mean sun which subtracted from the mean motion of the sun gives the instantaneous daily motion of the superior planet. After one day the arc S'M would increase by $(n'-n)$ and let $MM'=(n'-n)$; through M' draw M'P' parallel to MP to cut the orbit at P'. Join EP', then $\angle$SEP' would be the elongation on the next day. Increase of elongation is the angle PEP'. This is found thus:—

Let SK be drawn parallel to EP. From P and P' draw perpendiculars on SK. Then $\llcorner$ PSK is the śīghra equation; $PP'=MM'=(n'-n)$. The difference of the perpendiculars from P and P' on SK is thus calculated

$$\llcorner PSK = \textit{Śīghra Phala}$$

$$\llcorner P'SK = \textit{Śīghra Phala} + (n'-n).$$

Suppose both the $\llcorner$s PSK, P'SK lie between two consecutive angles of which the sines are given in the table. The increase of the sine for the increase $(n'-n)$ in the angle

$$= \frac{\text{Tab. Difference of sines at Śīghra Phala} \times (n'-n)}{225} \text{ min.,}$$ the

sines being tabulated at intervals of 225 min.

This increase of the sine is here the difference of the perpendiculars from P' and P on SK and is equal to the perpendicular from P' on EP.

$$\therefore \text{R sin P' EP} = \frac{(n'-n) \times \text{R} \times \text{Tab. difference at } \textit{Śīghra Phala}}{225 \times \text{EP'}}$$

$\therefore$ the rate of increase of elongation

$$= \frac{(n'-n) \times \text{Tab. difference at } \textit{Śīghra Phala} \times \text{R}}{214\sqrt{\{\text{R}+p\cos(\odot-\text{M})\}^2+\{p\sin(\odot-\text{M})\}^2}} \text{ min.,}$$

EP' here is taken equal to EP in the limit and

[1] $\text{EP} = \sqrt{(\{\text{R}+p\cos(\odot-\text{M})\}^2+\{p\sin(\odot-\text{M})\}^2}$ since the arc $\text{S'M}=\odot-\text{M}$

[1] Here $\text{R}=\text{ES'}=\text{SS''}$, $p=\text{ES}$,

$\therefore$ the instantaneous daily motion of the planet

$$= n' - \frac{(n'-n) \times \text{Tab. difference at } \textit{Śīghra Phala} \times \text{R}}{214\sqrt{\{\text{R}+p\cos(\odot-\text{M})\}^2+\{p\sin(\odot-\text{M})\}^2}}$$

If now for the tabular difference, the function cosine is brought in; since R : R cos E_s = 214 : Tab. difference

$\therefore$ Tabular difference $= \frac{214 \times \text{R}\cos \text{E}_s}{\text{R}}$, where E_s represent the *Śīghra Equation.*

Hence the instantaneous daily motion becomes

$$= n' - \frac{(n'-n) \times \text{R}\cos \text{E}_s}{\sqrt{\{\text{R}+p\cos(\odot-\text{M})\}^2+\{p\sin(\odot-\text{M})\}^2}}$$

This is exactly the form in which Bhāskara states the result.

We now consider another example of differentiation that was first done by Bhāskara (1150 A.D.). It arose in connection with finding the complement of the angle at which the ecliptic cuts a secondary to the celestial Equator. The 'sine' of the angle between the eastward direction of the ecliptic at any point of it and the eastward direction of the equator is known as the *Āyana Valana.* If any point of the ecliptic has a celestial longitude l, and δ be the corresponding declination of that point, we have the well-known equation :

Sin δ = Sine $l \times$ sin ω, where ω is the obliquity of the ecliptic.

Bhāskara recognises that $\frac{d\delta}{dl}$ is here the sine of the required angle and actually proves it—

$$\text{R}\frac{d\delta}{dl} = \frac{\text{R}\cos l \times \text{R}\sin\omega}{\text{R}\cos\delta}$$

I give below a free translation of Bhāskara's own commentary on the stanzas 69-74, Chapter VIII of the

"Golādhyāya," which shows the method by which he arrived at the correct result.

"At the place (S) of the sun in the ecliptic (YSC), show the disc like a coin; draw the diurnal circle (ASA′) of the sun; then the north-south (perpendicular) distance of the ecliptic from the point of intersection of the circumference of the disc and the diurnal circle, is the *jyā*[1] of *valana* in the disc; that is the difference between the declination of the centre of the sun and the declination of the point of the ecliptic which is ahead of the centre by the semi-diameter of the disc. Hence this is being found; multiply the semi-diameter of the disc (in minutes) by the tabular difference that has to be used in taking the proportional part in finding the *jyā* of the sun's longitude, and divide by 225 (the interval in min. at which the *jyās* are given in the table), the result is the difference between the *jyās* (*i.e.*, of the longitude if the sun's centre and of that longitude *plus* the semi-diameter). It is however necessary to find the true tabular difference by proportion:[2] if for a *koṭijyā* equal to the radius, the tabular difference is 225, what is the tabular difference for this *koṭijyā* of the longitude. The result is the true tabular difference $\left(=\frac{225\times \mathrm{R}\cos l}{\mathrm{R}}\right)$. Multiply the radius of the sun's disc by this and divide by 225 $\left(\frac{r\times 225\times \mathrm{R}\cos l}{\mathrm{R}\times 225}\right.$ $\left.=\frac{r\,\mathrm{R}\cos l}{\mathrm{R}}\right.$ where r is the angular radius (in min.) of the disc$\Big)$. This is the difference between *jyās*. Then apply a proportion to find the difference of declinations: if for a *jyā* equal the radius, the *jyā* of the declination become the *jyā* of[3] 24°,

[1] *Jyā* means the Indian sine, and "*Koṭijyā*" means the Indian cosine.

[2] Bhāskara here assumes that $\mathrm{d}(\sin\theta)=d\theta\cos\theta$.

[3] Bhāskara takes ω, the obliquity of the ecliptic to be $=24°$.

what difference of (the *jyās* of) declinations is obtained by this difference of longitudes ? (Bhāskara here commits a slip, which as we shall see, he corrects afterwards; he takes the difference between the *jyās* of declinations as the difference of declinations, *i.e.*, takes $\cos\delta\, d\delta = d\delta$). The result is the difference of declinations, which is the *valana* in a circle whose radius is equal to the radius of the disc. Then take another proportion: if this be the *valana* in the circle of the disc, what is the *valana* in a circle of radius 3438 ?

$$\left(\frac{r}{\mathrm{R}}\frac{\mathrm{R}\cos l}{} \times \frac{\mathrm{R}\sin 24^\circ}{\mathrm{R}} \times \frac{\mathrm{R}}{r} = \frac{\mathrm{R}\cos l \times \mathrm{R}\sin 24^\circ}{\mathrm{R}} = \textit{Āyana valana}\right).$$

Here the multiplier and the divisor are each equal to R and also to the radius of the disc, which being cancelled, the *jyā* of 24° is the multiplier and the radius, the divisor of the *koṭijyā*, which is evidently the *jyā* of declination for the *koṭi* $(90^\circ - l)$ of the longitude and is equal to *Āyana valana* in a great circle. But this will only be when the planet is on the equator, for in such a case alone, the radial line joining the position of observer (on the earth's equator) to the centre of the disc at the zenith, becomes like the stick of an umbrella, and above it the disc is everywhere equally inclined (*i.e.* perp. to it) like an umbrella. Hence the great circle in which the *jyā* of *valana* is constructed, coincides with the horizon; so that the *Āyana valana* becomes what has been obtained before. When the planet is at the end of the sign Aries, the zenith distance of the disc is north and equal to the declination of that point. Then the radial line becomes the hypotenuse, the perpendicular drawn from the centre of the disc to the axis of the armillary sphere is the *koṭijyā* of the declination, which is the base while the *jyā* of the declination becomes the perpendicular. This is like the case of an umbrella which when the stick is held like a hypotenuse to the vertical direction, becomes the

hypotenuse to the horizontal direction. Here also the *jyā* of *Āyana valana* is the hypotenuse. Now the expression for the difference of declinations (really the difference of the *jyās* of declinations) above found is perpendicular to the radius of the diurnal circle and is in the form of the base of a right-angled triangle. In order to find the corresponding hypotenuse we use a proportion thus: if for a base equal to the *koṭijyā* of the declination, the hypotenuse is the radius, what is the hypotenuse for this base? In the expression as determined above, the *jyā* of 24° is the multiplier and the divisor is the radius; now the new multiplier and the divisor are the radius and the *koṭijyā* of the declination $\left(\frac{\text{R cos } l \times \text{R sin } 24^\circ}{\text{R}} \times \frac{\text{R}}{\text{R cos } \delta} = \bar{A}yana\ valana\right)$; hence cancelling the radius which occurs both as a multiplier and a divisor, we find that the *koṭijyā* is multiplied by the *jyā* of 24° and divided by the *koṭijyā* of the declination as before, as the expression for the *Āyana valana*."

4. **Integration.**—In ancient mathematics, integration that is met with is a mode of summation. The beginning of this type of integration is to be ascribed to that unknown mathematician who first found the area of the circle to be $= \frac{\odot ce \times r}{2}$.

The most important case of integration was the method by which the surface and the volume of the sphere were determined. In India the volume of a sphere was first correctly expressed by *Śrīdhara* in his *Triśatikā*, 56. The methods are met with in Bhāskara's Golādhyāya. The following is a free translation of Bhāskara's own commentary on this point. Here also the terms *Jyā*, *jyārdha* are to be taken as equivalent to the Indian 'Sine.'

" In order to make the point clear to the beginner the teacher should demonstrate it on the surface of a sphere. Make a model of the earth in clay or wood and suppose its

circumference to contain as many units of length as there are minutes of arc in a whole circle, *i.e.*, 21600 units. Mark a point on the surface, with that point as the centre with $\frac{1}{96}$ of the circumference as the (arcual) radius describe a circle. Again with the same centre with twice that thread draw another circle; with three times that, another circle and continue in this operation till with 24 times that thread, the 24th circle is described. Of these circles the radii will be the *jyārdhas* (*jyās*), *viz.*,[1] 225, 449, etc. From these by proportion the lengths of the circles are obtained. Here the length of the last circle is 21600 units and its radius=3438. If the *jyārdhas* (*jyās*) be multiplied by 21600 and divided by 3438 (or more correctly multiplied by 3927 and divided by 1250) we get the lengths of the circles. Between any two circles there is an annular figure and there are 24 such figures, more if more than 24 *jyās* are used. In each figure the larger, lower circle may be taken as the base and the upper smaller circle as the opposite side, while the perpendicular is 225. Hence by the rule "लम्बगुणं कुमुखयोगार्द्धमिति" (*i.e.*, the rule for finding the area of a trapezium) the area of each ring may be found. The sum of all these areas=the surface of half of a sphere; twice that equals the surface of the whole sphere. This is equal to the product of the diameter and the circumference."

Note.—Let the areas of the rings from any point on the sphere, be denoted by $A_1, A_2, A_3, \ldots\ldots A_{24}$.

$$\text{Then } A_1 = \frac{225 \times \text{circumference of the 1st circle}}{2},$$

$$= \frac{225 \times 3927 \times 2J_1}{1250 \times 2} \text{ where } J_1 \text{ denotes the 1st } jyā \text{ R sin } 225',$$

$$= \frac{225 \times 62832}{10000} \times \frac{J_1}{2}.$$

[1] The Indian "sines" as given by Bhāskara :—225, 449, 671, 890, 1105, 1315, 1520, 1719, 1910, 2093, 2267, 2431, 2585, 2728, 2859, 2977, 3084, 3177, 3256, 3321 3372, 3409, 3431, and 3438. *Grahaganita*, VIII, 3.6.

Similarly

$$A_2=\frac{225\times 62832}{10000}\times\frac{J_1+J_2}{2},\ A_3=\frac{225\times 62832}{10000}\times\frac{J_2+J_3}{2}\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_{24}=\frac{225\times 62832}{10000}\times\frac{J_{23}+J_{24}}{2}.$$

$\therefore$ the surface of the hemisphere

$$=\frac{225\times 62832}{10000}\left(J_1+J_2+J_3+\ldots\ldots J_{23}+\tfrac{1}{2}\times J_{24}\right)$$

$$=\frac{225\times 62832\times 52513}{10000}$$

$$=\frac{21600}{96}\times 6{\cdot}2832\times 52513$$

$$=21600\times 3437.$$

$\therefore$ the surface of the whole sphere $=21600\times 3437\times 2$

$$=\odot ce\times \textit{Diameter}.$$

The result is not in perfect agreement with theory because of the fact that only 24 *jyās* are used.

Thus Bhāskara having proved that the surface of the sphere to be equal to $=\odot ce\times$ diameter, says—

"This is as I have said in my Arithmetic :—

'The area of a circle is equal to the product of the circumference by one-fourth of the diameter. That result multiplied by 4 gives the surface of the sphere which is like the net surrounding a hand ball; the same (surface of a sphere) multiplied by the diameter and divided by six becomes invariably the volume of the sphere.'"

(Symbolically

$$(1)\ \text{Area of a circle}=\frac{\odot ce\times\text{diameter}}{4}$$

$$=\pi r^2$$

(2) Surface of a sphere = area of a circle $\times$ 4

$$= 4\pi r^2$$

(3) Volume of a sphere = Surface of a sphere

$$\times \frac{\text{diameter}}{6}$$

$$= \frac{4\pi r^2 \times 2r}{6}$$

$$= \tfrac{4}{3}\ \pi r^3.)$$

That is one-sixth of the surface of the sphere multiplied by the diameter is the volume of the sphere.

"*Proof.*—Suppose the sphere to be divided into as many pyramids as there are units of area in the surface, the height of each of which is equal to the radius. All the vertices of the pyramids will meet at the centre of the sphere. In this way the sum of the volumes of all these pyramids is the volume of the sphere, from which the rule for finding the volume readily follows."

The methods followed, as shown above, are allied to modern methods of integration. Although we miss here the highly ingenious method of Archimedes in summing up a trigonometrical series there can be no question that the Indian method is perfectly original.

Paper 24

A. N. Singh (1936)

On the Use of Series in Hindu Mathematics

Osiris, Vol. 1, Jan. 1936, pp. 606–628 (28 pages). (Published by The University of Chicago Press)

Notes

In his 1834 paper on the Hindu quadrature of the circle, Whish gave several infinite series for computing π and also the power series expansion of inverse-tan function all discovered by the members of the Kerala school of astronomy and mathematics. These series were quoted by De Morgan in 1843 while castigating them as a 'clever hoax' hoisted by Whish. The scholarly world had to wait for another eighty years to see a mention of these series in print and being discussed about.

The significance of this paper by A. N. Singh was that it was for the first time that the power series expansions of the sine and cosine functions as they had been given in the texts of the Kerala school as well as the attributions of their discovery to the mathematicians of the Kerala school were published in a modern scholarly journal.

A. N. Singh

A. N. Singh was a Professor of Mathematics at the Lucknow University. His fields of specialization were Theory of Functions and History of Hindu Mathematics. He played a key role in establishing the Govt Astronomical Observatory, Benares in 1953 and he was appointed as the Honorary Director of the Observatory. The Observatory was later moved to Nainital in 1955. In July 1954, during the course of his inspection tour to the Observatory, Singh expired at Varanasi.

The book *History of Hindu Mathematics: A Source Book* authored by Bibhutibhushan Datta and A. N. Singh and originally published in two parts in 1930's is a treatise on the history of Indian mathematics. The book has since been reissued in one volume by Asia Publishing House in 1962. The treatise has been a standard reference for the history of Indian mathematics for many years.

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On the use of series in Hindu mathematics

INTRODUCTION.—The object of the present paper is to give an account of the use of series in Hindu Mathematics. For this purpose the Hindu literature, mathematical as well as non-mathematical, has been examined. Particular instances of the arithmetic and geometric series have been found to occur in Vedic literature as early as 2000 B.C. From Jaina literature it appears that the Hindus were in possession of the formulae for the sum of the arithmetic and the geometric series as early as the fourth century B.C., or earlier.

The earliest known Hindu work on mathematics available to us is a mutilated copy of what is known as the *Bakhshālī Manuscript* which contains a work on *Pātīgaṇita* composed about 300 A.D. (1) Amongst other works which contain an account of series may be mentioned the *Āryabhaṭīya* (499 A.D.), the *Brāhma-sphuṭa Siddhānta* (628 A.D.), the *Triśatikā* (750 A.D.), the *Gaṇita-sāra-saṁgraha* (850 A.D.), the *Mahāsiddhānta* (950 A.D.), the *Līlāvatī* (1150 A.D.), the *Gaṇita Kaumudī* (1350 A.D.), etc. In all these works series is treated as one of the fundamental operations *(śreḍhī vyavahāra)*, and a separate section is generally devoted to the formulae and problems relating to series. In Europe too series was looked upon as one of the fundamental operations (2), evidently due to Hindu influence through the Arabs. Besides the arithmetic and geometric series a number of other types of series, e.g., the series of sums, the series of the squares or cubes of the natural numbers, the arithmetico-geometric series, the series of polygonal or figurate numbers etc. occur in some

(1) The manuscript was edited and published by G. R. KAYE, who assigns a much later date to the work. KAYE's conclusions are unreliable. See B. DATTA, *Bull. Cal. Math. Soc.*, Vol. XXI (1929), pp. 1-60.

(2) See SMITH, *History of Mathematics*, Vol. II, p. 497.

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of the works mentioned above. There is, however, no mention of the harmonic series.

Evidence of the use of infinite series in India is found in the fifteenth century. The infinite series appear to have been obtained in connection with the evaluation of the trigonometrical functions and the computation of π. Noteworthy achievements of the Hindu mathematicians in this theory are the expansions of Sin x, Cos x, and Tan ^{-1}x in powers of x, obtained long before they were known in Europe or anywhere else.

No account of the use of series in Hindu mathematics has been published by any previous writer and as such, I hope, the present discussion will be found useful by the historian of mathematics.

ORIGIN AND EARLY HISTORY.—Series of numbers developing according to certain laws have attracted the attention of people in all times and climes. The Egyptians are known to have used the arithmetic series about 1550 B.C. (3) Arithmetic as well as geometric series are found in the Vedic literature of the Hindus (*c.* 2000 B.C.). In the *Taittirīya Saṁhitā* (4) we find the following series :

(i) 1, 3, 5,... 19, 29,... 99.
(ii) 2, 4, 6,... 20.
(iii) 4, 8, 12,...
(iv) 10, 20, 30,...
(v) 1, 3, 5,... 33.

In the *Vājasaneyī Saṁhitā* (5), we find the *yugma* (even) and *ayugma* (odd) series :

(vi) 4, 8, 12, 16,... 48.
(vii) 1, 3, 5, 7,... 31.

The *Pañcaviṁśa Brāhmaṇa* (6) has the following geometric series :

(viii) 12, 24, 48, 96,... 196608, 393216.

Another geometric series occurs in the *Dīgha Nikāya* (7). It is :

(3) In the Ahmes Papyrus. *Cf.* PEET, *Rhind Papyrus*, p. 78; SMITH, *l.c.*
(4) See vii. 2. 12-17; iv. 3. 10.
(5) See xvii. 24. 25.
(6) xviii. 3. Compare also *Lāṭyāyana Śrauta sūtra*, viii. 10. 1 *et seq. Kātyāyana Śrauta sūtra*, xxii. 9. 1-6.
(7) T. W. RHYS DAVIDS, *Dialogues of the Buddha*, III, (1921), pp. 70-72.

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(ix) 10, 20, 40,... 80000.

The Hindus must have obtained the formula for the sum of an arithmetic series at a very early date, but when exactly they did so can not be said with certainty. It is, however, definite that in the fifth century B.C. they were in possessior of the formula for the sum of the series of natural numbers; for in the *Brhaddevatā* (8) (500-400 B.C.) we have the result

$$2 + 3 + 4 + \ldots + 1000 = 500499.$$

In the *Kalpa-sūtra* of BHADRABĀHU (*c.* 350 B.C.) we have the sum of the following geometric series

$$1 + 2 + 4 + \ldots + 8192$$

given correctly as 16383, showing that the Hindus possessed some method of finding the sum of the geometric series in the fourth century B.C.

KINDS OF SERIES.—It thus appears that the Hindus studied the arithmetic and the geometric series at a very early date. ĀRYABHAṬA I (499), BRAHMAGUPTA (628) and other posterior writers considered also the cases of the sum of the sums, the squares and the cubes of the natural numbers. MAHĀVĪRA (850) gave a rule for the summation of an interesting arithmetico-geometric series, *viz.*,

$$\sum_{1}^{n} t_m$$

where $t_1 = a$ and $t_m = rt_{m-1} \pm b$. $(m \geqslant 2)$; and NĀRĀYAṆA (1356) considered the summation of the figurate numbers of higher orders.

TECHNICAL TERMS.—The Sanskrit term for a series is *śreḍhī* meaning literally " progression," " any set or succession of distinct things," or śreṇī (or śreṇi), literally " line," " row," " series " or " succession "; hence in relation to mathematics it implies " a series or progression of numbers." Thus it is clear that the modern terms progression and series are analogous to the Hindu terms, and they seem to have been adopted in the West under Hindu influence, in preference to the Greek term ἔκθεσις (ekthesis) which literally means a setting forth. The Sanskrit name

(3) *Bṛhaddevatā* edited in original Sanskrit with English translation by MACDONELL, Harvard, (1904).

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for a term of a series is *dhana* (9) (literally, " any valued object "). The first term is called *ādi-dhana* (" first term "), and any other term *iṣṭa-dhana* (" desired term "). When the series is finite, its last term is called *antya-dhana* (" last term ") and the middle term *madhya-dhana* (" middle term "). Often, for the sake of abridgement, the second words of these compound names are deleted, so that we have the terms *ādi*, *iṣṭa*, *madhya* and *antya* in their places. The first term is also called *prabhava* (" initial term ") or *mukha* (" face ") and its synonyms. The technical names for the common difference in an arithmetic series are *caya* or *pracaya* (from the root *caya* " to go," hence meaning " that by which the term goes," that is, " increment "), *uttara* (" difference," " excess "), *vṛddhi* (" increment ") etc. The common ratio in a geometric series is technically called *guṇa* or *guṇaka* (" multiplier ") and so this series is distinguished from the arithmetic series by the specific name *guṇa-śreḍhī*. The number of terms in a series is known as *pada* (" step," meaning " the number of steps in the sequence ") or *gaccha* (" period "). The sum is called *sarva-dhana* (" total of terms "), *śreḍhī-phala* (" result of progression ") *śreḍhī gaṇita* (or simply *gaṇita*, because the sum of the series is obtained by computation), *śreḍhī-saṁkalita* (or in short *saṁkalita*, " sum of the series ").

The above mentioned technical terms occur commonly in almost all the known Hindu treatises on arithmetic, from the so called *Bakhshālī* treatise (*c.* 300) onwards. But in the latter, series has also been designated by *varga*, meaning " group." Occasionally we meet with the terms *paṁkti* (10) and *dhārā* (11) which signify " continuous line or series." NĀRĀYAṆA has used the special term *āya* (literally, " income ") for a sum of natural numbers.

SUM OF AN A. P.—Problems on the summation of an arithmetic series are met with in the earliest available work on Hindu mathematics, the *Bakhshālī Manuscript*. The statement of the formula for the sum begins with the word *rūpoṇā*, so that summa-

(9) In mathematics *dhana* means an affirmative quantity or *plus*. This probably explains the use of this term to denote the elements of a series which have to be added together.

(10) See chapter xiii of the *Gaṇita Kaumudī* (MS.) of NĀRĀYAṆA.

(11) For instance, see the *Triloka-sāra* of NEMĪCANDRA (*c.* 975).

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tion is indicated by the terms *rūpoṇā karaṇena* (" by the operation *rūpoṇā* etc.") throughout the work. In the statement of the solution of problems, the first term, the common difference and the number of terms are written together and the resulting sum after these, as follows :

" | *ā* 1 / 1 | *u* 1 / 1 | *pa* 19 / 1 |

rūpoṇā karaṇena phalam | 190 / 1 | "

In the above *ā* stands for *ādi* (" first term "), *u* for *uttara* (" common difference ") and *pa* for *pada* (" number of terms "). The above quotation may be translated thus (12) :

" The first term is 1/1, the common difference is 1/1 and the number of terms is 19/1, therefore, performing *rūpoṇā* etc., the sum is 190/1."

ĀRYABHAṬA I (499) states the formula (13) for finding the middle term, the *r*th term and the sum of any desired number of terms as follows :

" The desired number of terms minus one, halved and multiplied by the common difference when added to the initial term gives the middle term. The number of terms that precede any desired term (*i.e.*, $r - 1$, if the *r*th term is desired) multiplied by the common difference and being added to the first term gives (the desired term). The middle term multiplied by the number of terms, or the sum of the first and last terms multiplied by half the number of terms gives the sum of the desired number of terms."

Let

$$a_1 + a_2 + a_3 + \ldots + a_r + a_{r+1} + \ldots + a_{n+r-1} + a_{n+r} + \ldots$$

(12) Note how the denominator 1 is written in the case of all the integral quantities. This is to show that the quantities involved may have fractional values also.

(13) *A*, ii. 19. The translation given here differs from that given by CLARK, (*Āryabhaṭīya*, Chicago, 1930, p. 35). The manuscript of the *Āryabhaṭīya* with the commentary of BHĀSKARA I (522), in our possession gives the reading *samukham madhyam* in the place of *samukhamadhyam* in KERN's edition. BHĀSKARA I remarks, " *atra bahūni sūtrāṇi* " *i.e.*, " this contains several rules." The commentators SŪRYADEVA and PARAMEŚVARA also make the same remark. They all agree with the interpretation given here.

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be a series in A. P., and let the common difference of the series be b. Then taking any n terms, say,

$$a_r + a_{r+1} + \ldots + a_{r+n-1},$$

the rule given above states :

(i) middle term $= \left\{ a_r + \frac{n-1}{2} b \right\} = \text{M (say)}$

(ii) nth term $= \{ a_r + (n-1) b \}$

(iii) Sum of the desired number of terms $= \text{M}.n$

$$= \frac{a_r + a_{r+n-1}}{2} \cdot n$$

BRAHMAGUPTA (14) says :

" The last term is equal to the number of terms minus one multiplied by the common difference, (and then) added to the first term. The middle term is half the sum of the first and the last terms; this multiplied by the number of terms is the sum."

BRAHMAGUPTA's results are the same as those of ĀRYABHAṬA I. Similar statements occur in the works of ŚRĪDHARA (15), ĀRYABHAṬA II (16), BHĀSKARA II (17) and others. MAHĀVĪRA points out that the common difference may be a positive or negative quantity (18).

The particular case

$$\sum_1^n r = \frac{n(n+1)}{2}$$

is mentioned in all the Hindu works (19).

ORDINARY PROBLEMS IN A. P.—The problems of finding out (1) the common difference, (2) any desired term and (3) the number of terms are common to all Hindu works. They occur first in the *Bakhshālī Manuscript* (20). The problem of finding the number of terms requires the solution of a quadratic

(14) *BrSpSi*, p. 186.
(15) *Triś*, p. 28.
(16) *MSi*, p. 158.
(17) *L*, p. 27.
(18) *GSS*, p. 102.
(19) It is sometimes mentioned in connection with addition, as in ŚRĪDHARA's *Triśatikā* and the *Gaṇita-sāra-saṁgraha* of MAHĀVĪRA.
(20) See p. 25 & p. 35, problem 9; and p. 36 problem 10. The solution of this problem is incorrectly printed.

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equation (21). Some indeterminate problems in which more than one of the above quantities are missing also occur in the *Bakhshālī Manuscript*, the *Gaṇita-sāra-saṁgraha* of MAHĀVĪRA and the *Gaṇita Kaumudī* of NĀRĀYAṆA. A typical example of such problems is the finding out of an arithmetic series that will have a given sum and a given number of terms (22).

As illustrations of some other types of Hindu problems on arithmetical progression may be mentioned the following :

(1) There were number a of *utpala* flowers representable as the sum of a series in arithmetical progression, whereof 2 is the first term and 3 the common difference. A number of women divided these flowers equally among themselves. Each woman had 8 for her share. How many were the women and how many the flowers (23)?

(2) A person travels with velocities beginning with 4 and increasing successively by the common difference 8. Again, a second person travels with velocities beginning with 10 and increasing successively by the common difference 2. When do they meet (24)?

(3) The continued product of the first term, the number of terms and the common difference is 12. If the sum of the series is 10, find it (25).

(4) A man starts with a certain velocity and a certain acceleration per day. After 8 days another man follows him with a different velocity and an acceleration of 2 per day. They meet twice on the way. After how many days do these meetings occur (26)?

GEOMETRIC SERIES.—MAHĀVĪRA gives the formula

$$S = \frac{a\,(r^n - 1)}{r - 1}$$

for the sum of a geometric series whose first term is a and common ratio r. He says :

" The first term when multiplied by the continued product

(21) Such problems occur in the *Bakhshālī Manuscript* also.
(22) For some indeterminate problems see COLEBROOKE, p. 293.
(23) *GSS*, vi. 295.
(24) *GSS*, vi. $32\frac{1}{2}$.
(25) *GK*, *śreḍhī vyavahāra*, ex. under rule 6.
(26) *Ibid.*, ex. under rule 9.

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of the common ratio, taken as many times as the number of terms, gives rise to the *guṇadhana*. And it has to be understood that this *guṇadhana*, when diminished by the first term and then divided by the common ratio lessened by one, becomes the sum of the series in geometric progression" (27).

The same result is stated by him in the following alternative form :

"In the process of successive halving of the number of terms, put zero or one according as the result is even or odd. (Whenever the result is odd subtract one). Multiply by the common ratio when unity is subtracted and multiply so as to obtain square (when otherwise, *i.e.*, when the half is even). When the result of this operation is diminished by one and is then multiplied by the first term and (is then) divided by the common ratio lessened by one, it becomes the sum of the series" (28).

If n be the number of terms and r the common ratio, the first half of the above rule gives r^n. This process of finding the nth power of a number was used by PIṄGALA (29) (*c.* 200 B.C.) to find 2^n. The second half of the rule then gives

$$S = \frac{a\,(r^n - 1)}{r - 1}.$$

The above formula for the sum is stated by PṚTHUDAKASVĀMĪ (30), ĀRYABHAṬA II (31) and BHĀSKARA II (32) in the second form which appears to be the traditional method of stating the result.

MAHĀVĪRA has given rules for finding the first term, the common ratio or the number of terms, one of these being unknown and the others as well as the sum being given (33).

As illustrations of problems on geometric series may be mentioned the following :

(1) Having first obtained 2 golden coins in a certain city, a man goes on from city to city, earning everywhere three times

(27) *GSS*, ii. 93.

(28) *GSS*, ii. 94, also vi. 311½, where the rule is applied to the case in which the common ratio is fractional.

(29) *Chandaḥ sūtra*, Calcutta, 1892, viii. 20 ff.

(30) *BrSpSi*, p. 186 Com.

(31) *MSi*, p. 159.

(32) *L*, p. 31.

(33) *GSS*, ii. 97-103.

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of what he earned immediately before. Say, how much he will make on the eighth day (34)?

(2) When the first term is 3, the number of terms 6 and the sum is 4095, what is the value of the common ratio (35)?

(3) The common ratio is 6, the number of terms is 5, and the sum is 3110. What is the first term here (36)?

(4) How many terms are there in a geometric series whose first term is 3, the common ratio is 5 and the sum is 22888183593 (37)?

SERIES OF SQUARES.—The series whose terms are the squares of the natural numbers, seems to have attracted attention at a fairly early date in India. The formula

$$\sum_{1}^{n} r^2 = \frac{n\,(n+1)\,(2n+1)}{6}$$

occurs in the *Āryabhaṭīya* (38) (499), where it is stated in the following form :

" The sixth part of the product of the three quantities consisting of the number of terms, the number of terms plus one and twice the number of terms plus one, is the sum of the squares." This formula occurs in all the known Hindu works (39).

MAHĀVĪRA gives the sum of a series whose terms are the squares of the terms of a given arithmetic series.

Let

$$a + (a+b) + \ldots + (a + \overline{r-1}\,b) + \ldots + (a + \overline{n-1}\,b)$$

be an arithmetic series. Then according to him

$$a^2 + (a+b)^2 + \ldots + (a + \overline{r-1}\,b)^2 + \ldots + (a + \overline{n-1}\,b)^2 = n\left[\left\{\frac{2n-1}{6}\,b^2 + ab\right\}(n-1) + a^2\right].$$

NĀRĀYAṆA (40) gives the above result in the following form :

(34) *GSS*, ii. 96.
(35) *GSS*, ii. 102 first half.
(36) *GSS*, ii. 102 second half.
(37) *GSS*, ii. 105 last part.
(38) *Ā*, p. 39.
(39) Except the *Triśatikā*. The rule along with many others must have been given in ŚRĪDHARA's bigger work of which the *Triśatikā* is an abridgment.
(40) *GK*, *śreḍhī vyavahāra*, 17½ and the first half of 18.

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$$\sum_{1}^{n} (a + \overline{r-1}b)^2 = a \sum_{1}^{n} \{ a + 2(r-1)b \} + b^2 \sum_{1}^{n-1} r^2.$$

SERIES OF CUBES.—ĀRYABHAṬA I states the formula giving the sum of the series formed by the cubes of the natural numbers as follows :

" The square of the sum of the original series is the sum of the cubes " (41).

Thus according to him

$$\sum_{1}^{n} r^3 = \left(\sum_{1}^{n} r \right)^2 = \left\{ \frac{n(n+1)}{2} \right\}^2.$$

The above formula occurs in all the Hindu works. The general case in which the terms of the series are the cubes of the terms of a given arithmetic series, has been treated by MAHĀVĪRA (42).

Let

$$S = \sum_{1}^{n} (a + \overline{r-1}b)$$

be an arithmetic series whose first term is a and the common difference b. Then according to MAHĀVĪRA

$$\sum_{1}^{n} (a + \overline{r-1}b)^3 = b \cdot S^2 \pm Sa\,(a \sim b)$$

according as $a >$ or $< b$.

NĀRĀYAṆA (43) has also given the above result in the same form as MAHĀVĪRA.

SERIES OF SUMS.—Let

$$N_n = 1 + 2 + 3 + \ldots + n,$$

then, the series

$$\sum_{1}^{n} N_r,$$

formed by taking successively the sum up to 1, 2, 3,... terms of the series of natural numbers, is given in all the Hindu works (44), beginning with that of ĀRYABHAṬA I, who says :

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(41) *Ā*, p. 39.
(42) *GSS*, vi. 303.
(43) *GK*, *śreḍhī vyavahāra*, 18½ f.
(44) With the exception of the *Triśatikā*.

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" In the case of an *upaciti* which has one for the first term and one for the common difference between the terms, the product of three terms having the number of terms (n) for the first term and one for the common difference, divided by six is the *citighana*. Or the cube of the number of terms plus one, minus the cube root of the cube (45), divided by six " (46).

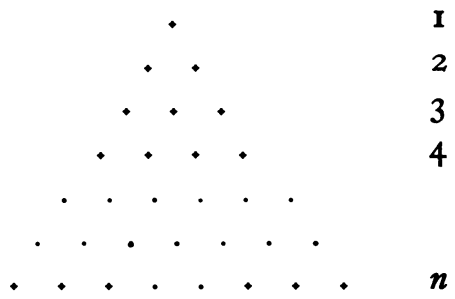
The above rule states that

$$\sum_{1}^{n} N_r = \frac{n\,(n+1)\,(n+2)}{6}$$

$$\text{or} \qquad = \frac{(n+1)^3 - (n+1)}{6}$$

The sum of the series $\sum_{1}^{n} N_r$ has been called by ĀRYABHATA *citighana* which means the " solid content of a pile in the shape of a pyramid on a triangular base." The pyramid is constructed as follows :

Form a triangle with $\sum_{1}^{n} r$ things arranged as below :



Total $= n(n+1)/2$.

Form a similar triangle with $\sum_{1}^{n-1} r$ things and place it on top of the first, then form another such trianlge with $\sum_{1}^{n-2} r$ things and place it on top of the first two. Proceed as above till there is one thing at the top. The figure obtained in this manner will be a pyramid of layers such that the base layer consists of $\sum_{1}^{n} r$

(45) This means $\{(n+1)^3\}^{\frac{1}{3}} = (n+1)$. Recourse is taken to this form of expression for the sake of metrical convenience.

(46) *Ā*, p. 39.

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things, the next higher layer consists of $\sum_{1}^{n-1} r$ things, and so on. The number of things in the solid pyramid

$$citighana = \sum_{1}^{n} N_r ,$$

where

$$N_r = \sum_{m=1}^{m=r} m.$$

The base of the pyramid is called *upaciti*, so that

$$upaciti = \sum_{m=1}^{n} m.$$

The above *citighana* is the series of figurate numbers. The Hindus obtained the formula for the sum of the series of figurate numbers as early as the fifth century A.D. and they knew the sum of the series of natural numbers in the fifth century B.C. It can not be said with certainty whether they used the representation of the sum by triangles or not. The subject of piles of shots, however, has been given great prominence in the Hindu works, all of which contain a section dealing with *citi* ("piles"). It will not be a matter of surprise if the geometrical representation of figurate numbers is traced to Hindu sources.

MAHĀVĪRA'S SERIES.—MAHĀVĪRA (850) has generalized the series of sums in the following manner :

Let $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n$,

be a series in arithmetical progression, the first term being α_1 and the common difference β, so that $\alpha_r = (\alpha_1 + \overline{r-1}\,\beta)$. MAHĀVĪRA considers the following series

$$\sum_{r=1}^{r=n} \left(\sum_{m=1}^{m=\alpha_r} m \right),$$

and gives its sum as

$$\frac{n}{2}\left[\left\{ \frac{(2n-2)\beta^2}{6} + \frac{\beta}{2} + \alpha_1 \beta \right\} (n-1) + \alpha_1 (\alpha_1 + 1) \right]. \quad (47)$$

NĀRĀYAṆA gives the above result in another form. According to him

$$\sum_{r=1}^{n} \left(\sum_{m=1}^{\alpha_r} m \right) = \left(\sum^{\alpha_1+\beta} m - \sum_{1}^{\alpha_1} m \right) \sum_{1}^{n-1} m + n \sum_{1}^{\alpha_1} m + \beta^2 \sum_{1}^{n-2} \left(\sum_{1}^{r} m \right).$$

(47) *GSS*, vi. 305-305½.

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Denoting by N_r the sum of r terms of the series of natural numbers, NĀRĀYAṆA's result may be written in the form

$$\sum_{r=1}^{n} N_{\alpha_r} = (N_{\alpha_1+\beta} - N_{\alpha_1})\, N_{n-1} + n\, N_{\alpha_1} + \beta^2 \sum_{1}^{n-2} N_r$$

$$= \left\{ \frac{(\alpha_1 + \beta)(\alpha_1 + \beta + 1)}{2} - \frac{\alpha_1 (\alpha_1 + 1)}{2} \right\} \frac{n(n-1)}{2} + \frac{n \cdot \alpha_1 (\alpha_1 + 1)}{2} + \beta^2 \frac{(n-2)(n-1) \cdot n}{6}$$

which can be easily reduced to MAHĀVĪRA's form.

NĀRĀYAṆA'S SERIES.—NĀRĀYAṆA has given formulae for the sum of a series whose terms are formed successively by taking the partial sums of other series in the following manner :

Let the symbol ${}^{n}V_1$ denote the arithmetic series of natural numbers up to n terms; *i.e.*, let

$${}^{n}V_1 = 1 + 2 + 3 + \ldots + n.$$

Let ${}^{n}V_2$ denote the series formed by taking the partial sums of the series ${}^{n}V_1$. Then

$${}^{n}V_2 = \sum_{r=1}^{n} {}^{r}V_1.$$

Similarly, let

$${}^{n}V_3 = \sum_{r=1}^{n} {}^{r}V_2,$$

$${}^{n}V_m = \sum_{r=1}^{n} {}^{r}V_{m-1}.$$

The series ${}^{n}V_m$ has been called by NĀRĀYAṆA m-*vāra-saṁkalita* (" *m*-order-series ") meaning therby that the operation of forming a new series by taking the partial sums of a previous series has been repeated m times. The number m may be called the order *(vāra)* of the series.

NĀRĀYAṆA states the sum ${}^{n}V_m$ as follows :

" The terms of the sequence beginning with the *pada* (number of terms, *i.e.* n) and increasing by one taken up to the order *(vāra)* plus one times are successively the numerators, and the terms of the sequence beginning with unity and increasing by one are respectively the denominators. The continued product of these (fractions) gives the *vāra-saṁkalita* (" sum of the iterated series of a given order ").

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Thus according to the above, n being the number of terms of the iterated series, and m the order, we get the following sequence of numbers

$$\frac{n}{1}, \frac{n+1}{2}, \frac{n+2}{3}, \ldots\ldots, \frac{n+m}{m+1}.$$

The sum of the series is the continued product of the above sequence, *i.e.*,

$${}^{n}V_{m} = \frac{n(n+1)(n+2)\ldots,(n+m)}{1 \cdot 2 \cdot 3 \cdot \ldots (m+n)}.$$

Putting $m = 1, 2, 3, \ldots$ we get

$${}^{n}V_{1} = \sum_{1}^{n} r = \frac{n(n+1)}{1.\ 2},$$

$${}^{n}V_{2} = \sum_{1}^{n} {}^{r}V_{1} = \frac{n(n+1)(n+2)}{1.\ 2.\ 3.},$$

$${}^{n}V_{3} = \sum_{1}^{n} {}^{r}V_{2} = \frac{n(n+1)(n+2)(n+3)}{1.\ 2.\ 3.\ 4.},$$

and so on.

NĀRĀYAṆA (1350) has made use of the numbers of the *vāra-saṁkalita* in the theory of combinations, in chapter XIII of his work, the *Gaṇita Kaumudī*. The series discussed above are the series of figurate numbers. They seem to have been first studied in the West by PASCAL (1665).

GENERALIZATION.—NĀRĀYAṆA has considered the more general series obtained in the same way as above from a given arithmetical progression.

Let

$${}^{n}S_{1} = \sum_{1}^{n} \alpha_{r} = \alpha_{1} + \alpha_{2} + \alpha_{3} + \ldots + \alpha_{n},$$

where $\sum_{1}^{n} \alpha_{r}$ is an arithmetic series whose first term is α_{1} and common difference β. As above, let us define the iterated series ${}^{n}S_{2}$, ${}^{n}S_{3}$,... ${}^{n}S_{k}$ as follows :

$${}^{n}S_{2} = \sum_{r=1}^{n} {}^{r}S_{1},$$

$${}^{n}S_{3} = \sum_{r=1}^{n} {}^{r}S_{2},$$

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$$ {}^{n}S_{k} = \sum_{r=1}^{n} {}^{r}S_{k-1}. $$

NĀRĀYAṆA states the formula for the sum of the series thus :

" The sum of the iterated series of the given order derived from the natural numbers equal to the given number minus one, is put down at two places. These become the multipliers. The order as increased by unity being divided by the given number of terms as diminished by unity is a multiplier of the first (of these multipliers). The first term and the common difference multiplied respectively by the two quantities and (the results) added together give the required sum of the iterated series."

Suppose it to be required to find ${}^{n}S_{m}$, where n is the *pada* (number of terms) and m is the *vāra* (order) of the iterated series. Let, as before, ${}^{n}V_{r}$ denote the iterated series of the rth order derived from the series of n natural numbers. Then, taking ${}^{(n-1)}V_{m}$ at two places, and multiplying the first of these by $\frac{m+1}{n-1}$ as directed, we get

$$ \frac{m+1}{n-1} {}^{(n-1)}V_{m} \quad \text{and} \quad {}^{(n-1)}V_{m}. $$

Multiplying the first term, α_1, and the common difference, β, by these two respectively and adding we get the required sum

$$ {}^{n}S_{m} = \alpha_1 \frac{m+1}{n-1} {}^{(n-1)}V + \beta \, {}^{(n-1)}V_{m}. $$

Rationale. The above formula has been evidently obtained by NĀRĀYAṆA as follows :

$$ \begin{aligned} {}^{n}S_{1} = \sum_{1}^{n} \alpha_r &= \alpha_1 + (\alpha_1 + \beta) + \ldots\ldots + (\alpha_1 + \overline{n-1}\,\beta). \\ &= n \left\{ \alpha_1 + \frac{n-1}{2} \beta \right\}, \\ {}^{n}S_{2} = \sum_{1}^{n} {}^{r}S_{1} &= \alpha_1 \sum_{1}^{n} r + \beta \sum_{1}^{n} \frac{r(r-1)}{2} \\ &= \alpha_1 \, {}^{n}V_{1} + \beta \, {}^{(n-1)}V_{2}, \\ {}^{n}S_{3} = \sum_{1}^{n} {}^{r}S_{2} &= \alpha_1 \sum_{1}^{n} {}^{r}V_{1} + \beta \sum_{1}^{n} {}^{(r-1)}V_{2} \\ &= \alpha_1 \, {}^{n}V_{2} + \beta \, {}^{(n-1)}V_{3} \end{aligned} $$

. . . . . .

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$$\therefore \quad {}^{n}S_{m} = \alpha_1 {}^{n}V_{m-1} + \beta {}^{(n-1)}V_{m}$$

$$\text{But } {}^{n}V_{m-1} = \frac{m+1}{n-1} {}^{(n-1)}V_{m}$$

$$\therefore \quad {}^{n}S_{m} = \alpha_1 \frac{m+1}{n-1} {}^{(n-1)}V_{m} + \beta {}^{(n-1)}V_{m}.$$

NĀRĀYAṆA'S PROBLEM.—The above series has been investigated by NĀRĀYAṆA in order to solve the following type of problems :

" A cow gives birth to one calf every year. The calves become young and themselves begin giving birth to calves when they are three years old. O learned man, tell me the number of progeny produced during twenty years by one cow."

Solution :

(i) The number of calves produced during twenty years by the cow is 20.

(ii) The first calf becomes a cow in 3 years and begins giving birth to calves every year, so that the number of its progeny during the period under consideration is (20 — 3) = 17. Similarly the second calf becoming a cow produces during the period under consideration (19 — 3) = 16 calves, and so on. Thus the total number of calves of the second generation is

$$\sum_{1}^{17} r = {}^{17}V_{1}.$$

(iii) The first calf of the eldest cow (of the group of 17) in (ii) produces during the period under consideration (17 — 3) = 14 calves; the second calf of the same group produces 13 calves, and so on. The total progeny (of the second generation) of the group of 17 in (ii) is (14 + 13 + 12 + ... + 1) = ${}^{14}V_{1}$. Similarly the total progeny of the group of 16 in (ii) is ${}^{13}V_{1}$, of the group of 15 in (ii) is ${}^{12}V_{1}$, and so on. Thus the total progeny of the third generation is

$$\sum_{1}^{14} {}^{r}V_{1} = {}^{14}V_{2}.$$

Similarly the total progeny of the fourth generation is

$$\sum_{1}^{(14-3)} {}^{r}V_{2} = {}_{3}{}^{11}V ;$$

and so on.

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The total number of cows and calves at the end of 20 years is

$$1 + 20 + {}^{17}V_1 + {}^{14}V_2 + {}^{11}V_3 + {}^{8}V_4 + {}^{5}V_5 + {}^{2}V_6.$$

$$= 1 + 20 + \frac{17.18}{1.2} + \frac{14.15.16}{1.2.3} + \frac{11.12.13.14}{1.2.3.4} + \frac{8.9.10.11.12}{1.2.3.4.5} + \frac{5.6.7.8.9.10}{1.2.3.4.5.6} + \frac{2.3.4.5.6.7.8}{1.2.3.4.5.6.7}$$

$$= 1 + 20 + 153 + 560 + 1001 + 792 + 210 + 8$$

$$= 2745.$$

After giving the solution of the problem NĀRĀYAṆA remarks: "An alternative method of solution is by means of the *Meru* used in the theory of combinations in connection with (the calculations regarding) metre. This I have given later on."

MISCELLANEOUS RESULTS.—The following results have been given by ŚRĪDHARA, MAHĀVĪRA and NĀRĀYAṆA:

(1) (48) $n^2 = 1 + 3 + 5 + \ldots$ to n terms,

(2) (49) $n^3 = \sum_1^n \left\{3r\,(r-1) + 1\right\}$

(3) $\qquad = 3\sum_1^n r\,(r-1) + n,$

(3) (50) $n^3 = n + 3n + 5n + \ldots$ to n terms,

(4) (51) $n^3 = n^2\,(n-1) + \sum_1^n (2r-1),$

(5) (52) $\left\{(n+3)\frac{n}{4} + 1\right\}(n^2+n) = \sum_1^n r + \sum_1^n r^2 + \sum_1^n r^3 + \sum_1^n \left(\sum_1^r m\right) = \sum_1^n r\left(1 + r + r^2 + \frac{r+1}{2}\right)$

(6) (53) $\sum_1^n r + n^2 = \sum_1^n \left\{2 + 3\,(r-1)\right\}$

(7) $\qquad \sum_1^n r + n^3 = \sum_1^n \left\{(n+1) + (2n+1)\,(r-1)\right\}$

(48) *Triś*, p. 5; *GSS*, ii. 29; *GK*, i. 18.
(49) *Triś*, p. 6; *GSS*, ii. 45; *GK*, i. 22.
(50) *GSS*, ii. 44; *GK*, *śreḍhī-vyavahāra*, 10-11.
(51) *Ibid.*
(52) *GSS*, vii. 309½.
(53) (6), (7) and (8) follow from a rule given by NĀRĀYAṆA, *GK*, *l.c.* 11.

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$$(8) \qquad n^2 + n^3 = \sum_{1}^{n} \left\{ (n+1) + 2(n+1)(r-1) \right\}$$

$$(9)\ (54)\ \mathrm{S} \pm \left(\frac{\mathrm{S}}{a} - n \right) \cdot \frac{m}{r-1} = a + (ar \pm m) + \left\{ (ar \pm m) r \pm m \right\} + \left[\left\{ (ar \pm m) r \pm m \right\} r \pm m \right] + \ldots \text{ to } n \text{ terms}$$

where $\mathrm{S} = a + ar + ar^2 + \ldots$ to n terms.

BINOMIAL SERIES.—The development of $(a + b)^n$ for integral values of n has been known in India from very early times. The case $n = 2$ was known to the authors of the *Śulba Sūtras* (1500 to 1000 B.C.). The series formed by the binomial coefficients

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

seems to have been studied at a very early date. PIṄGALA (*c.* 200 B.C.) a writer on metrics knew the sum of the above series to be 2^n (55). This result is found also in the works of MAHĀVĪRA (56) (850), PṚTHUDAKASVĀMĪ (57) (860) and all later writers.

PASCAL TRIANGLE.—The so-called PASCAL triangle was known to PIṄGALA who explained the method of formation of the triangle in short aphorisms *(sūtra)*. These aphorisms have been explained by the commentator HALĀYUDHA thus :

" Draw one square at the top; below it draw two squares so that half of each of them lies beyond the former on either side of it. Below them, in the same way, draw three squares; then below them four; and so on up to as many rows as are desired : this is the preliminary representation of the *Meru*. Then putting down 1 in the first square, the figuring should be started. In the next two squares put 1 in each. In the third row put 1 in each of the extreme squares, and in the middle square put the sum of the two numbers in the two squares of the second row. In the fourth row put 1 in each of the two extreme squares : in an intermediate square put the sum of the numbers in the two squares of the previous row which lie just above it. Putting

(54) *GSS*, vi. 314.
(55) PIṄGALA, *Chandah Sūtra*, viii. 23-27.
(56) *GSS*, ii. 94.
(57) *BrSpSi*, xii. 17 commentary.

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down of numbers in the other rows should be carried on in the same way. Now the numbers in the second row of squares show the monosyllabic forms : there are two forms each consisting of one long and one short syllable. The numbers in the third row give the disyllabic forms : in one form all syllables are long, in two forms one syllable is short (and the other long), and in one all syllables are short. In this row of the squares we get the number of variations of the even verse. The numbers in the fourth row of squares represent trisyllabic forms. There one form has all syllables long, three have one syllable short, three have two short syllables, and one has all syllables short. And so on in the fifth and succeeding rows; the figure in the first square gives the number of forms with all syllables long, that in the last all syllables short, and the figures in the successive intermediate squares represent the number of forms with one, two, etc. short syllables."

Thus according to the above, the number of variations of a metre containing n syllables will be obtained from the representation of the *Meru* as follows :

Number of syllables		Total number of variations
	1	
1 . . .	1 1	. . $2 = 2^1$
2 . . .	1 2 1	. . $4 = 2^2$
3 . . .	1 3 3 1	. . $8 = 2^3$
4 . . .	1 4 6 4 1	. . $16 = 2^4$
5 . . .	1 5 10 10 5 1	. . $32 = 2^5$
6 . . .	1 6 15 20 15 6 1	. . $64 = 2^6$

(Meru prastāra)

From the above it is clear that PIṄGALA knew the result

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n.$$

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INFINITE SERIES.—The results given above regarding finite series indicate the extent to which the theory of series had advanced in Northern India by the middle of the fourteenth century. The mathematicians of South India, especially those of Kerala seem to have made great progress and appear to have been in possession of the infinite geometric series and other simple infinite series by the end of the fourteenth century. Sometime during the first half of the fifteenth century they discovered what is now known as GREGORY's series. Use of this series seems to have been made for the calculation of π, and in astronomy. As the works of this period are not available, it is not possible to trace the gradual evolution of the infinite series in India.

TALAKULATTURĀ'S SERIES.—TALAKULATTURĀ NAMBUTIRĪ (1432) discovered an infinite series for the arc of a circle in terms of its sine and cosine and the radius of the circle. He says :

" By the method stated before for the calculation of the circle, the arc corresponding to a given value of the sine can be found. Multiply the given value *(iṣṭa)* of the sine *(jyā)* by the radius and divide by the cosine *(koṭijyā)*. The result thus obtained is the first quotient. Then operating again and again with the square of the (given) sine as the multiplier and the square of the cosine as the divisor, obtain from the first quotient other quotients. Divide the successive quotients by the odd numbers 1, 3, 5, etc., respectively. Now subtract the even order of quotients from the odd ones. The remainder is the arc required " (58).

That is to say, if r denote the radius of a circle, α an arc of it and θ the angle subtended at the centre by that arc, then

$$r\theta = \alpha = \frac{r \operatorname{Sin} \theta}{1.\operatorname{Cos} \theta} - \frac{r.\operatorname{Sin}^3\theta}{3.\operatorname{Cos}^3\theta} + \frac{r.\operatorname{Sin}^5\theta}{5.\operatorname{Cos}^5\theta} - \frac{r \operatorname{Sin}^7\theta}{7.\operatorname{Cos}^7\theta} + \cdots$$

The series is convergent if $\sin \theta < \cos \theta$, i.e., $\theta < \frac{\pi}{4}$. But if $\theta > \frac{\pi}{4}$, the series is divergent and so the rule appears to fail. If in that case, however, we take $\sin\left(\frac{\pi}{2} - \theta\right)$ as given instead of $\sin \theta$, then in accordance with the rule, we get the series

(58) The occurrence of this series in the *Tantra saṁgraha* by NAMBUTIRĪ was first pointed out by WHISH, *Trans. Royal Asiatic Soc.*, III (1835).

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$$\frac{r\pi}{2} - \alpha = \frac{r \operatorname{Sin}\left(\frac{\pi}{2} - \theta\right)}{1.\operatorname{Cos}\left(\frac{\pi}{2} - \theta\right)} - \frac{r \operatorname{Sin}^3\left(\frac{\pi}{2} - \theta\right)}{3.\operatorname{Cos}^3\left(\frac{\pi}{2} - \theta\right)} - \frac{r \operatorname{Sin}^5\left(\frac{\pi}{2} - \theta\right)}{5.\operatorname{Cos}^5\left(\frac{\pi}{2} - \theta\right)} - \ldots$$

$$\text{or} \quad \frac{r\pi}{2} - \alpha = \frac{r \operatorname{Cos}\theta}{1.\operatorname{Sin}\theta} - \frac{r \operatorname{Cos}^3\theta}{3.\operatorname{Sin}^3\theta} + \frac{r \operatorname{Cos}^5\theta}{5.\operatorname{Sin}^5\theta} - \ldots$$

which is convergent. Knowing the value of $\frac{r\pi}{2} - \alpha$, we can easily calculate the value of α. Thus the rule gives the desired result even in case $\theta > \frac{\pi}{4}$. Hence the author remarks:

" Of the arc and its complement, take the sine of the smaller as given *(iṣṭa)*; then performing repeatedly the operations (described), one obtains a similar result."

The above series is stated also by PATHUMANA SOMAYĀJĪ (1733) and ŚAṄKARAVARMAṆA (*c.* 1700?). The former writes :

" Find the first quotient by dividing by the cosine the given sine as multiplied by the radius. Then get the other quotients by multiplying the first and those successively resulting by the square of the sine and dividing them in the same way by the square of the cosine. Now dividing these quotients respectively by 1, 3, 5, etc., subtract the sum of even ones (in the series) from the sum of the odd ones. Thus the sine will become the arc " (59).

ŚAṄKARAVARMAṆA says :

" Divide the product of the radius and the sine by the cosine. Divide this quotient and others resulting successively from it on repeated multiplication by the square of the sine and division by the square of the cosine, by 1, 3, 5, etc., respectively. Then subtract the sum of the even quotients (in the series) from the sum of the odd ones. The remainder is the arc (required) " (60).

Introducing the modern tangent function, the above series can be written as

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots$$

The series was rediscovered by JAMES GREGORY in 1671 and

(59) *Karaṇapaddhati* (MS.), vi. 18.
(60) *Sadratnamālā* (MS.), iv. 11.

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then by LEIBNITZ in 1673. It is now generally attributed to the former. But rightly speaking the series should be credited to TALAKULATTURĀ NAMBUTIRĪ.

SERIES FOR THE SINE AND COSINE OF AN ARC.—The Hindus discovered series for the sine and cosine of an angle in powers of its circular measure. For example, PATHUMANA writes :

" In the series of quotients obtained by dividing an arc of a circle severally by 2, 3, etc., times the radius, multiply the arc by the first (term); the resulting product by the second (term); this product again by the third (term); and so on. Put down the even terms of the sequence so obtained after the arc and the odd ones after the radius, and subtract the alternate ones. The remainders will respectively be the sine and cosine of that arc " (61).

Let α be an arc of a circle of radius r, and let it subtend an angle θ at the centre. Proceeding according to the above rule we get the series of quotients

$$\frac{\alpha}{2.r}, \frac{\alpha}{3.r}, \frac{\alpha}{4.r}, \frac{\alpha}{5.r} \quad \ldots\ldots \quad (A)$$

Then, multiplying as directed we get the following sequence :

$$\frac{\alpha^2}{2.r}, \frac{\alpha^3}{2.3.r^2}, \frac{\alpha^4}{2.3.4.r^3}, \ldots\ldots \quad (B)$$

Putting down even terms of (B) after the arc (α), we have

$$\alpha, \frac{\alpha^3}{3!r^2}, \frac{\alpha^5}{5!r^4}, \frac{\alpha^7}{7!r^6}, \ldots\ldots \quad (C)$$

and putting down odd terms after the radius (r), we get

$$r, \frac{\alpha^2}{2!r}, \frac{\alpha^4}{4!r^3}, \frac{\alpha^6}{6!r^5}, \ldots\ldots \quad (D)$$

Then in (C) subtracting alternate terms we get

$$\textit{Jyâ}\ \alpha = r \operatorname{Sin}\theta = \alpha - \frac{\alpha^3}{3!r^2} + \frac{\alpha^5}{5!r^4} - \ldots$$

$$\text{or} \quad \operatorname{Sin}\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

(61) *Karanapaddhati*, vi. 12 f.

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Similarly from (D) we have

$$Kotijyâ\ \alpha = r \operatorname{Cos} \theta = r - \frac{\alpha^2}{2!\,r} + \frac{\alpha^4}{4!\,r^3} - \frac{\alpha^6}{6!\,r^5} + \ldots$$

or $$\operatorname{Cos} \theta = 1 - \frac{\theta^2}{2!} \quad \frac{\theta^4}{4!} \quad \frac{\theta^6}{6!} + \ldots$$

These series appear also in the works of ŚAṄKARAVARMAṆA (62).

List of Abbreviations

1. *Ā* — *Āryabhaṭīya*, edited by H. KERN, Leiden, 1874.
2. *BrSpSi* — *Brâhma-sphuṭa Siddhānta*, edited by S. DVIVEDI, Benares, 1902.
3. COLEBROOKE — *Algebra with arithmetic and mensuration from the Sanscrit of Brahmegupta and Bhascara*, London, 1817.
4. *GK* — *Gaṇita Kaumudī* (unpublished) of NĀRĀYAṆA.
5. *GSS* — *Gaṇita-sāra-saṁgraha*, edited by M. RAṄGĀCĀRYA, Madras, 1912.
6. *L* — *Līlāvatī*, edited by S. DVIVEDI, Benares (reprint), 1912.
7. *MSi* — *Mahāsiddhānta*, edited by S. DVIVEDI, Benares, 1910.
8. *Triś* — *Triśatikā*, edited by S. DVIVEDI, Benares, 1899.

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(62) *Sadratnamālā*, iv. 5

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Paper 25

K. Mukunda Marar (1940)

Derivation of the Value of π - How the Ancients Did It

The Teachers' Magazine: The Journal of the Cochin Teachers' Association, Vol. XV, no. 1, 1940, pp. 28–34.

Notes

It can be safely presumed that the modern assessment of Mādhava's work began with this paper. However, the journal in which it was published might have had a circulation only in a very small geographical region in India (the name of the journal suggests that it was a publication of the teachers' association of the erstwhile princely state of Cochin) it would be very unlikely that it caught the attention of the scholarly world outside of that small geographical region.

The paper presents in modern notations the proof of the Mādhava-Leibniz series as given in *Yuktibhāṣa*.

K. Mukunda Marar

According to the American genealogy website geni.com, K. Mukunda Marar (Mukundan Kottilil) was born in the year 1922 as the son of Ramavarma Maruthampuran (1894–1974) and Parukutty Kottilil. So Mukunda Marar must have been an eighteen-year old student when he published this historical paper. It was Ramavarma Maruthampuran who, along with A. R. Akhileswara Aiyer, published for the first time Jyeṣṭhadeva's Malayalam *Yuktibhāṣā* (Part I) with notes and explanations in modern mathematical notations in the year 1948. Mukunda Marar was married to Rashmi Dighe and had two children, Ramachandran (Ramu) and Meena. Mukunda Marar died in the year 2000.

Source: Internet Archive

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DERIVATION OF THE VALUE OF π — HOW THE ANCIENTS DID IT

BY

(*K. MUKUNDA MARAR, TRICHUR.*)

[What is given below is a geometrical method of deriving the value of π taken from the ancient Hindu work, *Yuktibhasa*]

The ancient Hindu mathematicians obtained the value of π from the fact that it is the ratio of the circumference of a circle to its diameter. By comparing the lengths of the circumference of any circle and its diameter the value of π is derived.

Construction. A circle, with centre O, is inscribed in a square X Y Z V. Let A Z, a half side of the square, be divided into a great number of equal parts at B, C, D, E, F.....(In the figure for the sake of illustration AZ is divided into only six equal parts.) Join O to A, B, C.....Let the circle cut OB, OC, OD.....at b, c, d, ...From A, B, C, D.....drop perpendiculars Ap, BQ, CR, DS, ET... on OB,OC, OD, OE, OF.....etc. From b, c, d, e.....drop perpendiculars bq, cr, ds, et, on OC, OD, OE, OF etc.

In a right angled triangle, according to the ancient Hindu terminology, the hypotenuse is called the കർണ്ണം, the base is the ഭുജം and the perpendicular to ഭുജം is known as കോടി.

Hence in the right angled triangles OAB, OAC, OAD.....etc.

the കോടി is OA in all cases

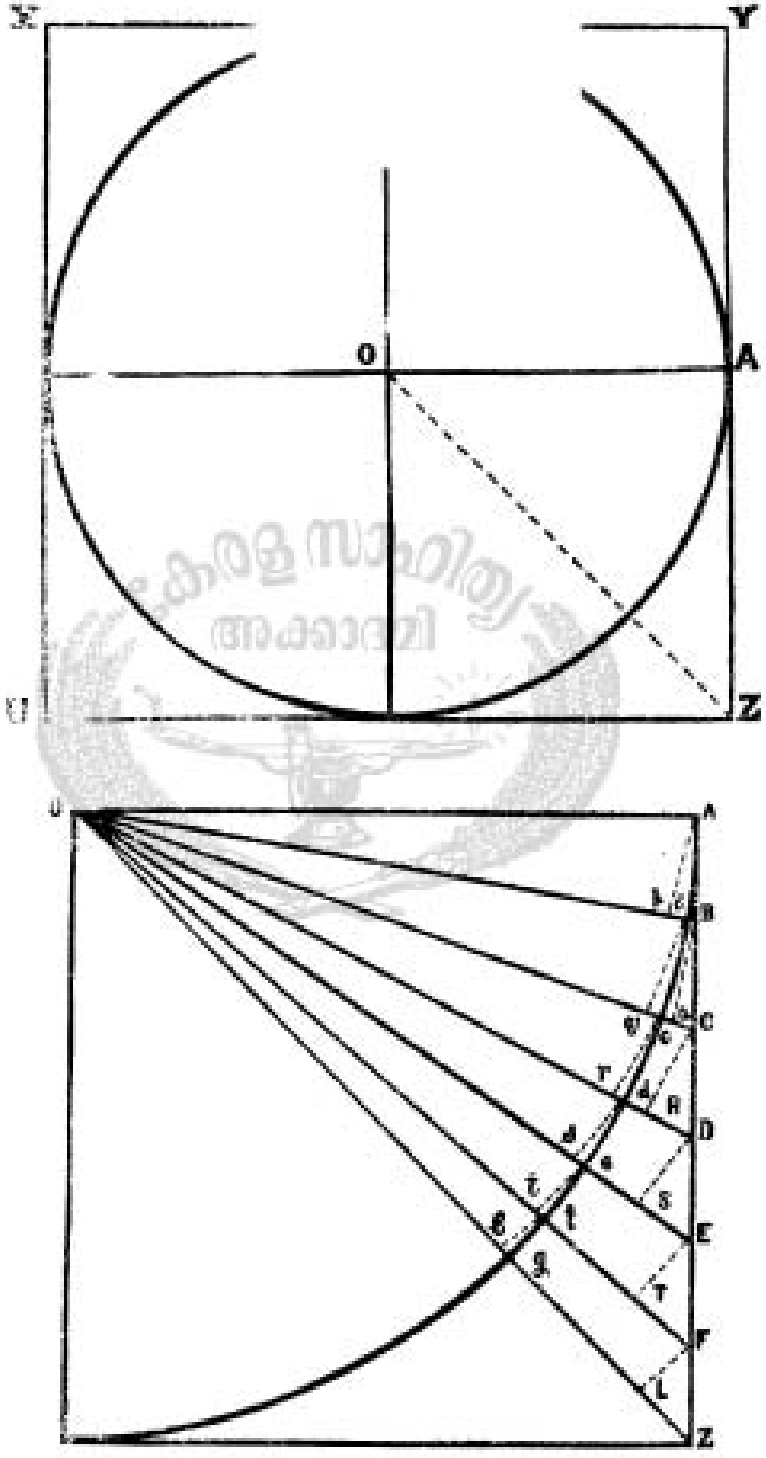
the കർണ്ണങ്ങൾ are OB, OC, OD OE.....

and the ഭുജങ്ങൾ AB, AC (= 2 AB), AD (= 3 AB). .....

In a right angled triangle ഭുജവർഗ്ഗവും കോടിവർഗ്ഗവും കർണ്ണവർഗ്ഗം. [This property of the right angled triangle can be very easily proved by Hindu Method]

$$\therefore \quad OB^2 = OA^2 + AB^2$$
$$OC^2 = OA^2 + AC^2$$
$$OD^2 = OA^2 + AD^2$$
$$OE^2 = OA^2 + AE^2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{ etc.}$$

DERIVATION OF THE VALUE OF π



It seems that the ancient mathematicians had not a good conception of angles and their measurements. Yet they had a good idea of similar triangles and their properties.

The perpendicular from a point on a st. line is called the [illegible]. (See figures on page 29.)

The triangles OAp and OAB are similar.

$$\therefore \frac{Ap}{AB} = \frac{OA}{OB}$$

$$\therefore Ap = \frac{OA.\ AB}{OB.} = \frac{OA^2.\ AB}{OA.\ OB}$$

Similarly,

$$\therefore BQ = \frac{OA.\ BC.}{OC.}$$

$$CR = \frac{OA.\ CD}{OD} \quad \text{.....and so on.}$$

Again triangles OBQ and Obq are similar.

$$\frac{bq}{BQ} = \frac{Ob}{OB}$$

$$\therefore bq = \frac{Ob.\ BQ}{OB}$$

$$bq = \frac{Ob}{OB} \cdot \frac{OA.\ BC}{OC} \quad \text{(substituting value of BQ)}$$

$$= \frac{OA^2.\ BC}{OB.\ OC} \quad \text{(Ob=OA both being radii of circle)}$$

Similarly, $Cr = \frac{OA^2.\ CD}{OC.\ OD}$

$$ds = \frac{OA^2.\ DE}{OD.\ OE}$$

$$et = \frac{OA^2.\ EF}{OE.\ OF}$$

$$fl = \frac{OA^2.\ FZ}{OF.\ OZ} \quad \text{and so on.}$$

Since AZ is divided into a great number of equal parts Ap, bq, cr, ds..... etc., may be taken as the small arcs intercepted between the respective [illegible]s. Also AB = BC = CD=DE.....

DERIVATION OF THE VALUE OF

Hence in the expressions far the value of Ap, bq, cr, ds...etc., the ഗുണ്യങ്ങൾ (i. e. multiplicants) BC, CD, DE.....are all equal. Also OA^2 (which is the ഗുണകം or multiplier) is the same in all cases. But the ഹരണങ്ങൾ or divisors are different.

Since AZ is divided into a great number of equal parts AB, BC, CD.....are exceedinlgy small so that (OB—OA) is a very small quantity. Hence $(OB-OA^2)$ is still smaller which is practically zero.

$$\therefore \quad OB^2 + OA^2 = 2\,OA.\,OB$$

$$\therefore \quad OA.\,OB = \frac{OA^2+OB^2}{2}$$

$$\therefore \frac{1}{OA.\,OB} = \frac{OA.\,OB}{OA^2OB^2} = \frac{OA^2+OB^2}{2\,OA^2OB^2} = \frac{1}{2}\left(\frac{1}{OA^2} + \frac{1}{OB^2}\right)$$

Similarly, $\frac{1}{OB.\,OC} = \frac{1}{2}\left(\frac{1}{OB^2} + \frac{1}{OC^2}\right)$

$$\frac{1}{OC.\,OD} = \frac{1}{2}\left(\frac{1}{OC^2} + \frac{1}{OD^2}\right)$$

.....and so on.

$$\frac{\text{Circumference of circle}}{8} = Ag$$

$$= Ap + bq + cr + ds.....$$

$$= \frac{OA^2.\,AB}{OA.\,OB} + \frac{OA^2.\,BC}{OB.\,OC} + \frac{OA^2.\,CD}{OC.\,OD} \cdots\cdots\cdots\cdots$$

$$= \frac{1}{2}\left\{OA^2.\,AB\left(\frac{1}{OA^2} + \frac{1}{OB^2}\right) + OA^2\,BC\left(\frac{1}{OB^2} + \frac{1}{OC^2}\right)\right.$$

$$\left. + OA^2.\,CD\left(\frac{1}{OC^2} + \frac{1}{OD^2}\right) \cdots\cdots\cdots\cdots\right\}$$

$$= \frac{1}{2}\left\{\frac{OA^2.\,AB}{OA^2} + \frac{OA^2.\,BC}{OB^2} + \frac{OA^2.\,CD}{OC^2} \cdots\cdots\cdots\cdots\right\}$$

$$+ \frac{1}{2}\left\{\frac{OA^2.\,AB}{OB^2} + \frac{OA^2.\,BC}{OC^2} + \frac{OA^2.\,CD}{OD^2} + \cdots + \frac{OA^2.\,AB}{2\,OA^2}\right\}$$

The last step is obtained by splitting each of the above terms and forming two series. The first term of the first series is AB and the last term of the last series is $\frac{AB}{2}$

But here $\frac{1}{2}\left(AB+\frac{OA^2.BC}{OB^2}+\frac{OA^2.CD}{OC^2}\cdots\cdots\right)$

$-\frac{1}{2}\left(\frac{OA^2.AB}{OB^2}+\frac{OA^2.BC}{OC^2}+\frac{OA^2.CD}{OD^2}+\cdots\cdots+\frac{AB}{2}\right)$

$=\frac{1}{2}AB-\frac{1}{4}AB$

$=\frac{1}{4}AB.$ Which is practically zero (since AB is very small).

$\therefore \frac{1}{2}\left(\frac{OA^2.AB}{OA^2}+\frac{OA^2.BC}{OB^2}+\frac{OA^2.CD}{OC^2}\cdots\right)=\frac{1}{2}\left(\frac{OA^2.AB}{OB^2}\right.$

$\left.+\frac{OA^2.BC}{OC^2}+\frac{OA^2.CD}{OD^2}\cdots\cdots\right)$

$\therefore \frac{\text{Circumference}}{8}=\frac{OA^2.AB}{OB^2}+\frac{OA^2.BC}{OC^2}+\frac{OA^2.CD}{OD^2}$

$+\frac{OA^2.DE}{OE^2}\cdots\cdots$

But $\frac{OA^2.AB}{OB^2}=\frac{OA^2.AB}{OA^2}-\frac{AB(OB^2-OA^2}{OB^2}$

$=\frac{OA^2.AB}{OA^2}-\frac{AB.AB^2}{OB^2}$

$=\frac{OA^2.AB}{OA^2}-\frac{AB.AB^2}{OA^2}+\frac{AB.AB^4}{OA^4}\cdots\cdots$

$=AB-\frac{AB.AB^2}{OA^2}+\frac{AB.AB^4}{OA^4}-\frac{AB.AB^6}{OA^6}\cdots$

Similarly,

$\frac{OA^2.BC}{OC^2}=BC-\frac{BC.AC^2}{OA^2}+\frac{BC.AC^4}{OA^4}-\frac{BC.AC^6}{OA^6}\cdots$

$\frac{OA^2.CD}{OD^2}=CD-\frac{CD.AD^2}{OA^2}+\frac{CD.AD^4}{OA^4}-\frac{CD.AD^6}{OA^6}\cdots$

$\frac{OA^2.DE}{OE^2}=DE-\frac{DE.AE^2}{OA^2}+\frac{DE.AE^4}{OA^4}-\frac{DE.AE^6}{OA^6}\cdots$

and so on for each of the other terms.

DERIVATION OF THE VALUE OF π

Here there is no limit to this series,

$$\frac{\text{Circumference}}{8} = Ap + bq + cr + ds \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{OA^2 \cdot AB}{OB^2} + \frac{OA^2 \cdot BC}{OC^2} + \frac{OA^2 \cdot CD}{OD^2} \ldots\ldots\ldots$$

$$= \left(AB - \frac{AB \cdot AB^2}{OA^2} + \frac{AB \cdot AB^4}{OA^4} - \frac{AB \cdot AB^6}{OA^6}\right) \ldots\ldots$$

$$+ \left(BC - \frac{BC \cdot AC^2}{OA^2} + \frac{BC \cdot AC^4}{OA^4} - \frac{BC \cdot AC^6}{OA^6}\right) \ldots\ldots\ldots$$

$$+ \left(CD - \frac{CD \cdot AD^2}{OA^2} + \frac{CD \cdot AD^4}{OA^4} - \frac{CD \cdot AD^6}{OA^6}\right) \ldots\ldots\ldots$$

$\ldots\ldots\ldots\ldots\ldots\ldots$ and so on.

$$(AB+BC+CD\ldots\ldots) - \left(\frac{AB \cdot AB^2}{OA^2} + \frac{BC \cdot AC^2}{OA^2} + \frac{CD \cdot AD^2}{OA^2}\right) + \left(\frac{AB \cdot AB^4}{OA^4} + \frac{BC \cdot AC^4}{OA^4} + \frac{CD \cdot AD^4}{OA^4} \ldots\ldots\right)$$

$$- \left(\frac{AB \cdot AB^6}{OA^6} + \frac{BC \cdot AC^6}{OA^6} + \frac{AC \cdot AD^6}{OA^6} \ldots\ldots\right) \ldots\ldots\text{etc.}$$

But since, $AC = 2\,AB$

$AD = 3\,AB$

$AE = 4\,AB$ and so on,

$$\frac{AB \cdot AB^2}{OA^2} + \frac{BC \cdot AC^2}{OA^2} + \frac{CD \cdot AD^2}{OA^2} \ldots\ldots\ldots\ldots$$

$$= \frac{AB^3}{OA^2}(1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots) = \frac{AB^3}{OA^2} \cdot \frac{n^3}{3}$$

for $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \dfrac{2n^3 + 3n^2 + n}{6}$

As n is very great n^2 and n are very small when compared to n^3.

Similarly we can prove that, if n is very great,

$$1^3 + 2^3 + 3^3 + 4^3 \ldots\ldots + n^3 = \frac{n^4}{4}$$

$$1^4 + 2^4 + 3^4 + 4^4 \ldots\ldots + n^4 = \frac{n^5}{5}$$

$$1^5 + 2^5 + 3^5 + 4^5 \ldots\ldots + n^5 = \frac{n^6}{6}$$

and so on.

$$\therefore \frac{\text{Circumference}}{8} = (AB + BC + CD + DE.....)$$

$$- \frac{AB^3}{OA^2}(1^2 + 2^2 + 3^2 .....) + \frac{AB^5}{OA^4}(1^4 + 2^4 + 3^4)$$

$$- \frac{AB^7}{OA^6}(1^6 + 2^6 + 3^6 .....) + \frac{AB^9}{OA^8}(1^8 + 2^8 + 3^8 .....)$$

.....etc.

$$= R - \frac{AB^3}{OA^2}\frac{n^3}{3} + \frac{AB^5}{OA^4}\frac{n^5}{5} - \frac{AB^7}{OA^6}\frac{n^7}{7} .....$$

where R is radius of circle

$$= R - \frac{(n.\,AB)^3}{3\,OA^2} + \frac{(n.\,AB)^5}{5\,OA^4} - \frac{(n\,AB)^7}{7\,OA^6} .....$$

$$= R - \frac{R^3}{3\,OA^2} + \frac{R^5}{5\,OA^4} - \frac{R^7}{7\,OA^6} .....$$

$$= R - \frac{R^3}{3\,R^2} + \frac{R^5}{5\,R^4} - \frac{R^7}{7\,R^6} ..... \text{for } OA = AZ = R.$$

$$= R - \frac{R}{3} + \frac{R}{5} - \frac{R}{7} .....$$

$$= R\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} .....\right)$$

Circumference $= 8R\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} ...\right)$

$$\text{Circumference} = \frac{\text{Circumference}}{\text{Diameter}} = \frac{\text{Circumference}}{2R}$$

$$= 4\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} .....\right)$$

But, $\frac{\text{Circumference}}{\text{Diameter}} = \pi$

$$\therefore \pi = 4\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + ...\right)$$

Thus the value of π is derived as,

$$\pi = 4\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + ...\right)$$

This value is the same as that obtained from Gregory's series.

$$\text{Tan}^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \text{ when } x = 1$$

$$\therefore \tan^{-1} 1 = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + ...$$

$$\text{i. e., } \frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + ...$$

$$\therefore \text{i. e., } \pi = 4\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + ...\right)$$

Paper 26

K. Mukunda Marar and C. T. Rajagopal (1944)

On the Hindu Quadrature of the Circle (with an appendix by K. Balagangadharan)

Journal of the Bombay Branch of the Royal Asiatic Society (New Series), Vol. 20, 1944, pp. 65–82.

Notes

According to David Edwin Pingree, an American historian of mathematics in the ancient world, it was this paper that started the process of the modern assessment of Mādhava's work. (David Pingree, The logic of non-Western science: mathematical discoveries in medieval India, Daedalus Fall 2003. Available at direct.mit.edu.)

It was the publications of C. T. Rajagopal and his co-workers during the period 1940–1950 that brought out in modern mathematical terms, all the important demonstrations for the infinite series for π, the sine and the cosine functions, as given in the Malayalam text *Yuktibhāṣā*.

C. T. Rajagopal

C. T. Rajagopal (1903–1978) taught mathematics at Madras Christian College from 1931 to 1951 and joined the Ramanujan Institute for Advanced Study in Mathematics in 1951 and was appointed as its Director in 1955. He retired from the Institute in June 1969 but continued his research work with a small financial assistance through the Institute. In mathematics, Rajagopal's areas of specialisation were sequences, series, summability, functions of a complex variable. Another topic great interest to him was the history of medieval Indian mathematics. M. S. Rangachari who was a student, colleague and collaborator of Rajagopal, observes: "Rajagopal took upon himself the task of establishing, to the satisfaction of mathematicians and mathematical historians, that the fundamentals of mathematical analysis were discovered and used in Kerala at least two centuries before Newton and Leibniz. He was assisted in this task by the late Prince Rama Varma of Cochin and his own students."

K. Mukunda Marar

For a biographical sketch of K. Mukunda Marar see p. 239.

K. Balagangadharan

Balagangadharan (1922–2012) hailed from a royal family, being the son of Ramavarma Kuttan Thampuran (a prince of the royal family of Cochin) and Kalyanikutty Nethyaramma. He obtained his Master's degree in Mathematics (1940–43) from the University of Madras. He continued to stay at Madras Christian College where he studied for his Master's degree to work as a tutor during 1943–45 and subsequently moved to Wadia College, Pune where he worked during the years 1945–51. Balagangadharan is said to have prepared an English translation of *Yuktibhāṣā* of Jyeṣṭadeva and given to C. T. Rajagopal during their mathematical interactions. It appears that the manuscript of the work has been irretrievably lost. Balagangadharan was with the Tata Institute of Fundamental Research (TIFR), Mumbai, during the period from 1951 till 1982. After retirement from TIFR, Balagangadharan worked as a professor of mathematics during the period 1984–1994.

Source: Internet Archive

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ON THE HINDU QUADRATURE OF THE CIRCLE

By K. Mukunda Marar *and* C. T. Rajagopal

"All history is the history of thought." There can be nothing else except thought that can be the object of historical knowledge.—R. G. Collingwood.

1. This paper is a sequel to an article bearing the same title contributed more than a hundred years ago to the *Transactions of the Royal Asiatic Society of Great Britain and Ireland*, by Charles M. Whish of the Hon. East India Company's Civil Service in the Madras Establishment.[1] The article of Whish has come to be accepted as one of our chief sources of information concerning Hindu achievements in "circle squaring", but the questions it raises with regard to the date of these achievements have still to be answered. If the date given by Whish is accepted, the first significant Hindu investigations of π (the ratio of the circumference of a circle to its diameter) must have preceded by some two centuries the efforts of Newton, Leibniz and Gregory in the same direction. Such a conclusion is no doubt provisional until we have established the antiquity as well as the indigenous origin of the four "śāstras" from which Whish draws his material: *Tantrasaṅgraha*, *Yukti-Bhāṣa*, *Karaṇa-Paddhati* and *Sadratnamālā*. Nevertheless the existence of these mathematical works,[2] with a main theme which is supposed to be five hundred years old—a supposition which has not been disproved thus far—must make one wary of any assertion that "after the time of Bhāskara (born A.D. 1114) no Indian mathematical work of historical value or interest is known."[3] Even when such statements issue from a historian of Indian mathematics, of the eminence of G. R. Kaye, one has the uneasy feeling that a difficult question of history has been dealt with brusquely, rather in the manner of the Scottish preacher who said: "My friends, we will look this difficulty boldly in the face—and pass on."

Whether we choose to defend Kaye or to attack him, we have to reckon with difficulties which cannot be merely stared out of countenance. Before we can hope to tackle them with any assurance, we need all the help which students of Sanskrit, historians of Malabar and Indologists in general can give us. Our purpose, in what follows, is not to offer a solution of the difficulties, but to bring them vividly to the fore. This we propose to do by filling in the details of a picture which Whish has given in outline—an undertaking which Whish himself contemplated in a sequel to his essay. After enunciating the rules, embodied in Sanskrit ślokās, for expressing π in infinite series of certain forms and for

[1] "On the Hindú Quadrature of the Circle, and the infinite series of the proportion of the circumference to the diameter exhibited in the four Śástras, the Tantrasangraham, Yucti-Bhásá, Carana Padhati and Sadratnamála," *TRAS.*, 3 (1835), 509–523. This article will hereafter be denoted by the abbreviation *TRAS* and the four "Śāstras" of Whish by *T*, *Y*, *K*, *S* respectively, wherever it is found convenient. These works, with the exception of the third, are available only in manuscript. Our study of the relevant portions of the works and the literature relating to the works has been helped not a little by the devoted labours of Prince Rāma Varma (12th prince of Cochin) and Mr. K. Balagangadharan, one of our colleagues. We wish to express our deep gratitude for all that these two gentlemen have done for us.

[2] *Karaṇa-Paddhati* has been recently published in the Trivandrum Sanskrit Series. Our references *K* (T.S.S.) in the following pages are to this volume. MSS. of *Tantrasaṅgraha*, *Yukti-Bhāṣā* and *Sadratnamālā* are available in the Government Oriental Manuscripts Library (Tirupati): (i) *T* (Sanskrit) with commentary, No. R 3205; (ii) *T* (Sanskrit), No. D 13426; (iii) *Y* (*Gaṇita-Yukti-Bhāṣa*-Malayalam) No. D 332; (iv) *S* (Sanskrit), No. R 4448; (v) *S*, Vyākhyā without the text (Malayalam), No. D 243. The Adyar Library contains (i) *T* (Sanskrit), No. 34 I. 3, (ii) *S* (Sanskrit), No. 21 B. 6. The Director of the Library has been kind enough to permit us to check some of Whish's quotations with reference to these MSS.

[3] G. R. Kaye, *Indian Mathematics* (Calcutta, 1915), 24.

calculating its value therefrom to any degree of approximation, he planned to publish the demonstrations of the rules as given in *Yukti-Bhāṣa*,[4] the Malayaḷam commentary on the mathematical texts of *Tantrasaṅgraha* and *Karaṇa-Paddhati*. But he died before he could complete the task he had set himself[5] and left a gap in the history of Indian mathematics accessible to the English-speaking world. In the hundred years from Whish's death until now there would appear to have been but a single attempt to fill the gap, viz., Mukunda Marar's essay, "Derivation of the value of π—how the ancients did it", in a recent number of the Journal of Cochin Teachers' Association.[6]

1.1. The first section of our paper gives the geometrical proof, substantially as it stands in *Yukti-Bhāṣa*, of Gregory's series for the inverse tangent which is the origin of all the series for π in Whish's article. To enable readers to study the proof in the proper historical perspective, we have indicated the lines along which Gregory and Leibniz, independently of each other, arrived at the series which has been named after the former.

1.2. Our second section is devoted exclusively to Gregory's series for π and the method employed by Hindu mathematicians, as it is set forth in *Yukti-Bhāṣa*, to convert this alternating series into others more rapidly convergent and consequently more useful in evaluating π. Stated in the language of modern mathematics, the Hindu method is seen to be sufficiently general to convert Gregory's alternating series for π into another in which the general term is $O(1/n^{2p+1})$ or $O(1/n^{2p+2})$ as $n \to \infty$, p being any positive integer.

1.3. In our concluding section we touch on the questions of date and authorship of Whish's manuscript sources. Since the answers to the questions are to be sought mainly in the field of Kerala literature and tradition, we have brought together our gleanings in this field for the sake of the non-Keralīya who may not have ready access to them from a single source.

I. *Yukti-Bhāṣa* proof of Gregory's series.

2. Gregory's series for the inverse tangent can be written thus: for $|\arctan t| \leqslant \pi/4$,

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots \qquad [K, S.]$$

It is formulated in certain Sanskrit ślokās which Whish quotes from the two manuscript works shown in brackets.[7] On the basis of the material furnished by Whish, we may say that *Tantrasaṅgraha* too reveals a knowledge of the series, implicitly if not explicitly. For it contains enunciations, in the form of ślokās, of two special cases of the series: the limiting case $t = 1$ [*T*, *K*][8] and the particular case $t = 1/\sqrt{3}$ [*T*, *S*].[9] *Yukti-Bhāṣa* gives a geometrical proof of the former case which, with a very slight and obvious alteration in the figure, is seen to cover all cases. This alteration is almost the only point of departure from its original of the proof given below.[10]

[4] *TRAS*, 523.

[5] *JRAS*, 3 (1836) contains a reference to "Mr. C. M. Whish of the Madras Civil Service, deceased", in connexion with the donation of his collection of manuscripts to the Society by his brother J. C. Whish (*vide* proceedings of the Society, July 2, 1836, p. lxxiii).

[6] *Teachers' Magazine*, 15 (1940), 28–34.

[7] *TRAS*, 518, 520. Any reference hereafter to the source of a theorem, consisting in the abbreviation for the source placed within square brackets [ ], may be taken to indicate the fact that Whish has given the enunciation of the theorem in śloka form as found in the source. Gregory's series is given in *K* (T.S.S.), 19, Chap VI, verse 18. To facilitate reference, Whish's ślokās, as well as one or two others, relevant for our purpose, are collected together in an Appendix by Mr. K. Balagangadharan.

[8] *TRAS*, 516. Also *K* (T.S.S.), 10, Chap. VI, verse 1.

[9] *TRAS*, 513*f*, 519.

[10] The reader is referred to Mukunda Marar's article in *Teacher's Magazine* for the actual proof in *Yukti-Bhāṣa*.

2.1. *Yukti-Bhāṣa* bases its proof on the two lemmas which follow. The first of the lemmas is obtained geometrically from similar triangles and the second is a deduction from the first.

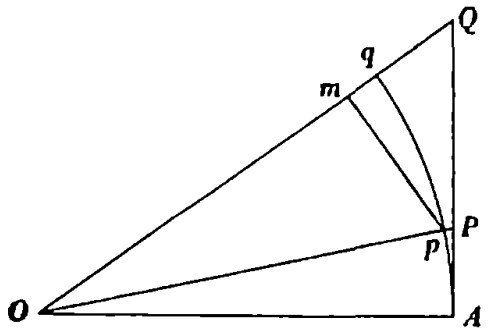

FIG. 1.

LEMMA 1. Let P, Q be points on the tangent at A to a circle of unit radius whose centre is O; let OP meet the circle in p. If m is the foot of the perpendicular from p to OQ, pm is given by

$$pm = \frac{PQ}{OP \,.\, OQ}.$$

LEMMA 2. If, in Lemma 1, PQ is small, the arc pq of the circle intercepted between OP and OQ is given by

$$\text{arc } pq = \frac{PQ}{OP^2} \quad \text{or} \quad \frac{PQ}{1+AP^2},$$

correct to a first approximation.

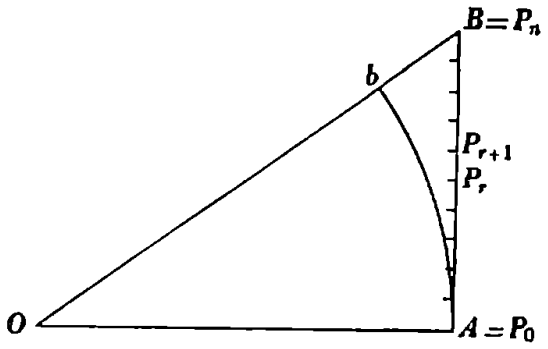

FIG. 2.

. PROOF OF GREGORY'S SERIES. Let the point B, on the tangent at A, be such that $A\hat{O}B \leqslant 45°$. Write $\tan A\hat{O}B = AB = t$ $(\leqslant 1)$. Divide AB into n equal parts, denoting the points of division and the end points by $P_0(=A)$, P_1, P_2, . . . , P_{n-1}, $P_n(=B)$, in the order in which they occur from A to B. Let OB meet the circle in b. Then we find, from Lemma 2,

$$\text{arc } Ab = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{P_rP_{r+1}}{1+AP_r^2} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t/n}{1+(rt/n)^2}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t}{n} \left\{ 1 - \left(\frac{rt}{n}\right)^2 + \cdots + (-1)^{\nu-1} \left(\frac{rt}{n}\right)^{2\nu-2} + \frac{(-1)^{\nu} \left(\frac{rt}{n}\right)^{2\nu}}{1+\left(\frac{rt}{n}\right)^2} \right\}.$$

Now making use of the result

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{r=0}^{n-1} r^p = \frac{1}{p+1},$$

we obtain

$$\text{arc } Ab = t - \frac{t^3}{3} + \ldots + (-1)^{\nu-1} \frac{t^{2\nu-1}}{2\nu-1} + (-1)^{\nu} R_\nu,$$

where $0 < R_\nu < 1/(2\nu+1)$; whence, letting $n \to \infty$, we are led to Gregory's series for $0 < t \leqslant 1$. The validity of the series for $-1 \leqslant t < 0$ becomes obvious when we change the sign of t.

2.11. There are two points which emerge from a consideration of the mathematical text presented here. In the first place, it employs relations which would appear not to have been noticed in Europe before the modern forerunners and followers of the calculus started their investigations. Thus the formula

$$\sum_{r=0}^{n-1} r^p \sim n^{p+1}/(p+1) \quad (n \to \infty)$$

first presented itself to Roberval (1634) and Fermat (1636) in connection with problems of quadrature,[11] while the relation

$$\text{arc tan } t = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t/n}{1+(rt/n)^2},$$

found its first explicit statement in the course of Euler's evaluation of π by an asymptotic series.[12]

Our second point is not unconnected with the first. The Hindu mathematicians achieved without the aid of the calculus results which for us are best treated by means of the calculus.[13] But the Europeans who first explored the possibilities of expressing a circular arc as an infinite series deliberately employed the calculus to guide them. Thus (i) Gregory (1761) would seem to have obtained his series for arc tan t by a direct use of the relation $d(\tan\theta) = \sec^2\theta\, d\theta$ and the subsequent integration in series of

$$\int d\,(\tan\theta)[1+\tan^2\theta]^{-1}.$$[14]

(ii) Leibniz (1673) obtained the same series by a calculation (involving integration) of the area bounded by the arc of a circular sector, the tangent at one end of the arc and the radius through the other end.[15] (iii) Newton (1669?)[16] derived his series for arc sin s by a method which is in effect the integration of the equality

$$d\theta = d\,(\sin\theta)/\sqrt{1-\sin^2\theta}.$$

The method may be freed from the explicit use of the calculus and brought into line with the method of limits used in *Yukti-Bhāṣa* by describing it as consisting in the

[11] D. E. Smith, *History of Mathematics* (1st ed.) 2, 688.

[12] Euler, *Commentarii Academiae Petropolitanae ad annum* 1730 cited by M. Cantor, *Vorlesungen uber Geschichte der Mathematik* (2nd ed.), 3, 672*f*. *Vide* also T. J. I'a. Bromwich, *An Introduction to the Theory of Infinite Series* (2nd ed.), 326.

[13] This is not to gainsay the fact that (i) the Hindus' proof of Gregory's series shows their awareness of the principle of integration as we ordinarily use it nowadays; (ii) their intuitive perception of small quantities like $O(1/n^p)$, $n \to \infty$, in a certain rôle, is as good as a practical knowledge of differentiation. (*Vide* Section II *infra*.)

[14] *James Gregory, Tercentenary Memorial Volume* published by the Royal Society of Edinburgh (1939), 357 and 463.

[15] Cantor, *op. cit.*, 80*f*.

[16] *Opuscula Newtoni* cited by Cantor, *op. cit.*, 159*f*.

subdivision into a large number n of equal parts of the orthogonal projection s of the arc Ab on the tangent at A. This leads to the relation

$$\text{arc sin } s = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{s/n}{\sqrt{1-(rs/n)^2}};$$

but the further proof depends on the use of the Binomial Theorem for a negative fractional index which our Hindu mathematicians do not seem to have known. Their manner of obtaining the expansion of $[1+(rt/n)^2]^{-1}$, viz., by repeated applications of the identity $[1+(rt/n)^2]^{-1} = 1-(rt/n)^2+(rt/n)^4[1+(rt/n)^2]^{-1}$, may be taken as some indication of their ignorance in this matter. Another pointer in the same direction is the absence from Hindu mathematical literature of Newton's series for arc sin s, but the presence of the approximate formula: arc sin $s \doteqdot s+s^3/3!$ for arc sin $s<3°\ 45'$, presumably obtained by the method described above, using the approximation

$$[1-(rs/n)^2]^{-\frac{1}{2}} \doteqdot 1+\frac{1}{2}(rs/n)^2$$

for small s.[17] Therefore, even if we post-date the Hindu researches in circle-rectification by a couple of centuries and more, to make their first appearance later than that of similar European researches, there is a *prima facie* case for asserting that the former were not directly inspired by the latter.

II. Hindu transformations of Gregory's series for π.

3. We discuss in this section the Hindu transformations of Gregory's series for π and certain rational approximations to π resulting therefrom. A fact not to be lost sight of in this connexion is the Hindus' realization of the irrationality of π which finds clear expression in a statement such as the following: "Why do we give for π an approximate value in place of the true value? For this reason: the ratio of the circumference to the diameter can never be expressed as the ratio of two integers."[18] It is not irrelevant to recall here that Lambert's paper demonstrating the irrationality of π was read before the Berlin Academy in the year 1761.

3.1. Issuing from Gregory's series for π,

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\cdots, \tag{1}$$

there is an endless sequence of approximations to $\pi/4$ in which every approximation is better than the preceding. For large values of n, the first three approximations are

$$\frac{\pi}{4} \doteqdot 1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{n}\mp\frac{1}{2(n+1)}, \tag{2a}$$ [19]

$$\frac{\pi}{4} \doteqdot 1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{n}\mp\frac{(n+1)/2}{(n+1)^2+1} \tag{3a}$$ [20] [T],

[17] K (T.S.S.), 19, Chap. VI, verse 19, which gives also sin $\theta \doteqdot \theta-\theta^3/3!$ for $\theta < 3°\ 45'$.

[18] This is a free translation of a passage from the work *Āryabhaṭīya of Āryabhaṭācārya with the Bhāṣya of Nīlakaṇṭhasomasutvan* (Part I), Trivandrum Sanskrit Series, which will be denoted in what follows by the abbreviation A I (T.S.S.); part II of the same work being denoted A II (T.S.S.). The passage in question is part of Nīlakaṇṭha's commentary on verse 10, chap. II, of Āryabhaṭīya.

[19] (2a), (2′a) and (4) are not explicitly mentioned by Whish, perhaps also by his sources.

[20] *TRAS*, 512*f*. (3a) and (3′a) are stated incorrectly by Whish with $(n-1)$ in place of $(n+1)$ in the last term. This is because he has given to the Sanskrit word ūrdhva in two ślokās of *Tantrasaṅgraha*, the incorrect meaning "antecedent" (to n) instead of the correct one "subsequent" (to n). These ślokās are to be found in chapter II, folios 10, 11 of the Adyar MS.

$$(4a)\qquad \frac{\pi}{4} \doteqdot 1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{n}\mp\frac{\frac{(n+1)^2}{4}+1}{\{(n+1)^2+4+1\}\frac{n+1}{2}} \quad [21] \qquad [T].$$

These approximations give rise to the following alternating series whose sums involve π linearly and in which the nth term is $O(1/n^{2p+1})$ as $n \to \infty$, $p = 1, 2, 3$.

$$(2)\qquad \frac{\pi-3}{4} = \frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\cdots \quad [22] \qquad [T],$$

$$(3)\qquad \frac{\pi}{16} = \frac{1}{1^5+4\cdot 1}-\frac{1}{3^5+4\cdot 3}+\frac{1}{5^5+4\cdot 5}-\cdots \quad [23] \qquad [T],$$

$$(4)\qquad \left(\frac{\pi}{4}-\frac{7}{9}\right)\Big/ 36 = \frac{1}{(3^3-3)(2^2+5)(4^2+5)}-\frac{1}{(5^3-5)(4^2+5)(6^2+5)}+\cdots.$$

In the concluding portion of a section bearing the name *Paridhi Vyāsam* in *Yukti-Bhāṣa* we find the method of deriving from (1) first (2*a*), (3*a*) and thence (2), (3). This method is explained in general terms in §§ 4, 4.1 below.

3.2. A slight modification of the method leads to a second sequence of approximations to π which can be derived from the alternating series

$$(1')\qquad \frac{\pi-2}{4} = \frac{1}{2^2-1}-\frac{1}{4^2-1}+\frac{1}{6^2-1}-\cdots \quad [24] \qquad [T],$$

in which the general term is $O(1/n^2)$. The derivation of this series itself is simple. In (1), grouping together two consecutive terms beginning with the first and then two consecutive terms beginning with the second, we obtain successively

$$\frac{\pi}{8} = \frac{1}{2^2-1}+\frac{1}{6^2-1}+\frac{1}{10^2-1}+\cdots \quad [25] \qquad [T],$$

$$\frac{4-\pi}{8} = \frac{1}{4^2-1}+\frac{1}{8^2-1}\quad\frac{1}{12^2-1}+\cdots \quad [26] \qquad [T],$$

whence by subtraction (1′) follows. The first two members of the sequence of approximations which can be derived from (1′), in much the same way as (2*a*) or (3*a*) can be derived from (1), are

$$(2'a)\qquad \frac{\pi-2}{4} \doteqdot \frac{1}{2^2-1}-\frac{1}{4^2-1}+\cdots\pm\frac{1}{n^2-1}\mp\frac{1}{2(n+1)^2},$$

$$(3'a)\qquad \frac{\pi-2}{4} \doteqdot \frac{1}{2^2-1}-\frac{1}{4^2-1}+\cdots\pm\frac{1}{n^2-1}\mp\frac{1}{2\{(n+1)^2+2\}} \quad [27] \qquad [T],$$

in the series corresponding to which the nth term will be shown to be $O(1/n^{2p+2})$, $p = 1, 2$ respectively (§5 *infra*).

4. In (1), let us denote the partial sum of the first $(2m+1)$ terms and that of the first $2m$ terms by $S\left(\frac{n+1}{2}\right)$ and $S\left(\frac{n-1}{2}\right)$ respectively, where $n = 4m+1$. These partial sums

[21] *TRAS*, 513.
[22] *TRAS*, 514. Also *K* (T.S.S.), 16, chap. VI, verse 2.
[23] *TRAS*, 514.
[24] *TRAS*, 515.
[25] *TRAS*, 515*f*. This series was also known to Leibniz, *vide* M. Cantor, *op. cit.*, 3, 83.
[26] *TRAS*, 515*f*.
[27] *TRAS*, 515.

are in fact approximate values of $\pi/4$ when n is large and can be improved upon by applying to them " corrections " $-f(n+1)$ and $f(n-1)$ respectively, the corrected values being

$$T\left(\frac{n+1}{2}\right) = S\left(\frac{n+1}{2}\right) - f(n+1),$$

$$T\left(\frac{n-1}{2}\right) = S\left(\frac{n-1}{2}\right) + f(n-1).$$

It will be noticed that the sign prefixed to the correction which is applied to any partial sum is the same as the sign of the total remainder following the partial sum in question; also, that the corrected value $T\left(\frac{n+1}{2}\right)$ of $S\left(\frac{n+1}{2}\right)$ is the partial sum of $\frac{n+1}{2}$ terms of a new series for $\frac{\pi}{4}$ more rapidly convergent than (1). Denoting by u_n the $\frac{n+1}{2}$ nd term of the new series, we have by subtraction

$$u_n = \frac{1}{n} - f(n+1) - f(n-1). \tag{5}$$

In (5) let us change n successively to $n-2$, $n-4$, . . . , 3 and eliminate $f(n-1)$, $f(n-3)$, . . . , $f(4)$ between the relations thus obtained and (5). Then we get

$$-u_3 + u_5 - \cdots + u_n = -\frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{n} + f(2) - f(n+1).$$

Hence if $f(n)$ satisfies the preliminary condition $f(n) = o(1)$ $(n \to \infty)$, we have the following series for π more rapidly convergent than (1).

$$\frac{\pi}{4} = 1 - f(2) - u_3 + u_5 - \cdots + u_n - \cdots \tag{6}$$

4.1. Now if we are to have $u_n = O(n^{-(2p+1)})$ $(n \to \infty)$, it is readily seen from (5) that a condition to be satisfied by f, more comprehensive than the one already mentioned, is $f(x) = O(1/x)$ $(x \to \infty)$. Therefore, in testing the suitability of various functions f in (5), we may confine ourselves to those of the type

$$2f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots \qquad (x \geqslant x_0). \tag{7}$$

With such a choice of f, (5) can be written:

$$\frac{1}{n} = 2f(n) + 2\frac{f''(n)}{2!} + 2\frac{f^{iv}(n)}{4!} + \cdots + \frac{1}{(2p)!}\{f^{(2p)}(n-\theta_1) + f^{(2p)}(n+\theta_2)\} + u_n \tag{8}$$

where $0 < \theta_1, \theta_2 < 1$ and the successive terms of the right-hand member (excluding u_n) are $O\left(\frac{1}{n}\right), O\left(\frac{1}{n^3}\right), \cdots, O\left(\frac{1}{n^{2p+1}}\right)$ respectively. Now introducing the condition $u_n = O\left(\frac{1}{n^{2p+1}}\right)$ in (8), we find that

$$\frac{1}{n} + O\left(\frac{1}{n^{2p+1}}\right) = 2f(n) + 2\frac{f''(n)}{2!} + \cdots + \frac{2}{(2p-2)!} f^{(2p-2)}(n). \tag{9}$$

Employing (7) in (9) we can calculate $a_1, a_2, \ldots, a_{2p}$ and verify that $a_2 = a_4 = \ldots = a_{2p} = 0$. Any function of the type (7), with the first $2p$ coefficients having these calculated values, can be used in (5) to get a series like (6) for π in which $u_n = O\left(\frac{1}{n^{2p+1}}\right)$.

4.11. Clearly a function $f(n)$ satisfying our requirements can be expressed in the form of a continued fraction,

$$2f(n) = \frac{b_1}{n+}\,\frac{b_3}{n+}\,\frac{b_5}{n+}\cdots\frac{b_{2p-1}}{n}, \tag{10}$$

where the b's can be calculated by expressing them in terms of the a's.

4.2. The method explained above gives as particular cases all the results mentioned in § 3.

First, $p = 1$ in (9) makes $2f(n) = 1/n$ and leads us to (2a). Now calculating u_n from (5) and employing it in (6), we get (2).

Next, $p = 2$ in (9) makes $a_1 = 1$, $a_3 = -1$, $(a_2 = a_4 = 0)$ and consequently $b_1 = b_3 = 1$. Hence (10) leads to $2f(n) = \dfrac{1}{n+}\,\dfrac{1}{n}$ and thence to (3a). Making use of this expression in (5), we obtain a value for u_n which substituted in (6) gives rise to the following series equivalent to (3).

$$\left(\frac{\pi}{4} - \frac{1}{5}\right) \Big/ 4 = -\frac{1}{3(2^2+1)(4^2+1)} + \frac{1}{5(4^2+1)(6^2+1)} - \cdots.$$

Finally, when $p = 3$, we obtain from (9), $a_1 = 1$, $a_3 = -1$, $a_5 = 5$, $(a_2 = a_4 = a_6 = 0)$ and hence $b_1 = b_3 = 1$, $b_5 = 4$. Therefore $2f(n) = \dfrac{1}{n+}\,\dfrac{1}{n+}\,\dfrac{4}{n}$ whence (4a) follows. The corresponding value of u_n calculated from (5) and used in (6) gives us (4).

4.3. Two observations will not be out of place here; the first on the evaluation of π by means of approximations such as those in § 3, the second on the kind of "proof" offered by *Yukti-Bhāṣa* for the approximations and the corresponding series.

(i) It seems likely that the remarkable approximations to π which Whish mentions[28]—correct to 9 decimal places [T], up to and correct to 10 decimal places [K],[29] up to and correct to 17 decimal places [S]—were got by using formulae such as (3a) or (4a). Some idea of the labour necessary to achieve such approximations may be obtained from a calculation very kindly communicated to us by Prince Rāma Varma (12th prince of Cochin). The prince has verified that, by taking $n = 55$, we can evaluate π correct to 10 decimal places from (4a), to 6 decimal places from (3a) and finally to the first decimal place only from (2a).

(ii) The reader can judge for himself how far our exposition is in accord with the spirit of the original by studying the arguments in *Yukti-Bhāṣa* leading up to any one of the approximations and the corresponding series, say (2a) and (2). The right choice of $f(n)$ in this case is preceded by "trials" in which $2f(n)$ is taken successively as

$$\frac{1}{n},\quad \frac{1}{n+\frac{1}{2}} \quad \text{and} \quad \frac{1}{n+\frac{1}{4n}};$$

the second choice of $f(n)$ being rejected as of a type more unsuitable than the first. The Hindu expositor thus reveals himself as a person gifted with a fine intuition in respect of certain orders of smallness which serves him in place of the analysis employed by us in 4.1.

5. We proceed to explain how, by slightly modifying the method in § 4.1, we can convert (1′) into another alternating series in which the nth term is $O(1/n^{2p+2})$. Denoting the partial sum of the first $n/2$ or $2m+1$ terms of the series (1′) by $S(n/2)$ and the corresponding partial sum of the new series by $T(n/2)$, let us write

$$T(n/2) = S(n/2) - f(n+1),$$

[28] *TRAS*, 511*f*.

[29] *K* (T.S.S.), 6, chap. VI, verse 7.

where $f(n)$ is to be determined so that u_n, the n/2nd term of the new series, is $O(1/n^{2p+2})$. This leads to the relation

$$(5') \qquad O(1/n^{2p+2}) = u_n = \frac{1}{n^2-1} - f(n+1) - f(n-1).$$

Since we have as a preliminary condition $f(n) = O(1/n^2)$, we can take

$$(7') \qquad 2f(x) = \frac{1}{x}\left\{\frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right\} \qquad (x \geqslant x_0).$$

This choice of f makes (5′) equivalent to

$$(9') \qquad \frac{1}{n^2-1} + O\left(\frac{1}{n^{2p+2}}\right) = \sum_{r=0}^{p-1} \frac{2}{(2r)!} f^{(2r)}(n),$$

whence we can calculate $a_1, a_2, \ldots, a_{2p}$ (verifying, as in § 4.1, that $a_2 = a_4 = \ldots = a_{2p} = 0$). Thus the correction function $f(n)$ which relates a partial sum of (1′) to the corresponding partial sum of the new series derived from (1′) is indeterminate save for its first $2p$ coefficients. It is however uniquely determined if restricted to the form

$$(10') \qquad 2f(n) = \frac{1}{n}\left\{\frac{b_1}{n+}\ \frac{b_3}{n+}\ \cdots\ \frac{b_{2p-1}}{n}\right\}.$$

5.1. Illustrations of § 5 are furnished by (2′a) and (3′a).

First consider the case $p = 1$. Then (9′) gives $2f(n) = 1/n^2$ which establishes (2′a) and, in conjunction with (5′), determines an alternating series with the same sum as (1′) and $u_n = O(1/n^4)$.

If $p = 2$ in (9′) we find that $a_1 = 1$, $a_3 = -2$, ($a_2 = a_4 = 0$) and hence in (10) $b_1 = 1$, $b_3 = 2$. This establishes at one stroke (3a) and an alternating series having the same sum as (1′) and $u_n = O(1/n^6)$.

The number of such illustrations is of course infinite.

6. Before concluding this section we may point out that a series for π in which all the terms are of the same sign and the nth term is $O(1/n^{2p+2})$ or $O(1/n^{2p+3})$ can be obtained by grouping together consecutive terms in the alternating series derived from (1) or (1′) which have the nth term $O(1/n^{2p+1})$ or $O(1/n^{2p+2})$. Thus from (2), grouping together consecutive terms beginning with the first, we get the series

$$\frac{\pi-3}{6} = \frac{2}{(2\,.\,2^2-1)^2-2^2} + \frac{1}{(2\,.\,4^2-1)^2-4^2} + \frac{1}{(2\,.\,6^2-1)^2-6^2} + \cdots .^{30} \qquad [K]$$

III. Date and authorship of the principal works cited.

7. Any discussion which seeks to date the first appearance of the Hindu investigations in circle-squaring must centre round the origin of *Tantrasaṅgraha* if not round that of *Karaṇapaddhati*. For tradition and scholarship alike support the statement of Whish that *Tantrasaṅgraha* was written about the year 4600 in Kali era or 1500 A.D., that is to say, long before either of the works *Yukti-Bhāṣa* and *Sadratnamālā*. While these two works are devoid of historical interest on account of their comparative recency, *Yukti-Bhāṣa* must nevertheless claim the attention of the historian of mathematics as being the only Hindu work on π which gives proofs of the theorems it states. *Karaṇapaddhati* has also claims, but of a different kind. Whether we assign to it the date 1733 A.D. (following Whish) or the date 1430 A.D. (following Vadakkumkur Rāja Rāja Varma), it is perhaps the earliest Hindu work which explicitly mentions Gregory's series, perhaps also the earliest which gives the first two terms in the series for the sine and the inverse

[30] *TRAS*, 517. Also *K* (T.S.S.), 17, chap. VI, verse 4.

sine. Our essay would therefore be incomplete without some observations on the date and authorship of the three works, *Tantrasaṅgraha*, *Karaṇapaddhati* and *Yukti-Bhāṣa*. As for the work *Sadratnamālā*, a starting point for a historical inquiry is furnished by the *Triennial Catalogue of Manuscripts* in the Madras Government Oriental Manuscripts Library which says that the work is "a treatise on astronomy: by Śaṅkaravarman, who wrote this at the instance of Rāmavarman, brother of Udayavarman, King of Kerala country (Kadattanād)".[31] It may be possible on the basis of this statement to confirm the supposition to which Whish seems to lead up, namely, that *Sadratnamālā* is a work of his own period.[32]

8. Among the other three works with which we are concerned, *Tantrasaṅgraha* seems to offer the clearest evidence of date and authorship. Every chapter of this work concludes with the statement that the author is one Nīlakaṇṭha of Kerala deśa and Gārga gotra.[33] Further the commentary attached to the work explains that the lines:

He Viṣṇo! nihitam kṛtsnam jagat tvayyeva kāraṇe
Jyotiṣam jyotiṣe tasmai namo nārāyaṇāya te,

from the introductory verse, and the lines:

Golaḥ kālakriyā cāpi dyotyatetra mayāsphutam
Lakṣmīśa nihitadhyānaiḥ iṣtam sarvam hi labhayate,

from the concluding verse contain two Kali chronograms representing the exact dates on which the work was begun and finished. The two chronograms (italicized in our quotations) are supposed to tell us that the actual composition of *Tantrasaṅgraha* occupied only five days—from the 1680548th day of Kali era (26th day of Mīna month of the Kali year 4601) to the 1680553rd day of Kali era (1st day of Meṣa month of the Kali year 4602).

8.1. The editor of Āryabhaṭīya with bhāṣya or commentary, in the Trivandrum Sanskrit Series, endorses this reading of the chronograms and supplements our information with the following biographical details about Nīlakaṇṭha.[34] The commentator who is none other than the author of *Tantrasaṅgraha*[35]—as shown by the fact that he acknowledges the latter as his composition—is a native of Kuṇḍa grāma (Ṭṛkkaṇḍiyūr in British Malabar) and a Somayājin (i.e. a Brahmin authorized to perform the Soma sacrifice) belonging to the Āśvalāyana Śākha of Gārga gotra. He is the son of a certain Jātaveda; he has a maternal uncle of the same name and a brother Śaṅkara who is an astronomer. Further he is a disciple of Dāmodara, son of Parameśvarācārya who (according to the Trivandrum editor) propounded the Dṛgganita system of astronomy, after fifty-five years of study, in 1353 Śaka.[36] Nīlakaṇṭha himself supplies this date in his *Āryabhaṭīya Bhāṣya* where we read:

Evam dṛgganitam Śake triṣu visvamite kṛtam.[37]

The Trivandrum editor closes his short sketch of Nīlakaṇṭha with the remark that the Somayājin shows himself a master of not only Jyotiṣa (a curious label for a subject which includes circle-squaring!) but also Mīmāṁsa, Nyāya, Vyākaraṇa and Vedānta—in brief "a solid mass of erudition submerged somehow or other under the current of time".

[31] *Triennial Catalogue*, 5, Part I (1932), 6518–20.

[32] Whish, however, is mistaken when he says that Śaṅkaravarma is the "younger brother of the present Raja of Kadattanāda near Tellicherry".

[33] Whish therefore has made a mistake in identifying the author with Talakuluttura Namḅūdiri, who, according to Mr. P. S. Puruṣottaman Nambūdiri (referred to later), is none other than the astrologer Govindabhaṭṭa eulogized in the introduction to *Sphuṭanirṇaya*.

[34] Introduction to *A* I (T.S.S.).

[35] *A* I (T.S.S.), 112, where Nīlakaṇṭha says, "so has been said by me in *Tantrasaṅgraha*".

[36] *A* I (T.S.S.), introduction, where it is said that Parameśvarācārya lived on the bank of the river Nilā near the seashore in North Kerala and was the author of numerous works including *Bhaṭadīpikā*, *Goladīpikā*, and a commentary on *Līlāvati*.

[37] Quoted in the Preface to *Goladīpikā* (T.S.S).

8.2. Mr. K. Balagangadharan has brought to our notice a significant piece of evidence which seems to corroborate the accepted view that Nīlakaṇṭha's period may be fixed between 1450 A.D. and 1550 A.D. Nīlakaṇṭha mentions in his *Āryabhaṭīya Bhāṣya*[38] that he has observed a total eclipse of the sun on the 'ahargaṇa' represented by 'hamsovihitamtāpa', that is to say, on the 1668478th day of Kali era; and an annular eclipse of the sun at Anantakṣetra (Trivandrum ?) on the 'ahargaṇa, grāsavṛdhyārdhitorkaḥ' or on the 1681072nd day of Kali era. These two days (in the order mentioned) fall approximately in the years 1467 A.D. and 1502 A.D.; so that a verification (by independent calculation) of the occurrence of the two eclipses will go a long way towards establishing the authenticity of the documents attributed to Nīlakaṇṭha.

9. *Karaṇapaddhati* differs from *Tantrasaṅgraha* in that it is to all intents and purposes an anonymous work. Its text furnishes no clue to the identity of the author beyond the statement that he is a Somayājin belonging to the family of 'Nūtanagṛha'[39] and born at Śivapura or Trichur. To this statement the editor of *Karaṇapaddhati* in the Trivandrum series adds his surmise that the author is a Vaiṣṇavite and, reviewing the internal evidence, concludes that the author is reluctant to reveal his identity. An old manuscript commentary we have been able to procure, by one Punnāga Kroḍa Bikṣu Varya, throws no further light on the question. However, Rāja Rāja Varma, in the first volume of his recently published History of Sanskrit Literature in Kerala,[40] takes the view that the period of the author may be placed between the years 550 and 650 of the Malabar era, corresponding roughly to 1375–1475 A.D. He quotes in support of his view a verse from a work called *Gaṇita sūcikā grantha* by one Govindabhaṭṭa. The verse runs:

Navīna vipine mahīmakhabhujām Somayā-
Jyudaragaṇa kotraya samabhavacca tenā munā
Vyalekhi sudṛguttama *Karaṇapaddhati* samskṛtā
Tripanca śati bhūmita pradhita Śaka samvatsare;

and may be rendered into English thus: A Somayājin and an expert astronomer, who was born in the Brahmin family of Navīnavipina (same as Nūtanagṛha or Pudumana) by him was composed this refined *Karaṇapaddhati*, the best in the Dṛg system, in 1353 Śaka era. If we accept the corresponding year in Christian era, 1430 A.D., as the date of *Karaṇapaddhati*, the Kali chronogram which Whish thinks he has discovered in the colophon of the work[41] becomes a find of dubious value.

10. If our information regarding the author of *Karaṇapaddhati* is scanty, that regarding the author of *Yukti-Bhāṣa* is scanty as well as uncertain. The paper manuscript of *Yukti-Bhāṣa* in the library of the Sanskrit College at Trippunithura, has the colophon:

Alekhi Yuktibhāṣa vipreṇa Brahmadattasaṃjñena ye
Golapathasthaḥ syuḥ kalirahitāḥ śodhayantaste.

which makes out (i) that the work was written by a Brahmin named Brahmadatta, (ii) according to Mr. K. Vāsudevan Mūsad of Trichur,[42] that the phrase "Yè Golapathasthaḥ syuḥ" followed by "kali" indicates the 1771931st day of Kali era and therefore suggests 4741 Kali era or 1639 A.D. as the date of the work. But the colophon quoted here is not to be found in certain palm-leaf manuscripts of the work; and, to add to our difficulties,

[38] *A* II (T.S.S.), 23.

[39] For this reason the author of *Karaṇapaddhati* is referred to as Pudumana Comatiri in Malayaḷam tradition.

[40] The book is in Malayaḷam and published by the Kamalālaya Book Depot, Trivandrum. The references given are to pages 529*f*.

[41] "The words Gaṇita Metatsamyae", says Whish, "written in numbers amount to 1765653 in terms of the days of the present age; which is found to agree with A.D. 1733, being 86 years and 271 days before the 31st of December 1819." *TRAS.*, 522.

[42] The views of Mr. Vāsudevan Mūsad and Mr. Puruṣottaman Nambūdiri were communicated to us by Mr. K. Balagangadharan.

A. K. T. K. M. Nambūdiripād, in a Malayaḷam essay contributed to a volume entitled *The Nambūdiris*, edited by Parayil Rāman Nambūdiri (1918), ascribes the authorship of *Yukti-Bhāṣa* to one Parangota of Ālattūr village. As if this were not enough, we have the statement of Whish that the author is Kelallūr Nambūdiri, and the statement of Mr. P. S. Puruṣottaman Nambūdiri—a well-known astronomer of Chengannur (Travancore) who has recently brought out an account of Hindu astronomy in Malayaḷam, called *Gaṇitanirṇaya*—that Kelallūr Nambūdiri is Nīlakaṇṭha himself!

10.1. What we can say with certainty about *Yukti-Bhāṣa* may be summed up thus. The introductory verse in the work is the same as the benedictory stanza in the commentary attached to *Tantrasaṅgraha*:

Pratyūha vyūha vihatikārakam paramamahaḥ
Antaḥkaraṇa śuddhim me vidadhātu sanātanam.

Strange to say, this verse is replaced by another in a Sanskrit version of *Yukti-Bhāṣa*, listed as No. R4382, with the elaborated title *Gaṇita-Yukti-Bhāṣa*, in the catalogues of the Madras Government Oriental Manuscripts Library. Except for this substitution and the omission of the reference to Brahmadatta, the Sanskrit version seems to agree closely with the Malayaḷam manuscript from Trippunittura. In both the Sanskrit and the Malayaḷam versions there is the explicit statement that the work is a systematic exposition of the mathematics of "planetary motions" after the manner of *Tantrasaṅgraha*.

11. With all this information to hand we have no further clue than Whish had to the identity of the "Brahmin who lived on the coast of the western ocean, and, having examined the heavens for twelve years, established what is laid down in *Tantrasaṅgraha*".[43] Is this Brahmin Parameśvara, or Nīlakaṇṭha, or the anonymous author of Karaṇapaddhati, or one whose incognito is even more difficult to pierce? The historian of mathematics faces this question as well as another no easier to answer. If we accept the indigenous origin of the Hindu work relating to π (whether in the fifteenth century or later), are we not up against the difficulty of having to regard the inadequate pre-Tantrasaṅgraha-Karaṇapaddhati mathematics as the foundation for the substantial superstructure raised on it? Even a casual glance at the foundation reveals its inadequacy. For instance, while Mahāvīra and Bhāskara are both familiar with the expressions for

$$\sum_{r=0}^{n-1} r^2 \quad \text{and} \quad \sum_{r=0}^{n-1} r^3,$$ [44]

their knowledge does not extend to the relation

$$\sum_{r=0}^{n-1} r^p \sim n^{p+1}/(p+1) \qquad (n \to \infty).$$

Again, while Āryabhaṭa and Mahāvīra are both interested in *Vyutkalita*, "the summation of a series after a certain number of initial terms have been cut off",[45] they go nowhere near the problem of expressing the remainder after a large number of terms of a slowly convergent series as a continued fraction to a certain degree of approximation. To mention one more instance, while Āryabhaṭa must be supposed to be familiar with the principle embodied in the approximation $\delta\theta \doteqdot \sin\delta\theta$ for small $\delta\theta$,[46] he is far from anticipating in any form the relation $d\theta = d(\tan\theta)/(1+\tan^2\theta)$ which the author of *Yukti-Bhāṣa* proves without bringing in trigonometrical considerations. In short, our study of the mathematical contributions of the Brahmin on the coast of the western ocean

[43] This is Whish's quotation from a work called *Dṛkkaraṇam* which we have been unable to trace.
[44] D. E. Smith, *History of Mathematics*, 2, 495n.
[45] D. E. Smith, *ibid.*, 1, 162. [46] D. E. Smith, *ibid.*, 2, 620.

reveals to us the gulf which separates his work from that of his predecessors. To vary the metaphor, we are confronted by a locked door the key to which seems to have been lost. Are we to seek for the key in some little-known corner of India's past where she came into vital contact with the West? Or in some unexplored mine of ancient lore not yet on the map of modern scholarship? Or in the sanctum of some long-forgotten Ācārya's mind?

APPENDIX.

By K. BALAGANGADHARAN

I

The main theorems of the foregoing article are collected below; underneath each theorem, its enunciations in the original śloka form and in English are given. In Theorems 3–12, C denotes the circumference of a circle whose diameter is D. The abbreviations employed to denote the references are all those of the article.

I may mention at the outset that the translations appended to the ślokas are not literal, aiming as they do at clarity rather than mere verbal faithfulness to their originals. As for the ślokas themselves, my sources have been the Trivandrum edition of *Karaṇa-paddhati* and the MSS. of *Tantrasaṅgraha* in the Trippunittura Sanskrit College Library and the Adyar Library. All my quotations from T, barring those under Theorems 6, 11, 12, 13, follow the Trippunittura MS., the quotations under these theorems following Whish and the Adyar MS.

THEOREM 1. $\arctan t = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \ldots \qquad (|\arctan t| \leq \pi/4)$

Ślokā :

व्यासार्धेन हतादभीष्टगुणत: कोटयाप्तमाद्यं फलं
ज्यावर्गेण विनिघ्नमादिमफलं तत्तत्फलं चाहरेत् ।
कृत्या कोटिगुणस्य तत्र तु फलेष्वेकत्रिपञ्चादिभि-
र्भक्तेष्वोजयुतैस्त्यजेत्समयुतिं जीवाधनुश्शिष्यते ॥

[*K* (T.S.S.), 19, chap. VI.]

Translation.—[Take any circular arc, as in the accompanying figure, whose "abscissa" is not less than its "ordinate".] Multiply the "ordinate" of the arc by the semidiameter and divide it by the "abscissa". This gives the first term. Multiply this term by the square of the "ordinate" and divide it by the square of the abscissa; a second term results. Repeat the process of multiplying by the square of the ordinate and dividing by the square of the abscissa. Thus obtain successive terms and divide them in order by the odd integers 1, 3, 5, . . . If now the terms whose order is odd are added to, and the terms whose order is even subtracted from the preceding, what remains is the circumference.

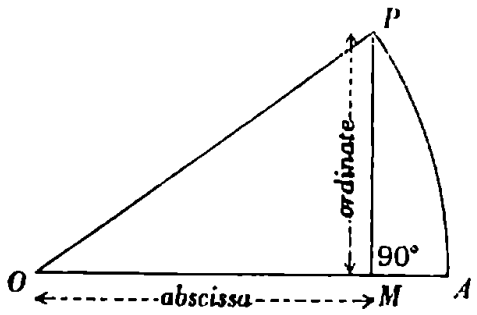

FIG. 3.

That is to say, in the figure (with $A\hat{O}P \leqslant 45°$).

$$\text{arc } AP = OP\left\{\frac{PM}{OM} - \frac{1}{3}\frac{PM^3}{OM^3} + \frac{1}{5}\frac{PM^5}{OM^5} - \ldots\right\}$$

Remarks.—(i) The words "ordinate" and "abscissa" and "arc" in the English translation do duty for the Sanskrit ज्या (literally bowstring) कोटि and धनुस् (bow) respectively. (ii) The restriction on the length of the arc is mentioned by the commentator.

THEOREM 2. *If, in Fig.* 3, $A\hat{O}P < 225'$, *then*

$$PM \doteqdot \text{arc } AP - \frac{1}{6}\frac{(\text{arc } AP)^3}{OP^2},$$

$$\text{arc } AP \doteqdot PM + \frac{1}{6}\frac{PM^3}{OM^2}.$$

Ślokā :

स्वल्पचापघनषष्ठभागतो विस्तरार्धकृतिभक्तवर्जितम् ।
शिष्टचापमिह शिञ्जिनी भवेत् तद्युतोऽल्पगुणोऽसकृद्धनुः ॥

[*K* * (T.S.S.), 19, chap. VI.]

Translation.—The arc, when small, diminished by the sixth part of its own cube divided by the square of the semidiameter, becomes the ordinate; and often, when small, the arc is equal to the ordinate increased by its cube divided by six times the square of the semidiameter.

Remarks.—(i) As in Theorem 1, the restriction on the arc is due to the commentator. (ii) The English "ordinate" now serves for the Sanskrit शिञ्जिनी.

THEOREM 3. $C = \sqrt{12}\, D\left\{1 - \frac{1}{3\,.\,3} + \frac{1}{5\,.\,3^2} - \frac{1}{7\,.\,3^3} + \ldots\right\}$

Ślokā :

व्यासवर्गाद्रविहतात्पदं स्यात्प्रथमं फलम् ।
ततस्तत्तत्फलाच्चापि यावदिच्छं त्रिभिर्हरेत् ॥
रूपाद्ययुग्मसंख्याभिर्लब्धेष्वेषु यथाक्रमम् ।
विषमाच्च युते त्यक्ते युग्मयोगे वृतिर्भवेत् ॥

[*T*, chap. II.]

Translation.—Extract the square root of twelve times the diameter squared. This is the first term. Dividing the first term repeatedly by 3, obtain other terms: the second after one division by 3, the third after one more division and so on. Divide the terms in order by the odd integers 1, 3, 5, . . . ; add the odd-order terms to, and subtract the even-order terms from, the preceding. The result is the circumference.

THEOREM 4. $C = 4D - 4D\left(\frac{1}{3} - \frac{1}{5}\right) - 4D\left(\frac{1}{5} - \frac{1}{7}\right) - \ldots$

* The same verse occurs in *T*, chap. II, with the last pāda replaced by स्पष्टता भवति चाल्पतावशात् ।

Śloka :

व्यासाच्चतुर्घ्नाद्बहुशः पृथक्स्थात् त्रिपञ्चसप्ताद्ययुगाहृतानि ।
व्यासे चतुर्घ्ने क्रमशस्त्वृणं स्वं कुर्यात्तदा स्यात्परिधिस्सुसूक्ष्मः ॥

[*K* (T.S.S.), 16, chap. VI.]

Translation.—Divide four times the diameter by each of the odd integers 3, 5, 7, . . . Take away every quotient whose order is even from the one preceding it. Subtract from four times the diameter the combined result of all such small operations. This gives the exact value of the circumference.

THEORRM 5. $C = 3D + 4D\left\{\frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \ldots\right\}$

Śloka :

व्यासाद्घनसंगुणितात् पृथगाप्तं त्र्याद्ययुग्विमूलघनैः ।
त्रिगुणव्यासे स्वमृणं क्रमशः कृत्वापि परिधिरानेयः ॥

[*K* * (T.S.S.), 16, chap. VI.]

Translation.—Divide four times the diameter separately by the cubes of the odd integers, from 3 onwards, diminished by these integers themselves. These quotients alternately add to and subtract from thrice the diameter. The circumference is again obtained.

THEOREM 6. $C = 16D\left\{\frac{1}{1^5+4.1} - \frac{1}{3^5+4.3} + \frac{1}{5^5+4.5} - \ldots\right\}$

Śloka :

समपञ्चाहतयो यरूपाद्ययुजाश्चतुर्घ्ना मूलयुताः ।
ताभिष्षोडशगुणिताद्व्यासात्पृथगाहृते तु विषमयते ॥
समफलयोगे त्यक्ते स्यादिष्टव्याससंभवः परिधिः ।

[*T*, chap. II.]

Translation.—Sixteen times the diameter is divided by the fifth powers of each of the odd integers 1, 3, 5,....., increased by these integers themselves. The quotients obtained in this order are added to, or subtracted from, the preceding, according as their order is odd or even. What remains after these operations is the circumference.

THEOREMS 7 & 8. $C = 8D\left\{\frac{1}{2^2-1} + \frac{1}{6^2-1} + \frac{1}{10^2-1} + \ldots\right\}$

$$C = 4D - 8D\left\{\frac{1}{4^2-1} + \frac{1}{8^2-1} + \ldots\right\}$$

Śloka :

द्व्यादेश्चतुरादेर्वा चतुरधिकानां निरेकवर्गास्स्युः ।
हाराः कुञ्जरगुणितो विष्कंभः स्वमितिकल्पितो भोज्यः ॥
फलयुतिराद्ये वृत्तं भाज्यदलं फलविहीनमन्यत्र ।

[*T*, chap. II.]

* This stanza again is found in *T*, chap. II, with two minor alterations: वनसंगुणित being replaced by वारिधिनिहत and त्रिगुण by त्रिघ्न ।

Translation.—One less than the squares of the successive integers starting with 2 or 4 and increasing steadily by 4, are the successive divisors. Eight times the diameter is divided separately by these and the results are added together to give the circumference in the first case. In the second case the similar sum is subtracted from four times the diameter to give the circumference.

THEOREM 9. $$C = 3D+6D\left\{\frac{1}{(2\,.\,2^2-1)^2-2^2}+\frac{1}{(2\,.\,4^2-1)^2-4^2}+\frac{1}{(2\,.\,6^2-1)^2-6^2}+\ldots\right\}$$

Śloka :

वर्गैर्युजां वा द्विगुणैर्निरेकै-
वर्गैर्विकृतैर्वर्जितयुग्मवर्गैः ।
व्यासं च षड्घ्नं विभजेत्फलं स्वं
व्यासे त्रिनिघ्ने परिधिस्तदा स्यात् ॥

[*K* (T.S.S.), 17, chap. VI.]

Translation.—Divide six times the diameter separately by the squares of, twice the squares of the even integers (2, 4, 6, ...) minus 1, diminished by the squares of the even integers themselves. The sum of the resulting quotients increased by thrice the diameter is the circumference.

THEOREM 10. $$C \doteqdot 4D\left\{1-\frac{1}{3}+\frac{1}{5}-\ldots\pm\frac{1}{n}\mp\frac{(n+1)/2}{(n+1)^2+1}\right\},$$

where n is odd and large.

Śloka :

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते ।
त्रिशरादिविषमसंख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात् ॥
यत्संख्ययात्र हरणे कृते निवृत्ता हृतिस्तु जामितया ।
तस्या ऊर्ध्वगतायास्समसंख्याया तद्दलं गुणोऽन्ते स्यात् ॥
तद्वर्गैः रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत् ।
तस्यामाप्तं स्वमृणे कृते धने शोधनञ्च करणीयम् ॥
सूक्ष्मः परिधिः सा स्यात् बहुकृत्वो हरणतोऽतिसूक्ष्मश्च ।

[*T*, chap. II.]

Translation.—Multiply the diameter by 4. Subtract from it and add to it alternately the quotients obtained by dividing four times the diameter by the odd integers 3, 5, 7, . . . Let the process stop at a certain stage giving rise to a "finite sum". Multiply four times the diameter by half the even integer subsequent to the last odd integer used as a divisor and then divide by the square of the even integer increased by unity. The result is the correction to be added to or subtracted from our finite sum, the choice of addition or subtraction depending on the sign of the last term in the sum. The final result is the circumference determined more exactly than by taking a large number of terms, i.e. terms going beyond the stage at which we stopped.

THEOREM 11. $$C \doteqdot 4D\left\{1-\frac{1}{3}+\frac{1}{5}-\ldots\pm\frac{1}{n}\mp\frac{\left(\frac{n+1}{2}\right)^2+1}{\left[(n+1)^2+4+1\right]\left(\frac{n+1}{2}\right)}\right\}$$

n being odd and large.

Śloka :

अस्मात्सूक्ष्मतरोऽन्यो विलिख्यते कश्चनापि संस्कारः ।
अन्ते समसंख्यादलवर्यस्सैको गुणस्स एव पुनः ॥
युगगुणितो रूपायुतस्समसंख्यादलहतो भवेद्धारः ।
त्रिशरादिविषमसंख्याहरणात्परमे तदेव वा कार्यम् ॥

[*T*, chap. II.]

Translation.—Next is given another correction more precise than the foregoing. The square of half the even integer next greater than the last odd-integer divisor, increased by unity, is a multiplier. This multiplier multiplied by 4, then increased by unity and then multiplied by the even integer already defined, gives a divisor. Multiply and divide four times the diameter by our multiplier and divisor respectively. The result is an improvement on our previous correction.

THEOREM 12. $C \doteqdot 2D+4D\left\{\frac{1}{2^2-1}-\frac{1}{4^2-1}+\cdots\pm\frac{1}{n^2-1}\mp\frac{1}{2[(n+1)^2+2]}\right\}$

n being even and large.

Śloka :

द्व्यादियुजां विकृतयो व्येका हारा द्विनिघ्ना विष्कंभे ।
धनमृणमन्तेऽन्त्योर्ध्वगतौजकृतिद्विसहितहारो द्विघ्निः ॥

[*T*, chap. II.]

Translation.—Divide four times the diameter separately by the squares of the even integers (2, 4, 6, . . .) diminished by unity. The quotients alternately add to and subtract from twice the diameter, the process terminating at a certain stage and defining a "finite sum". Take the odd integer subsequent to the last even integer squared, square it, add 2 to the square, double the sum, and with the result thus obtained, divide four times the diameter. This quotient added to or subtracted from the finite sum defined, leads to a corrected value of the circumference.

II

The passage from the work *Āryabhaṭīya of Āryabhaṭācārya with the Bhāṣya of Nīlakaṇṭhasomsutvān*, referred to in footnote 18 of the article is part of the following.

कुतः पुनर्वास्तवीं संख्यामुत्सृज्यासन्नैवेहोक्ता । उच्यते । तस्या वक्तुमशक्यत्वात् । कुतः । येन मानेन मीयमानो व्यासो निरवयवः स्यात्, तेनैव मीयमानः परिधिः पुनः सावयव एव स्यात् । येन च मीयमानः परिधिर्निरवयवस्तेनैव मीयमानो व्यासोऽपि सावयव एव, इत्येकेनैव मानेन मीयमानयोरुभयोः क्वापि न निरवयवत्वं स्यात् । महान्तमध्वानं गत्वाप्यल्पावयवत्वमेव लभ्यम् । निरवयवत्वं तु क्वापि न लभ्यमिति भावः ॥ [*A* I (T.S.S.), 41*f*.]

Translation.—Why then is it that discarding the exact value, only the approximate one has been mentioned here? This is the answer: because it (the exact value) cannot be mentioned. If the diameter, measured with respect to (by comparison with) a particular unit of measurement, is commensurable, with respect to that same unit of measurement, the circumference is incommensurable (the circumference cannot be *exactly* measured by the same unit); and if with respect to any unit the circumference is commensurable, then, with respect to that same unit, the diameter is incommensurable. Thus there will

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never be commensurability for both with respect to the same unit of measurement. Even by going a long way, only the "degree of commensurability" can be made very small, absolute commensurability can never be attained.

III

Prince Rāma Varma (12th prince of Cochin) has brought to light a fact which goes some way towards substantiating the conjecture that the proofs in *Yukti-Bhāṣa* are almost as old as *Tantrasaṅgraha*. He has in his possession a Sanskrit manuscript commentary on Bhaskara's *Līlāvati*, by an unknown Keralīya. This commentary, which bears the title *Kriyākramakarī*, contains the proofs of many of our theorems in forms which lead one to suppose that they are the originals of the proofs in *Yukti-Bhāṣa*. There is a verse in the commentary :

नारायणं जगदनुग्रहजागरूकं
श्रीनीलकण्ठमपि सर्वविदं प्रणम्य ।
व्याख्यां क्रियाक्रमकरीं रचयामि लीला-
वत्याः कथञ्चिदहमल्पधियां हिताय ॥

the first two pādas of which are identical with those of the third benedictory stanza in the commentary attached to *Tantrasaṅgraha*. This identity corroborates the statement in the verse that the author of *Kriyākramakarī* is a student of Nīlakaṇṭha and its Kali day of compilation (suggested by the underlined chronogram in the verse) is the 1681915th. Thus *Kriyākramakarī* would seem to be a work belonging to the same period as *Tantrasaṅgraha* (4602 Kali era) ; and it is not unlikely that a close study of it will lead to valuable conclusions regarding the origin of the mathematical arguments in *Yukti-Bhāṣa*.

Paper 27

C. T. Rajagopal (1949)

A Neglected Chapter of Hindu Mathematics

Scripta Mathematica, Vol. 15, no.1, pp. 201–209.

Notes

Scripta Mathematica was a quarterly journal published by Yeshiva University devoted to the philosophy, history, and expository treatment of mathematics. It was said to be, at its time, "the only mathematical magazine in the world edited by specialists for laymen." The journal was established in 1932 under the editorship of Jekuthiel Ginsburg, a professor of mathematics at Yeshiva University, and its first issue appeared in 1933. It ceased publication in 1973.

C. T. Rajagopal

For a biographical sketch of C. T. Rajagopal see p. 247.

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A NEGLECTED CHAPTER OF HINDU MATHEMATICS

By C. T. RAJAGOPAL

"By the grace of the One I worship, launching my bark of understanding on the ocean of knowledge—which has its regions of darkness as well as of light—I have at last brought forth from the depths this spotless gem of thought."—ARYABHAṬA I

A STRANGE destiny has fashioned India as a land of mystery entrenched against alien intruders behind its philosophies as well as its superstitions. Mystery shrouds even Hindu mathematics. That Hindu mathematics had a notable ancestry is today widely admitted; but it is little suspected even by those with an intimate knowledge of Indian history that there is a vast and almost unexplored continent in Hindu mathematics. A little over a century has elapsed since the first attempt was made to mark on the map of modern scholarship this virgin continent. The person who sighted the unknown coast was, by an odd trick of time, an English civilian of the Hon. East India Company, Charles M. Whish by name. Whish's paper carrying the abbreviated title "On the Hindu Quadrature of the Circle" (5),[1] submitted to the Royal Asiatic Society of Great Britain and Ireland on the 15th of December 1832, did not advertise his importance as the discoverer of a strange hinterland. There was little in the title of the paper to assure its readers that the material offered to them had with difficulty been drawn from that stock of mixed mathematics which the children of Keraḷa[2] had till then looked upon as their exclusive property; there was nothing in it which suggested that the author had overcome the exclusivism of the Keraḷīyas with the help of their pundits and princes—a help by no means easy to secure then, for, as we know today, the companions of our author in the civil service of the Hon. East India Company were "fortune-hunting adventurers lost to all sense of public morality"[3] who did much to alienate the sympathies of the natives.

[1] Arabic numerals in parentheses indicate references given in full at the end.

[2] Kerala is the name given to the southwest seaboard of India comprising British Malabar and the native states of Cochin and Travancore.

[3] The quotation is from a thesis, *Civil Service in India under the East India Company*, by Akshoy Kumar Ghosal, approved for the degree of Doctor of Philosophy of the London University. This thesis has recently been published by the Calcutta University.

Charles Whish, our intrepid investigator, acknowledged, in the amplified title of his paper, the four Kerala sources which had contributed to his composition. Three of them were manuscripts in Sanskrit entitled *Karaṇapaddhati* [Operational Technique],[4] *Tantrasaṅgraha* [Science Digest] and *Sadratnamālā* [A String of Lustrous Gems]; they contained, among other things, the infinite-series expressions for circular arcs which we associate with the names of Gregory, Leibniz, and Halley. The fourth was a manuscript called *Yukti-Bhāṣa* [Expository (Mathematics)], written in a highly Sanskritized form of Malayāḷam, the vernacular now in use along the Kerala coast. It furnished demonstrations, more or less complete, of the mathematical propositions expressed in neat verse in the three Sanskrit works generally and in *Tantrasaṅgraha* particularly. According to Whish, the last-named work was the oldest of the manuscripts discovered by him, its period being that of the first Portuguese landing in India. More recent findings, while confirming Whish's conclusion with regard to the period, favor a still-living tradition which makes out *Karaṇapaddhati* to be earlier than *Tantrasaṅgraha*, earlier by some seventy years. However this may be, it is now generally accepted, that the latter work was produced about the year 1500 A.D. by a great Brahmin savant, Nīlakaṇṭha, whose genius was as much at home in the subtleties of Indian logic as in the severities of mathematical analysis. Such a view, coupled with the fact that Nīlakaṇṭha's work contains not only Gregory's series for $\pi/4$ but also rational approximations to the series by formulae peculiar to this author, invites history to accommodate a new possibility—the possibility, namely, that India anticipated by at least a century and a half the significant contributions of the Newtonian era towards the squaring of the circle.

The prospect thus revealed has not been greeted with enthusiasm by the antiquaries who have been studying the history of Indian thought. This is borne out by the fact that the references to the discoveries of Whish are scanty even in the histories of Indian mathematics, written in India; and also by the fact that more than a hundred years after the appearance of his essay only one of the Sanskrit works mentioned by him (*Karaṇapaddhati*) is available in printed form. For the other three works the investigator has even today to turn to the collections of manuscripts in the various public and private libraries of India. In face of such discouraging circumstances two young friends of mine were persuaded to undertake a preliminary resurvey

[4] The translations of the titles of Hindu works are, needless to say, not literal; they are merely attempts to convey some idea of what the Hindus meant to suggest by their titles.

of the coast discovered by Whish, with special reference to certain features of it which had been left out in his sketch-map. The results of this resurvey are put together in an article recently published in the *Journal of the Bombay Branch, Royal Asiatic Society* (4).

The mathematical part of the article is mostly a reconstruction of the arguments which led the Hindus to the series for the inverse tangent and to certain rational approximations to the limiting value of the series. This reconstruction aims at preserving, in an appropriate setting, the mathematics which inspired *Yukti-Bhāṣa*. A reconstruction with such an aim may not appeal to the orientalist imbued with a spirit of pure literalism, but may have some interest for the modern historian of mathematics. For this reason a presentation of it in these pages may not be altogether out of place; especially, one designed to bring out a new feature of the Hindu method of approximating to π, viz., its ability to yield approximations to log 2 and, in fact, to the sum of any series of the type

$$\sum_{n=1}^{\infty} (-1)^{n-1}/(an + b) \qquad (a > 0).$$

I

All the general approximation formulae for π to be found anywhere in Hindu mathematical literature, are stated without proof by Nīlakaṇṭha. Their common basis being the series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots,$$

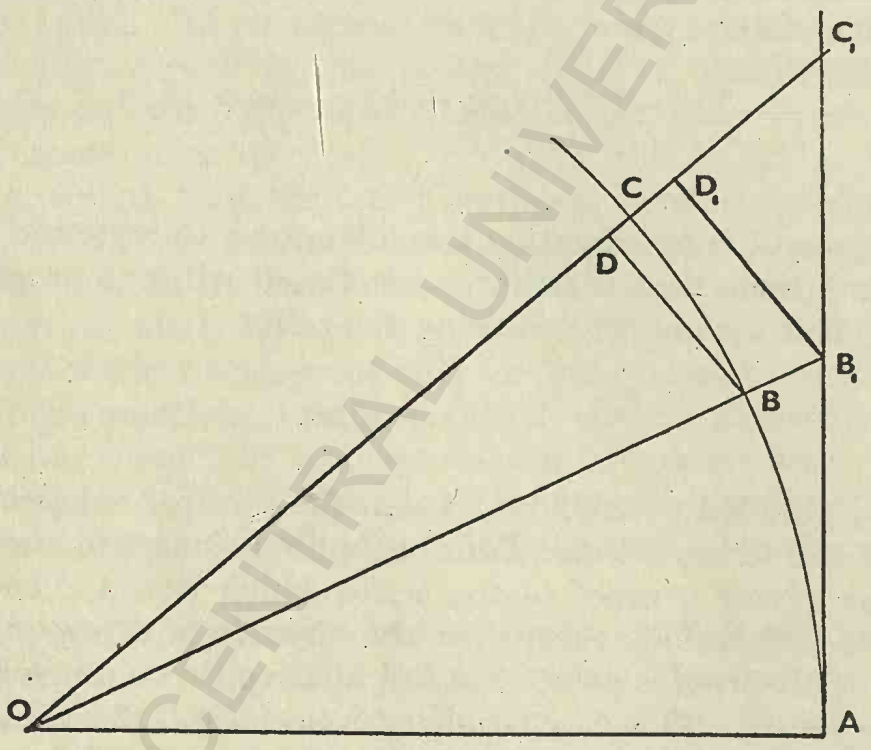
one is inevitably led to ask how Nīlakaṇṭha (or his predecessors) came to think of the series. *Yukti-Bāhṣa* supplies a complete answer to the question by giving a proof of the series which would even today be regarded as satisfactory except for the absence of a few justificatory remarks. This proof is printed in full in a recent volume of *Teacher's Magazine* (Journal of Cochin Teachers' Association) (3), while a résumé of it (applicable to the general series for the inverse tangent) appears in the publication of the Royal Asiatic Society already referred to (4). For the benefit of readers who may not have either of these journals ready at hand, I shall give here an outline of the proof, splitting it into three lemmas. The first of the lemmas is equivalent to the formula $d\theta = d(\tan\theta)/(1 + \tan^2\theta)$ which seems to have been used by Gregory (1671) to establish the series for the inverse tangent.[5] Of the other two lemmas, the second was first noticed in Europe by Euler (1739) and the third by Roberval (1634), also independently,

[5] See reference 2, p. 357, 463.

by Fermat (1636). Seeing that the Indian origin of these lemmas could not have been later than *Yukti-Bhāṣa* which is supposed to have been composed about 1639 A.D., one is led to conclude, at least provisionally, that two of them must have been known in India before they were known in Europe.

LEMMA 1. *Let BC be a small arc of a circle of unit radius whose center is O. If OB, OC meet the tangent at any point A of the circle in B_1, C_1, respectively, then an approximation to arc BC (first approximation in fact) is given by*

$$\text{arc } BC \overset{\text{app.}}{=} \frac{B_1C_1}{1 + AB_1^2}.$$



This is proved in *Yukti-Bhāṣa* by taking first a *finite* arc BC and drawing perpendiculars BD, B_1D_1 to OC. By considering successively the pairs of similar triangles (ΔOBD, ΔOB_1D_1), ($\Delta B_1C_1D_1$, ΔOC_1A), we find

$$\frac{BD}{B_1D_1} = \frac{OB}{OB_1} = \frac{1}{OB_1},$$

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$$\frac{B_1D_1}{B_1C_1} = \frac{OA}{OC_1} = \frac{1}{OC_1}$$

whence

$$BD = B_1C_1/OB_1 \cdot OC_1;$$

and so when BC is small, we have the approximate relations

$$\text{arc } BC \overset{\text{app.}}{=\!=} BD \overset{\text{app.}}{=\!=} B_1C_1/OB_1^2 = B_1C_1/(1 + AB_1^2).$$

LEMMA 2.

$$\text{arc tan } t = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t/n}{1 + (rt/n)^2}, \qquad |\text{arc tan } t| \leq \pi/4.$$

This is established by first dividing into n equal parts the intercept on the tangent at one end of the arc, between the bounding radii of the arc, and then applying Lemma 1 to each of the parts.

LEMMA 3.

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{r=0}^{n-1} r^p = \frac{1}{p+1}.$$

The lemma seems to have been proved for $p = 1, 2, 3, 4$, and assumed for all positive integral p.

Expressing Lemma 2 in the form

$$\text{arc tan } t = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t}{n} \left\{1 - \left(\frac{rt}{n}\right)^2 + \ldots + (-1)^{\nu-1}\left(\frac{rt}{n}\right)^{2\nu-2} + R_\nu\right\}$$

$$R_\nu(-1)^\nu\left(\frac{rt}{n}\right)^{2\nu} / \left\{1 + \left(\frac{rt}{n}\right)^2\right\},$$

and making use of Lemma 3 to evaluate each of the limits on the right-hand side, *Yukti-Bhāṣa* proves

THEOREM A.

$$\text{arc tan } t = t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots, \qquad |t| \leq 1,$$

of which the first explicit mention in Hindu literature seems to be in *Karaṇapaddhati*. For, *Sadratnamālā* which also contains the theorem would seem to be a work of the 19th century, that is to say, a work which, even if we accept Whish's hypothesis of a late origin for *Karaṇapaddhati*, cannot have preceded this work.

An examination of the proof outlined above brings out two points: (1) The passage from Lemma 2 to Theorem A is equivalent to a term-wise integration of the relation $\theta = \int_0^\theta d(\tan\theta)\,[1 + \tan^2\theta]^{-1}$. There-

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fore the proof of Theorem A in *Yukti-Bhāṣa* is, in substance, the same as the one which Gregory is supposed to have given. However, the former is, on the face of it, a product of the pre-calculus era, since geometrical and algebraical considerations alone have molded its form. (2) The Hindu proof is not a by-product of quadrature investigations connected with figures other than the circle, as for instance, Leibniz's proof of the series for arc tan 1.[6] The proof accomplishes with marvelous economy of means neither more nor less than what it has set out to do—which is the rectification of a circular arc to serve the immediate needs of astronomical computation.

II

In Nīlakaṇṭha and his commentator (the author of *Yukti-Bhāṣa*), we find rational approximations to π such as

$$\frac{\pi}{4} \overset{\text{app.}}{=\!=} 1 - \frac{1}{3} + \frac{1}{5} - \ldots \pm \frac{1}{n} \mp \frac{(n+1)/2}{(n+1)^2+1} \qquad (n \text{ large}),$$

$$\frac{\pi}{4} \overset{\text{app.}}{=\!=} 1 - \frac{1}{3} + \frac{1}{5} - \ldots \pm n \mp \frac{(n+1)^2/4 + 1}{[(n+1)^2 + 4 + 1](n+1)/2} \qquad (n \text{ large}),$$

whose remarkable feature is that, as far as we know, similar approximations are not to be found elsewhere in the mathematical literature of the world. Since Nīlakaṇṭha has given at least three of these approximations, it is at once suggested that either he or a predecessor might have been led to them by reasoning of a certain general type, without however either of them being very clearly aware of the fact. When we try to follow the lines along which *Yukti-Bhāṣa* seeks to prove the first of these approximations quoted above, we find that this is indeed the case, the entire proof being based on an intuitive estimate of certain orders of smallness represented by powers of n^{-1}. This point having been discussed in great detail elsewhere (4), it will be sufficient for our present purpose to isolate the principle which guided the intuitive faculty in question and present it in the form of

THEOREM B. *Hypotheses:*

(i) $$s = \frac{1}{\alpha+\beta} - \frac{1}{3\alpha+\beta} + \frac{1}{5\alpha+\beta} - \ldots \qquad (\alpha > 0);$$

(ii) $$2f(x) = \sum_{r=1}^{2p} (r-1)!\, a_r/x^r,$$

[6] See reference 1, p. 80–81.

where the a's satisfy the recurrence formula:

$$a_r + \frac{a_{r-2}}{2!} + \frac{a_{r-4}}{4!} + \ldots + \left\{\text{last term} = \frac{a_1}{(r-1)!} \text{ or } \frac{a_2}{(r-2)!}\right\} =$$

coefficient of x^{r-1} in the expansion of $\frac{1}{\alpha} e^{-\beta x/\alpha}$;

(iii) $$u_n = \frac{1}{\alpha n + \beta} - f(n+1) - f(n-1) \qquad (n = 3.5, \ldots),$$

$$u_1 = \frac{1}{\alpha + \beta} - f(2);$$

(iv) $$\sigma_n = \frac{1}{\alpha+\beta} - \frac{1}{3\alpha+\beta} + \ldots \pm \frac{1}{n\alpha+\beta} \mp f(n+1)$$

$(n = 1, 3, \ldots).$

Conclusion:

(v) $u_1 - u_3 + u_5 - \ldots$ *is a transformation of the given series for S in which*

$$u_n = O(1/n^{2p+1}) \, ds \, n \to \infty;$$

(vi) *in general,* σ_n *is an approximation to S satisfying the condition:*

$$|\sigma_n - S| = O(1/n^{2p+1}).$$

Proof: For all sufficiently large n (iii) is equivalent to

$$u_n = \sum_{r=0}^{\infty} \frac{(-1)^r \beta^r}{\alpha^{r+1} n^{r+1}} - \sum_{r=0}^{p-1} \frac{2f^{(2r)}(n)}{(2r)!} - \frac{1}{(2p)!}\{f^{(2p)}(n+\theta_1) + f^{(2p)}(n-\theta_2)\}$$

where, in the right-hand member, $0 < \theta_1, \theta_2 < 1$ and the last two terms are each $O(1/n^{2p+1})$. Hence, in this member, the coefficient of

$$\frac{1}{n^r} = \frac{(-1)^{r-1}\beta^{r-1}}{\alpha^r} - (r-1)!\left\{a_r + \frac{a_{r-2}}{2!} + \ldots + (\ \ )\right\}$$

$$= 0 \qquad (1 \leq r \leq 2p);$$

and our expansion of u_n begins with n^{-2p-1} multiplied by the (generally non-zero) coefficient

$$\frac{\beta^{2p}}{\alpha^{2p+1}} - (2p)!\left\{\frac{a_{2p-1}}{2!} + \frac{a_{2p-3}}{4!} + \ldots + \frac{a_1}{(2p)!}\right\} - a_1$$

This shows that, for all large n, $\{u_n\}$ is a monotonic null sequence. It also proves (v) when taken along with the fact

$$u_1 - u_3 + \ldots \pm u_n = \sigma_n \to S.$$

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To prove (vi) and complete our demonstration, we have only to notice that

$$|\sigma_n - S| = |1 \mp u_{n+2} \pm u_{n+4} \ldots| < |u_{n+2}| = O(1/n^{2p+1}).$$

The particular cases of σ_n mentioned at the outset of this section are now seen to result from two observations: (1) In the case $\beta = 0$, $f(n)$ does not contain even powers of n^{-1}. (2) In any case, $f(n)$ can be taken to be a convergent series of powers of n^{-1} whose first $2p$ coefficients satisfy the given recurrence formula. Consequently, in the case $\beta = 0$, we can represent $f(n)$ by a finite continued fraction,

$$2f(n) = \frac{b_1|}{|n} + \frac{b_3|}{|n} + \ldots \frac{b_{2p-1}}{n}$$

and calculate the b's in terms of the a's. Nīlakaṇṭha's approximations already mentioned are obtained with $p = 2$ and $p = 3$ ($\alpha = 1$, $\beta = 0$).

To illustrate the measure of approximation which can be secured by means of σ_n, when n is small, we can take the series for log 2 (which is the case $\alpha = \beta = {}^1/_2$). The partial sum of 10 terms of the series is (correct to 7 places of decimals) 0.6456348, while the same sum, after applying to it the correction $+f(20)$ in which $p = 2$, becomes 0.6931473 which is the value of log 2 correct to 6 places of decimals. The same work, when repeated with the series for $\pi/4$, gives π correct to 5 places of decimals, furnishing a second illustration of the efficacy of our approximations.

It is well to remember that Nīlakaṇṭha's commentator, the writer of *Yukti-Bhāṣa*, has set forth the essential content of Theorem B. In his proof of the case $\alpha = 1$, $\beta = 0$, $p = 2$ of Theorem B, he starts from σ_n as defined in the theorem and obtains the coefficients in $f(n)$ by a "trial and error" method, concluding with the derivation of the series $\sum (-1)^{n-1} u_{2n-1}$. Thus it is reasonable to suppose that the substance of Theorem B is as old as *Tantrasaṅgraha*, very likely not older since we are unable to trace it to any previous work. In other words, there is some justification for naming Theorem B after Nīlakaṇṭha.

A question confronting us at this point is whether it is possible in the same way, that is, arguing from the evidence furnished by mathematical tests, to name Theorem A after any particular Hindu mathematician. Such a question cannot yet be answered; one wonders sometimes whether it will ever be. One can, however, say, without any violence to the available data, that Theorem A seems to have originated with a group of Hindu scholars who were either contemporaries of Leonardo da Vinci, or separated from him by not more

than two generations. Looking down the aisles of time we can pick out at least three figures in this group—the gifted Nīlakaṇṭha who could go so far afield as to attract a pupil like Tuṇcath Rāmānujan Ezuthassan, that bright "morning star" in the vault of Kerḷaa literature; the austere Paramesvara who gave up a whole lifetime to the service of astronomy and must undoubtedly have acted as a beacon to his son's distinguished pupil, Nīlakaṇṭha; and finally that enigmatic genius who produced *Karaṇapaddhati* and wrapped himself up in a cloak of anonymity which to this day has resisted all attempts to pierce it. There is some difficulty about attributing finally to any one of these men the first discovery of the series which now bears the name of Gregory. This discovery must remain, unless we can push our historical inquiry further, the achievement of an unidentified "Brahmin who lived on the coast of the western ocean, and having examined the heavens for twelve years," laid down the foundations of the mathematics in Nīlakaṇṭha's digest. But even if we fail to give this Brahmin a name and local habitation, he will for ever be a type of that ancient spirit which is India—a man in loincloth standing on the edge of an illimitable past; a man looking frail and futile in a heaving, toiling, suffering mass of humanity; a man almost like any other in that primitive throng . . . almost, but not quite; for *Yukti-Bāhṣa* gives us a glimpse of him as he voyages alone through those strange seas of thought which could not be charted except by a Newton and a Leibniz.

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Paper 28

C. T. Rajagopal and A. Venkataraman (1949)

The Sine and Cosine Power-series in Hindu mathematics

Journal of the the Royal Asiatic Society of Bengal, Vol. 15, no. 1, 1949.

Notes

According to the authors, the paper discusses the significant achievement of the mathematicians of the Kerala school, namely, "a *tour de force* of analytical reasoning which led them to the sine and cosine power-series some two hundred years before Newton amd Leibniz claimed them for occidental mathematics."

C. T. Rajagopal

For a biographical sketch, see p. 247.

A. Venkataraman

Unfortunately, little biographical details of A. Venkataraman are available in the world of the internet.

Source: Internet Archive

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Vol. XV, No. 1, 1949

THE SINE AND COSINE POWER-SERIES IN HINDU MATHEMATICS

By C. T. RAJAGOPAL *and* A. VENKATARAMAN

Madras Christian College, Tambaram, S. India

(Communicated by Prof. P. C. Sengupta)

(*Received September 17, 1947*)

An account was given, in three earlier papers [5, 6, 7], * of some of the notable achievements of the Kerala Hindus of the middle ages in the field of mathematical analysis. In the present paper we propose to discuss another significant achievement of the Keraḷīyas—a *tour de force* of analytical reasoning which led them to the sine and cosine power-series some two hundred years before Newton and Leibniz claimed them for occidental mathematics.†

The Hindu mathematical literature which has come down to us does not associate these series with the name of any individual mathematician. In fact we are as much in the dark about the identity of the person(s) who first obtained the two series, as we are about the Hindu discoverer(s) of the power-series for the inverse tangent. We can, however, say with some certainty that all the three series came to be accepted as part of the repertory of a mathematician about the year 1500 A.D. when Nīlakaṇṭha published his volume of 'Science Abstracts' ‡ (*Tantrasaṅgraha*). We can also point to external evidence which makes it highly probable that the proofs of the three series were known to Nīlakaṇṭha's compeers and transmitted to his successors either orally or in writing [4, Appendix III] for perhaps two hundred years and more before they appeared in Malayāḷam, the vernacular of the Kerala country, in a small book of expository mathematics called *Yukti-Bhāṣa* (*circa* 1750). The proofs of the sine and cosine series which we offer to our readers are taken from this Malayāḷam work, while the enunciations of the series are based on ślokas or verses in *Tantrasaṅgraha* and another Sanskrit work, *Karaṇapaddhati*, supposed to be anonymous and produced a few decades before Nīlakaṇṭha's compendium.

* Numbers in heavy type within square brackets refer to the bibliography given at the end of this paper.

† Moritz Cantor [1] gives these facts about the European origin of the series. (1) A letter from Collins to Gregory dated December 24, 1670, communicated the series for the sine, the cosine and the arc sine as the discoveries of Newton. (2) About the end of July 1676, Oldenburgh sent to Leibniz a copy of a letter of Newton containing the binomial, the sine and the cosine series. (3) One of the problems occupying the attention of Leibniz in 1693 was the integration of differential equations in series, in particular that of the equation giving rise to the sine series: $a^2d^2x/dy^2+x = 0$ which is, in the notation of the present paper, $\rho^2d^2y/ds^2+y = 0$. The last-mentioned fact would appear to suggest that Leibniz's approach to the series was different from that of the Hindus. As regards Newton's approach to the series, Cantor does not offer us any material on the basis of which we can make up our minds.

‡ *Tantrasaṅgraha* is more correctly translated as 'a collection of *Tantras*'. The Tantras are a special class of Sanskrit astronomical literature in contradistinction to *Siddhāntas*. (P. C. Sengupta.)

Hindu Terminology

Choose on a given circle (of radius ρ) any point between the East and the North points. Then the arc (s) of the circle, having its *mūlam* (= base or beginning) at the East point and its *agram* (= summit or extremity) at the chosen point, is called *iṣṭacāpam* (which means literally 'chosen bow'). The perpendiculars from the chosen point to the East-West line and the North-South line are termed respectively the *bhujājyā* (y, say) and the *koṭijyā* (x, say) through the point*; the distance of the foot of the former perpendicular from the East point ($\rho - x = \bar{x}$, say) is referred to as the *śaram* (meaning literally 'arrow') of the chosen point. When it is found necessary to stress the associations of the *jyā* and the *śaram* with the chosen arc, we may prefix to each of them the adjective *iṣṭa*.

An arc of the circle whose extremities are any two points between the East and the North points may be called *śiṣṭacāpam* (literally 'part of bow'). The difference between the *bhujājyās* through the extremities of the arc is called the *bhujakhaṇḍam* of the *śiṣṭacāpam*; the difference between the *koṭijyās* of the extremities and that between the *śarams* of the extremities are called *koṭikhaṇḍam* and *śarakhaṇḍam* respectively.

The chord of any arc (whether it is an *iṣṭacāpam* or *siṣṭacāpam*) is termed its *samastajyā*.

Enunciations of the Sine and Cosine Series

Tantrasaṅgraha gives an enunciation which is not self-contained and runs thus:

> Nihatya cāpavargeṇa cāpam tattatphalāni ca
> Haretsamūlayugvargaistrijyāvargahataiḥ kramāt
> Jīvāptyai sangrahosyaiva vidvānityādina kṛtaḥ
> Nihatya cāpavargeṇa rūpam tattatphalāni ca
> Haredvimūlayugvargaistriyāvargahataiḥ kramāt
> Śarāptyai sangrahosyaiva stenastrītyādinā kṛtaḥ.

An English rendering of the passage may be given as follows. *One should (obtain a sequence of terms as follows:) repeatedly multiply the arc by the square of itself and divide by the square of each and every even number increased by that number and then multiplied by the square of the radius. This is (the starting point of) the process of getting at the* **(bhujā)jyā** *(of the arc), which is fully discussed in (the well-known verse) 'Vidvān, etc.'. One should (next obtain a second sequence thus:) repeatedly multiply the unit of measurement (which is the radius) by the square of the arc and divide by the square of each and every even number decreased by that number and then multiplied by the square of the radius. This is (the first step in) the process of getting at the* **śaram** *(of the arc), summarized in the verse 'Stenastrī, etc.'.*

The first sequence is

$$\frac{s^{2n+1}}{(2^2+2)(4^2+4)\ldots.(4n^2+2n)\rho^{2n}} \quad (n=1, 2, 3\ldots.) \qquad .. \qquad (1)$$

---

* The terms *cāpam*, *bhujājyā* and *koṭijyā* are generally used for double the lengths thus designated in our definitions. However, these definitions are in conformity with the usage in the particular context of *Yukti-Bhāṣa* which concerns us. Incidentally it may be noticed that *jyā* is a general term for either a chord parallel to one of the primary directions (East-West, North-South) or one half of such a chord. The particular half chords of this kind which figure prominently in our propositions are *bhujājyās*, but since there are similar propositions for *koṭijyās*, *Yukti-Bhāṣa* uses the general term *jyā* for the half chords, merely clarifying the term wherever necessary, by a descriptive epithet such as *piṇḍam* (= section) or *paṭhitam* (= object of study).

and the second

$$\frac{\rho s^{2n}}{(2^2-2)(4^2-4)\ldots(4n^2-2n)\rho^{2n}} \quad (n = 1, 2, 3 \ldots) \quad \ldots \quad \text{(II)}$$

both of them being included in the sequence,

$$\frac{s^n}{n!\ \rho^{n-1}} \quad (n = 2, 3, 4 \ldots) \ldots \quad \ldots \quad \ldots \quad \text{(III)}$$

The verses commencing with 'Vidvān' and 'Stenastrī', referred to in the above quotation, are the 14th and the 15th of Chapter VI of *Karaṇa-paddhati*. They explain how one can determine the *iṣṭabhujājyā* and *iṣṭaśaram* from certain codified values of the members of (I) for $n = 1, 2, \ldots 5$, and the members of (II) for $n = 1, 2, \ldots 6$ worked out on the assumption that s = a quarter of the circumference and $\rho = 3437$ *kalas* $+44$ *vikalas* $+48\frac{2}{60}$ *talparas*.* The method of constructing the series for the *iṣṭabhujājyā* and the *iṣṭaśaram* from the sequences (I) and (II) is explained not in these verses, but in the two immediately preceding these verses:

Cāpācca tattat phalatopi tadvat
Cāpāhatādvyādi hatat trimaurvyā
Labdhāni yugmāni phalānyathotah
Cāpādayugmāni ca vistarārthāt

Vinyasya coparyupari tyajettat
Śeṣau bhujākoṭiguṇau bhavetām

which may be translated thus. *Obtain the results of repeatedly multiplying the arc by itself and then dividing by* 2, 3, 4, ... *multiplied by the* **trimaurvyā** *or the radius. Write down, below the radius (in a column) the even results (i.e. results corresponding to* $n = 2, 4, 6 \ldots$ *. in* $s^n/n!\ \rho^{n-1}$), and below the radius (in another column) the odd results (corresponding to $n = 3, 5, 7, \ldots$ *. in* $s^n/n!\ \rho^{n-1}$). *(After writing down a number of terms in each column) subtract the last term of either column from the one next above it, the remainder from the term next above (the one last take into account) and so on (until the last subtraction is made from the radius in the first column and from the arc in the second). The two (final) remainders are respectively the* **koṭi(jyā)** *and the* **bhujā (jyā,** *to a certain degree of approximation).*

We have therefore the series:

$$x = \rho - \frac{s^2}{2!\ \rho} + \frac{s^4}{4!\ \rho^3} - \ldots + (-1)^n \frac{s^{2n}}{(2n)!\ \rho^{2n-1}} + o(1),\ n \to \infty,$$

$$y = s - \frac{s^3}{3!\ \rho^2} + \frac{s^5}{5!\ \rho^4} - \ldots + (-1)^n \frac{s^{2n+1}}{(2n+1)!\ \rho^{2n}} + o(1),\ n \to \infty.$$

Yukti-Bhāṣa regards each of these series as the result of successive corrections, $(-1)^n s^{2n}/(2n)!\ \rho^{2n-1}$ or $(-1)^n s^{2n+1}/(2n+1)!\ \rho^{2n}$ applied to ρ or s

* Here *kala* = an arc subtending at the centre an angle of 1 minute, *vikala* = an arc of 1 second, *talpara* = an arc of 1/60 of a second. Since a quarter circumference is 5,400 minutes of arc, it follows that a particular value of $\pi/2$ is assumed in this context. The codo words used for the corresponding values of $(\pi/2)^{11}\ \rho/111$ and $(\pi/2)^{12}\ \rho/121$ are *vidvān* and *stena* respectively. We are indebted to Prof. P. C. Sengupta for pointing out that the value assumed for π in these evaluations corresponds to a value of the radian remarkably close to the modern approximation 3437′·74677078 = 3437′−44″−48‴−22iv·49 (in Indian units), and distinct from Āryabhaṭa's approximation 3437′−44″−19‴·425.

as the case may be, every correction being numerically greater than the true correction necessary. Thus the Malayāḷam commentary makes it clear to us that by taking into account more and more terms in the series for x and y, we approach the true values of x and y more and more closely. The idea of approximation which appears parenthetically in our translation of the *Karaṇapaddhati* verses, is therefore not an interpolation.

Some Lemmas from *Yukti-Bhāṣa*

Certain lemmas will now be stated and proved in the order in which they appear in *Yukti-Bhāṣa*. The right-angled triangle whose hypotenuse is the radius through the mid-point of a *śiṣṭacāpam* and sides are the *bhujājyā* and the *koṭijyā* through the mid-point is similar to the right-angled triangle whose hypotenuse is the *samastajyā* of the *śiṣṭacāpam* and sides are the *koṭikhaṇḍam* and the *bhujakhaṇḍam* of the *śiṣṭacāpam*, for the two triangles have their corresponding sides (in the order mentioned) mutually perpendicular. Hence we have the

Fundamental Lemma

$$\frac{\textit{Samastajyā of śiṣṭacāpam}}{\text{radius}} = \frac{\textit{koṭi khaṇḍam of śiṣṭacāpam}}{\textit{bhujājyā of middle of śiṣṭacāpam}} = \frac{\textit{bhujakhaṇḍam of śiṣṭacāpam}}{\textit{koṭijyā of middle of śiṣṭacāpam}}$$

In the figure, $\overset{\frown}{P_{-1}P_1}$ is the *śiṣṭacāpam*, belonging to a circle with centre o and radius ρ. Its mid-point is P and its *samastajyā* ($P_{-1}P_1$) is α; the *bhujājyās* of P_{-1}, P_1, P are y_{-1}, y_1, y respectively and their *koṭijyās* x_{-1}, x_1, x respectively. Consequently,

$$\frac{\alpha}{\rho} = \frac{x_{-1}-x_1}{y} = \frac{y_1-y_{-1}}{x}; \quad .. \quad .. \quad .. \quad (1)$$

or, in central-difference notation,

$$\delta x = -\frac{\alpha}{\rho}y, \quad \delta y = \frac{\alpha}{\rho}x.$$

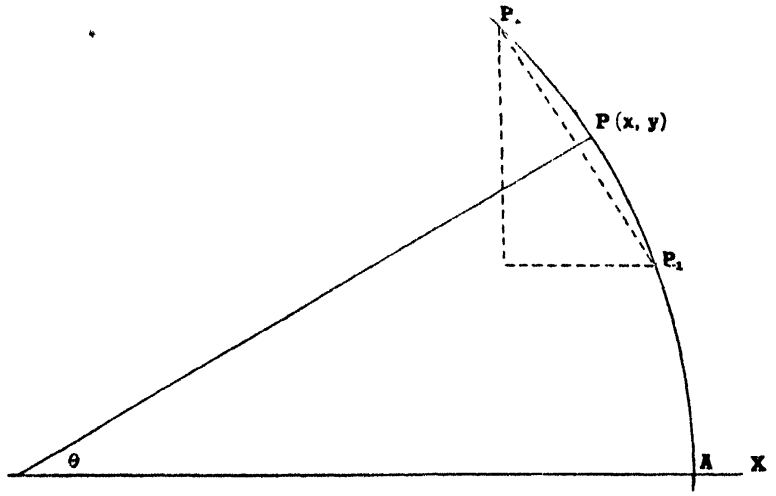

Fig. 1.

If θ, $\theta \pm \delta\theta$ be the angular distances of P, $P_{\pm 1}$ from the East point A of the circle, then $\alpha = \text{app. } 2\rho\delta\theta$ provided $\delta\theta$ is small, in which case the above relations yield:

$$\cos(\theta+\delta\theta) - \cos(\theta-\delta\theta) = \text{app. } -2\delta\theta \sin\theta,$$

$$\sin(\theta+\delta\theta) - \sin(\theta-\delta\theta) = \text{app. } 2\delta\theta \cos\theta.$$

Note.—(i) The last result was known to Bhāskara (twelfth century) who enunciated it in the form:

$$\text{Instantaneous change in } \sin\theta = \frac{\text{corresponding change in } s}{\rho} \times \cos\theta$$

or

$$\sin(\theta+\delta\theta) - \sin\theta = \frac{\delta s}{\rho}\cos\theta.$$

Bhāskara used a special term *tātkālika bhogya-khaṇḍa* for what we have called the 'instantaneous change' and his argument was much the same as that in *Yukti-Bhāṣa* [3 *].

(ii) The first explicit reference to our fundamental lemma in occidental literature seems to occur in an essay which forms part of a long letter written by Pascal at the beginning of October 1658 and is reproduced by J. M. Child in one of his historical works [2, p. 204, foot-note 20]:

The radius, ρ, is to the ordinate, y, as the hypotenuse, δs, of the infinitesimal right-angled triangle, whose sides are δx, δy, is to its base, δx.

NOTATION. In the remaining lemmas we shall use the notation explained below.

$\widehat{AP}$ is an *iṣṭacāpam* of a circle centre o and radius ρ. It is of length s and divided into n equal *khaṇḍams* or parts by points of division P_r ($r = 1, 2, \ldots . n-1$); the mid-point of any *cāpakhaṇḍam* $\widehat{P_{r-1}P_r}$ ($P_0 = A$, $P_n = P$) is $P_{r-\frac{1}{2}}$. The *bhujājyā* associated with any of the *cāpams* $\widehat{P_0P_r}$ ($r = 1, 2, \ldots . n$) may, for our present purpose, be referred to as a *piṇḍajyā* or a *paṭhitajyā* and denoted by y_r; the corresponding *koṭijyā* and *śaram* being denoted by x_r and $\bar{x}_r$ respectively. The last *piṇḍajyā* (y_n), *koṭijyā* (x_n) and *śaram* ($\rho - x_n$) are the same as the *iṣṭabhujājyā* (y), *iṣṭakoṭijyā* (x) and *iṣṭaśaram* ($\rho - x = \bar{x}$) respectively. The *cāpakhaṇḍasamastajyā* (α, say) is, of course, equal to the *samstajyā* of any *śiṣṭacāpam* such as $\widehat{P_{r-\frac{1}{2}}P_{r+\frac{1}{2}}}$ ($r = 1, 2, \ldots . n-1$).

LEMMA 1. *The* **koṭikhaṇḍam** *of the* **śiṣṭacāpam** $\widehat{P_{r-\frac{1}{2}}P_{r+\frac{1}{2}}}$ *whose extremities are the mid-points* ($P_{r-\frac{1}{2}}$, $P_{r+\frac{1}{2}}$) *of consecutive* **cāpakhaṇḍams**, *is obtained by multiplying the* **bhujājyā** *through the common point* (P_r) *of the* **cāpakhaṇḍams** *by the* **cāpakhaṇḍasamastajyā** and then dividing by the radius:

$$x_{r+\frac{1}{2}} - x_{r-\frac{1}{2}} = -\frac{\alpha}{\rho} y_r \quad (r = 1, 2, \ldots . , n-1). \qquad .. \quad (2)$$

* The article referred to is based on a note in the *Journal of the Asiatic Society of Bengal*, Vol. 27 (1858), pp. 213–216 and another in the *Journal of the Royal Asiatic Society*, Vol. 17 (1860), pp. 221-2.

Further, multiplying the above **koṭikhaṇḍam** *by the* **cāpakhaṇḍasamastajyā** *and then dividing by the radius, we get the difference between the* **bhujakhaṇḍams** *of the consecutive* **cāpakhaṇḍams** $\overset{\frown}{P_rP_{r+1}}$, $\overset{\frown}{P_{r-1}P_r}$; *that is to say,*

$$(y_{r+1}-y_r)-(y_r-y_{r-1}) = \frac{\alpha}{\rho}(x_{r+\frac{1}{2}}-x_{r-\frac{1}{2}}) \quad (r = 1, 2, \ldots, n-1). \qquad (3)$$

Hence, if we multiply the **piṇḍajyā** *through any point of division,* P_r, *by the square of the* **cāpakhaṇḍasamastajyā** *and divide by the square of the radius, the result is the difference between the* **bhujakhaṇḍams** *of the* **cāpakhaṇḍams,** $\overset{\frown}{P_{r-1}P_r}$, $\overset{\frown}{P_rP_{r+1}}$, *on either side of the point of division:*

$$(y_{r+1}-y_r)-(y_r-y_{r-1}) = -\frac{\alpha^2}{\rho^2}y_r \quad (r = 1, 2, \ldots, n-1).^* \ldots \qquad (4)$$

This difference between *bhujakhaṇḍams* (or, as they are sometimes referred to, *khaṇḍajyās*) is called *jyākhaṇḍāntaram*. The rule (4) for its evaluation is a consequence of (2) and (3) which are themselves consequences of (1). In central-difference notation we can state the rule in the form

$$\delta^2 y_r = -\frac{\alpha^2}{\rho^2}y_r.$$

Now multiply the relations (4) beginning with the last (for which $r = n-1$) by 1, 2, 3, ... respectively and add them. We then get

LEMMA 2. *The difference between the last* **piṇḍajyā** *and n times the first* **piṇḍajyā** *is equal to the 'second sum' of the* **piṇḍajyās** *(excluding the last) multiplied by the square of the* **cāpakhaṇḍasamastajyā** *and divided by the square of the radius. Or, briefly,*

$$y_n - ny_1 = -\frac{\alpha^2}{\rho^2}[y_{n-1}+2y_{n-2}+\ldots.+(n-1)y_1]$$

$$= -\frac{\alpha^2}{\rho^2}[y_1+(y_1+y_2)+\ldots.+(y_1+y_2+\ldots.+y_{n-1})].$$

When n is indefinitely large, the *piṇḍajyā* y_1 = app. the *cāpakhaṇḍam* s/n = app. the *cāpakhaṇḍasamastajyā* α, and so the above result gives the *iṣṭajyacāpāntaram* (i.e. difference between the *iṣṭajyā* and the *iṣṭacāpam*):

$$y-s = -\frac{1}{\rho^2}\lim_{n\to\infty}\left(\frac{s}{n}\right)^2[(n-1)y_1+(n-2)y_2+\ldots.+y_{n-1}]. \qquad (5)$$

Next, adding the equalities (2), we obtain

LEMMA 3. *The difference between the* **koṭijyā** *of the mid-point,* $P_{\frac{1}{2}}$, *of the first* **cāpakhaṇḍam** *and that of the mid-point,* $P_{n-\frac{1}{2}}$, *of the last* **cāpakhaṇḍam** *is the sum of the* **piṇḍajyās** *(excluding the last) multiplied by the* **cāpakhaṇḍasamastajyā** *and divided by the radius. In the notation we have adopted, this means that*

$$x_{n-\frac{1}{2}}-x_{\frac{1}{2}} = -\frac{\alpha}{\rho}(y_1+y_2+\ldots.+y_{n-1}).$$

* A referee has kindly drawn our attention to the following facts. (i) What is presumably the first appearance of the difference rule (4) is in stanza 12 of the Gaṇita section of *Āryabhaṭīya*, a translation of which appears in the *Calcutta University Journal of Letters*, Vol. 16 (1927). (ii) A later statement of the same rule may be found in *Sūryasiddhānta*.

When n is indefinitely large, *koṭijyās* $x_{n-\frac{1}{2}}$, $x_{\frac{1}{2}}$ may be replaced by $x_n = x$, $x_0 = \rho$ respectively and hence we have an expression for the *iṣṭaśaram*:

$$-\bar{x} = x - \rho = -\frac{1}{\rho}\lim_{n\to\infty}\left(\frac{s}{n}\right)(y_1 + y_2 + \ldots . y_{n-1}). \quad .. \quad (6)$$

LEMMA 4. *Let the* **cāpam** *with* **mūlam** *at the East point,* P_0, *and* **agram** *at a point of division,* P_r, *be of length* s_r (*so that* $s_n = s$). *Straighten the entire* **iṣṭacāpam**, P_0P_n. (i) *On the base* $P_{r-1}P_r$ ($r = 1, 2, \ldots, n$) *construct a rectangle of height* s_r. *Denote the total area of the rectangles up to* P_r *by* $s_r^{(1)}$.* (ii) *Next, on the base* $P_{r-1}P_r$ ($r = 1, 2, \ldots, n$) *construct a rectangle of height* $s_r^{(1)}$. *Let the total area of the new rectangles up to* P_r *be* $s_r^{(2)}$. *Define in this way* $s_r^{(3)}$, $s_r^{(4)}$, *etc. Then the sum of all the rectangles in* (i), (ii) .... *are* $s_n^{(1)}$, $s_n^{(2)}$ .... *respectively. When* n *is indefinitely large, they are called first* **cāpasaṅkalitam**, *second* **cāpasaṅkalitam**, *etc., with respect to the* **cāpam**, *and may be denoted by* $I^{(1)}(s)$, $I^{(2)}(s)$, *etc. Thus*

$$I^{(1)}(s) = \lim_{n\to\infty}\left(\frac{s}{n}\right)[s_1 + s_2 + \ldots . + s_n]$$

$$= \lim_{n\to\infty}\frac{s(s+s/n)}{2!} = \frac{s^2}{2!}\dagger$$

$$I^{(2)}(s) = \lim_{n\to\infty}\left(\frac{s}{n}\right)[s_1^{(1)} + s_2^{(1)} + \ldots . + s_n^{(1)}]$$

$$= \lim_{n\to\infty}\left(\frac{s}{n}\right)^2[s_1 + (s_1+s_2) + \ldots . + (s_1+s_2+\ldots . + s_n)]$$

$$= \lim_{n\to\infty}\left(\frac{s}{n}\right)^2[ns_1 + (n-1)s_2 + \ldots . + s_n]$$

$$= \lim_{n\to\infty}\frac{s(s+s/n)(s+2s/n)}{3!} = \frac{s^3}{3!}\dagger$$

Using the notation of the integral calculus we can write these results in the form:

$$I^{(1)}(s) = \int_0^s s\,ds, \quad I^{(2)}(s) = \int_0^s\int_0^s s(ds)^2, \text{ etc.},$$

---

* *Yukti-Bhāṣa* conceives this area as being made up of a single square on P_0P_1, two squares on P_1P_2, etc., taking n to be a very large number such as *parārdham* (10^{17}).

† Calling the upper limit of the variable s with which we are concerned *padam*, *Yukti-Bhāṣa* sums up the results in the form:

$$\text{First } saṅkalitam = \lim_{n\to\infty}\frac{Padam\ (Padam + khaṇḍam)}{2!} = \frac{(Padam)^2}{2!}$$

$$\text{Second } saṅkalitam = \lim_{n\to\infty}\frac{Padam\ (Padam + khaṇḍam)\ (Padam + 2\ khaṇḍams)}{3!} = \frac{(Padam)^3}{3!}$$

The word *saṅkalitam* is used here in the sense of an integral; but, in other places, *Yukti-Bhāṣa* uses the same word to denote a sum [e.g. 6, Lemma 3(*a*), (*b*)].

so that, writing $s = I^{(0)}(s)$, we have

$$I^{(p)}(s) = \int_0^s I^{(p-1)}(s)\, ds = \lim_{n\to\infty} \left(\frac{s}{n}\right) \times \left[I^{(p-1)}\left(\frac{s}{n}\right) + I^{(p-1)}\left(\frac{2s}{n}\right) + \ldots + I^{(p-1)}\left(\frac{ns}{s}\right)\right]$$

$$= \frac{s^{p+1}}{(p+1)!} \quad (p = 1, 2, \ldots)$$

The last result is tacitly assumed in the course of the discussion in *Yukti-Bhāṣa*. This is not to be wondered at since we can actually prove the result, using induction and the limit theorem:

$$\lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} = \frac{1}{p+1} *$$

which follows at once from another theorem given in *Yukti-Bhāṣa*.

ALTERNATIVE FORMS OF LEMMAS 2, 3. These lemmas can now be expressed more compactly thus.

Iṣṭajyacāpāntaram = $1/\rho^2$ (second *piṇḍajyasaṅkalitam* with respect to *cāpam*),

or
$$s - y = \frac{1}{\rho^2} \int_0^s \int_0^s y\, (ds)^2. \qquad \ldots \qquad \ldots \quad (7)$$

Iṣṭaśaram = $1/\rho$ (first *piṇḍajyasaṅkalitam* with respect to *cāpam*),

or
$$\rho - x = \frac{1}{\rho} \int_0^s y\, ds. \qquad \ldots \qquad \ldots \quad (8)$$

From (7) we have at once

$$s - y = \frac{1}{\rho^2} \int_0^s \int_0^s s\, (ds)^2 - \frac{1}{\rho^2} \int_0^s \int_0^s (s-y)\, (ds)^2$$

$$= \frac{I^{(2)}(s)}{\rho^2} - \frac{1}{\rho^2} \int_0^s \int_0^s (s-y)\, (ds)^2 \quad \ldots \qquad \ldots \quad (9)$$

whence, integrating both sides p times, with respect to s from 0 to s we find that, for $p = 1, 2, \ldots,$

$$\int_0^s \int_0^s \ldots \int_0^s (s-y)\, (ds)^p = \frac{I^{(p+2)}(s)}{\rho^2} - \frac{1}{\rho^2} \int_0^s \int_0^s \ldots \int_0^s (s-y)\, (ds)^{p+2}. \quad (10)$$

* The proof of this theorem, as it appears in *Yukti-Bhāṣa*, has been discussed in another paper [**6**, Lemma 3(*a*), (*b*)].

Proofs of the Sine and Cosine Series

We are now in a position to state, in the language of modern mathematics, the proofs of the two series in question. If, in (9), we employ (10) with $p = 2, 4, \ldots (2n-2)$ successively, we obtain

$$s-y = \frac{I^{(2)}(s)}{\rho^2} - \frac{I^{(4)}(s)}{\rho^4} + \ldots + (-1)^{n-1} \frac{I^{(2n)}(s)}{\rho^{2n}} + \frac{(-1)^n}{\rho^{2n}} \int_0^s \int_0^s \ldots \int_0^s (s-y)\,(ds)^{2n}$$

which gives, when $n \to \infty$,

$$s-y = \frac{I^{(2)}(s)}{\rho^2} - \frac{I^{(4)}(s)}{\rho^4} + \ldots + (-1)^{n-1} \frac{I^{(2n)}(s)}{\rho^{2n}} + \ldots \qquad (11)$$

Next (8) gives

$$\rho-x = \frac{1}{\rho}\int_0^s s\,ds - \frac{1}{\rho}\int_0 (s-y)\,ds$$

$$= \frac{I^{(1)}(s)}{\rho} - \frac{1}{\rho}\int_0^s (s-y)\,ds.$$

Using (10) with $p = 1, 3, \ldots, (2n-3)$ successively, in the above relation, we get

$$\rho-x = \frac{I^{(1)}(s)}{\rho} - \frac{I^{(3)}(s)}{\rho^3} + \ldots + (-1)^{n-1} \frac{I^{(2n-1)}(s)}{\rho^{2n-1}} + \frac{(-1)^n}{\rho^{2n-1}} \int_0^s \int_0^s \ldots \int_0^s (s-y)\,(ds)^{2n-1}$$

whence, letting $n \to \infty$, we find that

$$\rho-x = \frac{I^{(1)}(s)}{\rho} - \frac{I^{(3)}(s)}{\rho^3} + \ldots + (-1)^{n-1} \frac{I^{(2n-1)}(s)}{\rho^{2n-1}} + \ldots \qquad (12)$$

Thus $s-y$ and $\rho-x$ are each a series of *samskāraphalams* (= results of our study) of the expressions for them in terms of s and y. The *samskāraphalams* are in fact the succession of terms obtained from these expressions by taking repeatedly s instead of y.

If $s = \rho\theta$, then $y = \rho \sin\theta$, $x = \rho \cos\theta$, $I^{(p)}(s) = \rho^{p+1}\,\theta^{p+1}/(p+1)!$ so that (11) and (12) reduce to Newton's power-series for the sine and the cosine:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots,$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots.$$

The proofs we have given of these two series, taken along with the lemmas on which they are based, are in the nature of a preliminary report on a chapter entitled 'Jyāprakaraṇam' (Theory of the *jyā*) in *Yukti-Bhāṣa*.

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ADDENDUM: NOTE ON THE STYLE OF *YUKTI-BHĀṢA*

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The problem of assigning a date to *Yukti-Bhāṣa* has not engaged the attention of any orientalist so far, and the attempts hitherto made to deal with the question have had to be based on the evidence supplied by a few stray texts. One of these texts describes *Yukti-Bhāṣa* as an exposition of the mathematics of 'planetary motions' after the manner of *Tantra-saṅgraha*, with the inevitable suggestion that the first-named work is of later origin than the second. Another text is a colophon which is found only in some (manuscript) copies of *Yukti-Bhāṣa*, and runs:

> Alekhi Yuktibhāsa vipreṇa Brahmadatta samjñena *ye Golapathasthaḥ syuḥ kalirahitāḥ* śodhayantaste.

This colophon is supposed to ascribe the authorship of *Yukti-Bhāṣa*, or at any rate, of a version of the work (presumably the one known to us) to a Brahmin named Brahmadatta and enshrine a Kali chronogram (shown in italics) giving the date of composition of the work as the 17,71,931st day of Kali era (or roughly 1750 A.D.).[1] As there is not much else which can be even provisionally admitted as internal evidence one is compelled to turn to the language and stylistic peculiarities of *Yukti-Bhāṣa* for further light on its origin. It is not desirable to anticipate any conclusion which only a close study of the language can establish. But at the same time the first impression made by the language on an unbiassed reader may be of some value in itself, and I wish to record it, illustrating my remarks by some rather obvious examples from the text.

The most striking thing about *Yukti-Bhāṣa*, apart from its mathematics, is its style which is reminiscent of the fifteenth-century classic *Kṛṣṇagādha*.[2] Although the mathematical terms used in it are almost all Sanskrit, the style is definitely pre-maṇipravāḷa.[3] No reader can resist the suggestion which is gradually borne in upon him as he picks his way through the

[1] The date 1639 A.D. given by K. Mukunda Marar and C. T. Rajagopal [4] is incorrect if we accept Prof. P. C. Sengupta's arguments (in the note which follows this addendum) for identifying the Kali Ahargaṇa 17,71,931 with the Julian Day 23,60,396 or the 11th of June, 1750 A.D. Of course, the question whether this date marks the completion of the original *Yukti-Bhāṣa* or of a later edition of the work (based on earlier ones) due to Brahmadatta, has still to be answered.

[2] This is a poem about Śrī Kṛṣṇa by Cerussēri Nambūdiri, for which T. K. Balakrishnan Nair suggests the date 1454 A.D. in the introduction to *Cerussēri Bhāratam*.

[3] During the sixteenth and seventeenth centuries an admixture of Malayāḷam and Sanskrit known as 'maṇipravāḷa' (maṇi = gem and pravāḷa = coral, standing respectively for Malayāḷam and Sanskrit) was the prevailing literary fashion.

mathematics of *Yukti-Bhāṣa*, that he has before him a sample of the earliest extant prose in Malayāḷam. A characteristic of such prose is that it has not yet weaned itself from the influence of poetry and shows a preference for constructions like 'Ennāluṇṭu cila eḻuppam' which we find in many places in *Yukti-Bhāṣa*. Such peculiarities of construction may prove to be of help in fixing an upper limit to the period likely to have given birth to *Yukti-Bhāṣa*, particularly if it can be shown that they do not occur in works produced after a date.

There is a line of inquiry, with this purpose in view, which must suggest itself to readers of *Yukti-Bhāṣa*, who know anything at all about the development of Malayāḷam literature. Caldwell,[1] after studying the formation of the demonstrative pronouns of the third person singular in the various Dravidian languages, laid down the now familiar principle: 'Four demonstrative bases are recognized by one or other of the Dravidian dialects, each of which is a pure vowel, viz. *a* the remote, *i* the proximate, and *u* the medial demonstrative, together with *e* which is the suffix of emphasis in most of the dialects.' This principle may be supplemented by another which has an immediate application to the inquiry on hand. Even as early as Eḻuttaccan, the most popular Malayāḷam writer of the sixteenth century, the medial demonstrative *u* had gone out of use, only the remote *a* and the proximate *i* continuing to be in service. One must therefore attach some significance to the fact that the base *u*, combined with 'tu', i.e. the usage 'utu' is favoured by both *Yukti-Bhāṣa* and *Kṛṣṇagādha* but seldom by Eḻuttaccan. In other words, one cannot proceed to date *Yukti-Bhāṣa* without taking note of the large number of words like 'Guṇiccūtākum', 'Guṇippūtayi', 'Irippūtum ceyyum', 'Kūtuṁnūtum', etc., which are common to this work and *Kṛṣṇagādha* but almost entirely absent from the writings of the latter half of the sixteenth century.

There are other stylistic peculiarities which stamp the language of *Yukti-Bhāṣa* as archaic. For instance, there is, in old Malayāḷam, a distinction between 'atu' (remote demonstrative pronoun, neuter, singular) on the one hand, and 'ava' (remote demonstrative pronoun, neuter, plural) on the other. This distinction is invariably respected by the author of *Yukti-Bhāṣa* who uses archaisms like 'samkhyakaḷāvunnava' but never once a modern combination such as 'cilatu' (= a few) or 'palatu' (= many) where strict grammar would require 'ava' in place of 'atu'. As examples of other kinds of archaisms one may mention 'Kalpikkēṇṭuvatu' which is preferred to its 'lōpa-sandhi' form 'Kalpikkēṇṭatu' in current use and 'Guṇikkilumām' which contains the termination 'ām' (short for 'ākum') found only in old works. A complete catalogue of such archaisms, including words like 'Ariyāyiṭṭirikkum', 'Guṇippū', 'Kalpippū' 'Kalpiyāykil' and various others, can in fact be made without much difficulty. But I do not pursue the task since my main concern in this appendix is to explain intelligibly my first reaction to *Yukti-Bhāṣa* considered purely as literature.

I am aware that what I have said has not led to any definite conclusion as regards the date to be assigned to *Yukti-Bhāṣa*. I believe nevertheless that linguistic evidence will ultimately incline scholars to the view that *Yukti-Bhāṣa*, although it might have been committed to writing in the seventeenth century, originated in the first half of the sixteenth century—that is to say, was planned to the last detail (so as to facilitate memorization!) at a time when the predominant literary influence was still that of Cerussēri and the older masters who preceded Eḻutaccan.

[1] *Comparative Grammar of Dravidian Languages* (1875), p. 314.

Paper 29

Amulya Kumar Bag (1966)

Trigonometrical series in the *Karaṇapaddhati* and the probable date of the text

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Source: Indian National Academy of Sciences

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TRIGONOMETRICAL SERIES IN THE *KARAṆAPADDHATI* AND THE PROBABLE DATE OF THE TEXT

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The paper discusses the trigonometrical series for π, sine, cosine and tan functions contained in the *Karaṇapaddhati*. The original texts and their translations are given. As to its date C. M. Whish attempted to fix it in the beginning of the eighteenth century. Whish's derivation is untenable and it is shown that the text is likely to be contemporaneous with, or even to antidate, the *Tantrasaṃgraha*, of Nīlakaṇṭha Somasuttvan (A.D 1465–1545).

1. INTRODUCTION

The *Karaṇapaddhati* is an important astronomical work in ten chapters by an unknown Kerala astronomer of uncertain date. Only this much is known about the author that he was a Brāhmin who took his abode in the village of Śivapura.[1] The text in Devanāgarī script is published by K. Sambaśiva Śāstrī,[2] but this edition misses the opening verse.[3] Apart from the usual elements and formulae characteristic of Hindu astronomy, the work gives, in the sixth chapter, trigonometrical π, sine, cosine and tan series. The mathematical importance of the text was first pointed out by C. M. Whish[4] who discussed π and tan functions given in the text and attempted to fix its date to the beginning of the eighteenth century A.D. In this connection Whish also drew attention to three texts, e.g. *Tantrasaṃgraha*, *Yuktibhāṣā* and *Sadratnamālā*, containing similar trigonometrical series. Some of these series have been studied by C. T. Rājagopal *et al.*[5] and T. A. Saraswati.[6] It is proposed to consider here all the four trigonometrical series given in the *Karaṇapaddhati*, including the sine and cosine not discussed by Whish, as also the question of date of the text.

2.1. π SERIES

This series has been discussed in chapter 6 of the text and a number of series have been given. The opening verse runs as follows:

vyāsāccaturghnād bahuśaḥ pṛthaksthāt tripañcasaptādyayugāhṛtāni |
vyāse caturghne kramaśastvṛṇaṃ svaṃ kurjāt tadā syāt paridhiḥ susūkṣmaḥ ||
(ch. 6, 1)

'Four times of the diameter is to be divided separately by each of the odd integers 3, 5, 7 . . . ; every quotient whose order is even is taken away from the one preceding it. Combined result of all such small operations, when subtracted from four times the diameter, gives the value of the circumference with progressively greater accuracy.'

If C be the circumference and D the diameter, the rule may be expressed as:

$$C = 4D - 4D\left(\frac{1}{3} - \frac{1}{5}\right) - 4D\left(\frac{1}{7} - \frac{1}{9}\right) - \ldots$$

$$\text{or,} \qquad \pi/4 = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$$

The next verse gives the series in a different form as follows:

vyāsād vanasaṅguṇitāt pṛthagāptaṃ tryādyayug-vimūlaghanaiḥ |
triguṇavyāse svamṛnaṃ kramaśaḥ kṛtvāpi paridhirāneyaḥ || (ch. 6, 2)

'Four times the diameter is divided separately by the cubes of the odd integers from 3 onwards, diminished by these integers themselves. The quotients thus obtained are alternately added to and subtracted from thrice the diameter. The result is the circumference.'
Thus,

$$C = 3D + 4D\left\{\frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \ldots\right\}$$

$$\text{or,} \qquad \pi = 3 + 4\left\{\frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \ldots\right\}$$

Another series for π is given as follows:

vargairyujāṃ vā dviguṇairnirekairvargīkṛtair-varjitayugmavargaiḥ |
vyāsaṃ ca ṣaḍghnaṃ vibhajet phalaṃ svaṃ vyāse trinīghne paridhistadā syāt ||
(ch. 6, 4)

'Six times the diameter is divided separately by the square of twice the squares of even integers (2, 4, 6 . . .) minus 1, diminished by the squares of the even integers themselves. The sum of the resulting quotients increased by thrice the diameter is the circumference.'
Thus,

$$C = 3D + 6D\left\{\frac{1}{(2.2^2-1)^2-2^2} + \frac{1}{(2.4^2-1)^2-4^2} + \frac{1}{(2.6^2-1)^2-6^2} + \ldots\right\}$$

$$\text{or,} \qquad \pi = 3 + 6\left\{\frac{1}{(2.2^2-1)^2-2^2} + \frac{1}{(2.4^2-1)^2-4^2} + \frac{1}{(2.6^2-1)^2-6^2} + \ldots\right\}$$

$$\text{or,} \qquad \pi = 3 + 6\left\{\frac{1}{1.3.3.5} + \frac{1}{3.5.7.9} + \frac{1}{5.7.11.13} + \frac{1}{7.9.15.17} + \ldots\right\}$$

2.2. Sine and Cosine Series

The rules for the expression of sine and cosine functions in the form of series are given in the same chapter:

cāpācca tattat phalato'pi tadvat cāpāhatāddvyādihatat trimaurvyā |
labdhāni yugmāni phalānyadhodhaḥ cāpādayugmāni ca vistarārdhāt ||
vinyasya coparyupari tyajet tat śeṣau bhūjākoṭiguṇau bhavetām |

(ch. 6, 12–13½)

'The arc repeatedly multiplied by itself (any number of times) is multiplied by the arc. The result is divided by the product of 2, 3 . . . , etc. (up to the same number), and the radius is repeatedly multiplied by itself (the same number of times). The quotients (thus obtained) corresponding to the even numbers (up to which the aforesaid multiplication is repeatedly done) are set one below the other (in one column) and likewise those corresponding to the odd numbers (in another column). From the first term is subtracted the term immediately below and so on, and the remainders for the even column are subtracted from the are and those for the odd column from the radius. The results are the *bhūjajyā* and *koṭijyā* respectively.'

To explain the rule in the circle (Fig. 1),

SP = *cāpa*, s
AP = *trimaurvā* or radius, r
PM = *bhujajyā*, $r \sin \theta$
AM = *koṭijyā*, $r \cos \theta$

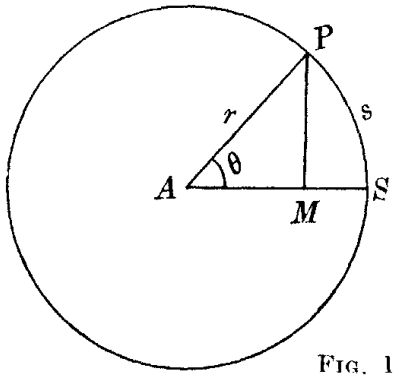

Fig. 1

If n be the number of times up to which the arc s be multiplied by itself, the first part of the rule gives the quotient as:

$$q_n = \frac{s^n \cdot s}{(2 \cdot 3 \ldots n \text{ terms}) \times r^n}$$
$$= \frac{s^{n+1}}{(n+1)!\, r^n}.$$

In the next step, the even quotients, q_2, q_4, q_6 . . . are set down in a column and differences $(q_2 - q_4)$, $(q_6 - q_8)$ . . . , etc., are determined. Similarly by arranging the odd quotients in another column, the differences $(q_1 - q_3)$, $(q_5 - q_7)$ . . . , etc., are obtained. Then *bhujajyā* is given by

$$r \sin \theta = s - (q_2 - q_4) - (q_6 - q_8) - \ldots$$
$$= s - \left(\frac{s^3}{3!\, r^2} - \frac{s^5}{5!\, r^4}\right) - \left(\frac{s^7}{7!\, r^6} - \frac{s^9}{9!\, r^8}\right) - \ldots$$

$$\text{or, } \sin \theta = \frac{s}{r} - \frac{1}{3!}\left(\frac{s}{r}\right)^3 + \frac{1}{5!}\left(\frac{s}{r}\right)^5 - \frac{1}{7!}\left(\frac{s}{r}\right)^7 + \frac{1}{9!}\left(\frac{s}{r}\right)^9 - \ldots$$

Koṭijyā is given by

$$r\cos\theta = r-(q_1-q_3)-(q_5-q_7)-\ldots$$

$$= r-\frac{s^2}{2!\,r}+\frac{s^4}{4!\,r^3}-\frac{s^6}{6!\,r^5}+\frac{s^8}{8!\,r^7}-\ldots$$

$$\text{or, } \cos\theta = 1-\frac{1}{2!}\left(\frac{s}{r}\right)^2+\frac{1}{4!}\left(\frac{s}{r}\right)^4-\frac{1}{6!}\left(\frac{s}{r}\right)^6+\frac{1}{8!}\left(\frac{s}{r}\right)^8-\ldots$$

From these, the modern forms can be readily obtained by putting $s = r\theta$ for small values of s or θ as follows:

$$\sin\theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots$$

$$\text{and } \cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots$$

In these forms, the series appeared for the first time in Europe in a letter which Newton sent to Oldenburg in 1676.[7] The same series, as well known, were derived later as a corollary to De Moivre's Theorem.

2.3. Tan Series

For the tan series, the verse runs as follows:

vyāsārdhena hatādabhiṣṭaguṇataḥ koṭyāptamādyaṃ phalaṃ
jyāvargeṇa vinighnamādimaphalaṃ tattatphalaṃ cāharet |
kṛtyā koṭiguṇasya tatra tu phaleṣvekatripañcādibhir-
bhakteṣvojayutaistajet samajutiṃ jīvādhanuśśiṣyate || (ch. 6, 18)

'Ordinate of the arc (*dhanuṣ*) is to be multiplied by the semi-diameter (*vyāsārdha*) and is divided by the abscissa (*koṭi*). This is the first term. Thus the result obtained is multiplied by the square of the ordinate (*jyā*) and is divided by the square of the abscissa. This is the second term. This process is repeated. The successive terms are divided by the odd integers 1, 3, 5, ... Now, the terms of odd order are added and the terms of even order are subtracted from the preceding, the circumference will be obtained.'

Abscissa* $AM \geqslant$ Ordinate PM, i.e. SAP $\leqslant 45°$ (Fig. 1), then according to the above translation, it can be written as:

$$\text{arc } SP = AP\left[\frac{PM}{AM}-\frac{1}{3}\frac{PM^3}{AM^3}+\frac{1}{5}\frac{PM^5}{AM^5}-\ldots\right]$$

$$\text{or, } s = r\left[\frac{r\sin\theta}{r\cos\theta}-\frac{1}{3}\left(\frac{r\sin\theta}{r\cos\theta}\right)^3+\frac{1}{5}\left(\frac{r\sin\theta}{r\cos\theta}\right)^5-\ldots\right.$$

$$\text{Hence, } \theta = \tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\ldots \text{ when } s = r\theta.$$

* This restriction is put by the commentator.

Karaṇapaddhati gives no proof of these series. These series also appear in the *Tantrasaṃgraha* to which Whish already drew attention.[8] *Yuktibhāṣā*, an exposition of Nīlakaṇṭha's *Tantrasaṃgraha*, gives proofs to all of these series.[9]

In the general history of mathematics, the π and the tan series are associated with the name of Gregory (A.D. 1638–1675).[10]

3. The Date of *Karaṇapaddhati*

Charles M. Whish, in his effort to fix the date of *Karaṇapaddhati*, quoted the concluding verse as follows:

iti śivapuranāma-grāmajaḥ ko'pi yajvā
kimapi karaṇapaddhatyāhvayaṃ tantrarūpam |
vyadhita gaṇitametat samyagālokya santaḥ
kathitamiha vidantaḥ santu santoṣavantaḥ ||

'A certain sacrificer born in the village named Śivapura composed (*vyadhita*) a treatise called *Karaṇapaddhati*. Let the good learned people after having studied this mathematical text thoroughly be satisfied from what has been said here.'

In his judgement, the words *gaṇitametat samyak* expressed 1765653 days of the Kaliyuga, that is, A.D. 1733 as the date of the text.[11] Presumably Whish used the *Kaṭapayadi* system of writing in arriving at the *Kali chronogram*, a system much in use in South India in the medieval times.

The essential features of the *Kaṭapayadi* alphabetical system of number writing are as follows:

(*a*) A consonant in association with a vowel takes significant values, whereas that without a vowel has no such numerical significance and should be disregarded. These values are:

k	*kh*	*g*	*gh*	*ṅ*	*c*	*ch*	*j*	*jh*	*ñ*
1	2	3	4	5	6	7	8	9	0
ṭ	*ṭha*	*ḍ*	*ḍh*	*ṇ*	*t*	*th*	*d*	*dh*	*n*
1	2	3	4	5	6	7	8	9	0
p	*ph*	*b*	*bh*	*m*					
1	2	3	4	5					
y	*r*	*l*	*v*	*ś*	*ṣ*	*s*	*h*		
1	2	3	4	5	6	7	8		

(*b*) In a conjoint consonant, only the last one denotes a number.

(*c*) In the final reading, the numbers are to be reckoned from right to left.

According to the above system, the words *gaṇitametat samyak* reduce to:

ga	—	*ṇi*	—	*ta*	—	*me*	—	*ta*	—	*t*	—	*sa*	—	*mya*	—	*k*	= 1765653
3		5		6		5		6		×		7		1		×	

Thus 1765653 days elapsed from the beginning of the *Kali* era till the date in question. Taking 365·25 days as the length of the Indian solar year, and the beginning of the *Kali* era as 17th February, 3102 B.C., the date of the text in the *Gregorian* calendar is obtained as under:

$$\begin{aligned} 1765653 \text{ days} &= (4834 \times 365{\cdot}25 + 34{\cdot}50) \text{ days} \\ &= 4834 \textit{ Kali} \text{ era (approx.)} \\ &= (3102 + 1733) \textit{ Kali} \text{ era (approx.)} \\ &= \text{A.D. } 1733 \text{ (since } \textit{Kali} \text{ era began at 3102 B.C.)} \end{aligned}$$

By this system, any word can be changed into a chronogram. A careful consideration of the use of the *Kaṭapayadi* systems in verses dealing with astronomical and mathematical subjects reveals that such alphabetical combinations are always readily distinguishable from those signifying accepted word meanings. Moreover, separate words are not used in the manner shown above. The following will illustrate the point:

dīpa	*balaṃ*	*java*	*bānā*	*netā*	*kṣāntiḥ*	*kathaṃ*	*mitho*	*dāsāḥ*
18	33	48	53	60	66	71	75	78

pāde	*guhā*	*bhuja*	*madaṃ*	*mudā*	*tadā*	*śukramandajīvaḥ*	*syuḥ* ‖
81	83	84	85	85	86	(are the sines of the *manda* (corrections) of the Venus)	

(*Dṛggaṇita*,[12] 2, 35)

Whish's selection of words with clear literary meanings is not only arbitrary, but his selection of separate words with a view to obtain a large number is not warranted by the usage of the system. If *gaṇitametat* and *samyak* are converted separately, these give 65653 and 17 which do not lead us anywhere. If the liberty assumed by Whish were possible, one would also be able to convert, say, *ko'pi yajvā kimapi* of the same verse into 1114151 days, that is A.D. 1043, an absurd conclusion as the text would then antidate Bhāskara II (*c.* 1150) whom it quotes at several places. Thus, no word of the concluding verse represents number or date and as such the derivation by Whish cannot be accepted.

Some light on the subject is now available from the studies of Rāja Rāja Varma and Ullur Parameśvara Iyer. Rāja Rāja Varma[13] has remarked that the period of the author of *Karaṇapaddhati* can be placed between the years 550 and 650 of the Malabar era, corresponding roughly to A.D. 1375–1475

and quoted in support of his view a verse from *Gaṇita Sūcikā Grantha* by Govinda Bhaṭṭa. The verse runs thus:

navīna vipine mahīmakhabhujām somayā-
jyudaragaṇa kotraya samabhavacca tenāmunā |
vyalekhi sudṛguttama Karaṇa-paddhati saṃskṛtā
tripañcaśati bhūmita pradhita śaka samvatsare ||

'A *somayājin* versed in astronomy was born in the brāhmin family of Navīna-vipina. By him was composed this refined *Karaṇapaddhati*, the best of the *Dṛg* (system) in 1353 Śaka era (A.D. 1431).'

Ullur Parameśvara Iyer[14] has given the beginning of the seventh century of the Malabar era (fifteenth century A.D.) as the date of the *Karaṇapaddhati*; but no basis for this assertion has been given.

In the circumstances, therefore, a more dependable placement of the time of the work should be attempted from an internal study of the work itself. *Karaṇapaddhati* contains discussions of a number of topics which also reappear in the *Tantrasaṃgraha* of Nīlakaṇṭha Somasuttvan. There is no uncertainty as to Nīlakaṇṭha's date, 1465–1545 A.D., inasmuch as this South Indian astronomer declared himself as the direct pupil* of Parameśvara, the indefatigable commentator and author of astronomical works.[15]

The rule for obtaining the π series already discussed, together with a rule for obtaining an approximate value for the last term, is given in the *Tantrasaṃgraha* as follows[16]:

vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate |
triśarādi-viṣama-saṃkhyā-bhakta-mṛṇaṃ svaṃ pṛthak kramāt kuryāt ||
yatsaṃkhyayātra haraṇe kṛte nivṛttā hṛtistu jāmitayā |
tasyā ūrdhvagatāyāssamasaṃkhyāyā taddalaṃ guṇo'nte syāt |
tadvargaiḥ rūpayuto hāro vyāsābdhighātataḥ prāgvat |
tasyāmāptaṃ svamṛṇe kṛte dhane śodhanañca karaṇīyam ||
sūkṣmaḥ paridhiḥ sā syāt bahukṛtvo haraṇato'tisūkṣmaśca | (ch. 2)

'Quotients obtained by dividing four times of the diameter by the odd integers 3, 5, 7 . . . are alternately subtracted from and added to 4D. The process stops at a certain stage giving rise to a finite sum. Four times of the diameter is to be multiplied by half the even integer and is divided by the square of the even integer increased by unity. This is the last term. The result is the correction to be added to or subtracted from our finite sum (the choice of addition or subtraction depends on the sign of the last term in the

* Nīlakaṇṭha in his *Āryabhaṭīya Bhāṣya* states '*tadeva paramācārya mamāha Parameśvara*' (*Gola*, 48). Parameśvara gives the dates of writing *Dṛggaṇita* and *Goladīpikā* as Śaka 1353 (A.D. 1431) and Śaka 1365 (A.D. 1443) respectively.

sum). The final result is the circumference determined more exactly than by taking a large number of terms.'

The above may be expressed as follows:

$$C = 4D\left[1-\frac{1}{3}+\frac{1}{5}-\ldots\pm\frac{1}{n}\mp\frac{\frac{n+1}{2}}{(n+1)^2+1}\right]$$

where n is odd and large.

As it is a convergent series, the author found it necessary to correct the last quotient.

In another verse of the *Tantrasaṃgraha*,[17] a rule has been given to obtain a still more accurate expression for the last term. The rule runs thus:

asmāt sūkṣmataro'nyo vilikhyate kaścanāpi saṃskāraḥ |
ante samasaṃkhyā-dala-vargassaiko guṇassa eva punaḥ ||
yugaguṇito rūpayutassamasaṃkhyādalahato bhavedhāraḥ | (ch. 2)

'Here another correction is given more precise than the foregoing. The square of half the even integer next (greater than the last odd integer divisor), increased by unity, is a multiplier. This multiplier, multiplied by four, then increased by unity and then multiplied by the even integer already defined, is the divisor.'

Evidently,

$$C = 4D\left[1+\frac{1}{3}-\frac{1}{5}-\ldots\pm\frac{1}{n}\mp\frac{\left(\frac{n+1}{2}\right)^2+1}{\left[\left\{\left(\frac{n+1}{2}\right)^2+1\right\}4+1\right]\left(\frac{n+1}{2}\right)}\right]$$

where n is odd and large.

This shows that the author of the *Tantrasaṃgraha* had definitely the knowledge of a slowly-converging series, of which there is no indication in the *Karaṇapaddhati*. From the much fuller and more refined treatment of these topics in the *Tantrasaṃgraha*, *Karaṇapaddhati* may be suspected to be a work, contemporaneous with, or even antidating, *Tantrasaṃgraha* of Nīlakaṇṭha (A.D 1465–1545).

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I am indebted to Sri S. N. Sen for going through this article and making valuable suggestions and to Professor D. M. Bose for preparation of the paper in its present form. Thanks are also due to Sri Nagendra Nath Vedāntatīrtha for checking the English rendering of the Sanskrit passages.

References

[1] The last verse of the tenth chapter reads:
'iti śivapura-nāma grāmjaḥ ko'pi yajvā'.

[2] *Trivandrum Sanskrit Series* (*T.S.S.*), No. 126.

[3] See *Madras Triennial Catalogue*, Vol. II, Part I. Sans. A., Catalogue No. *1310*.

[4] 'On the Hindu quadrature and the infinite series of the proportion of the circumference to the diameter exhibited in the four śāstras: *Tantrasaṃgraha, Yuktibhāṣā, Karaṇapaddhati* and *Sadratnamālā*.' *Transaction of the Royal Asiatic Society* (*TRAS*) of Great Britain and Ireland, **3**, Part III, 509–523, **1834**.

[5] 'On the Hindu Quadrature of the Circle', *Journal of the Bombay Branch of the Royal Asiatic Society* (*JBBRAS*), **20** (n.s.), 65–82, 1944; 'A neglected chapter of Hindu Mathematics', *Scripta Mathematica*, **15**, 201–209, 1949; 'The Sine and Cosine Power Series', *JASB*, **15**, Science Section, 1–13, 1949; 'On the Hindu proof of Gregori's Series', *Scripta Mathematica*.

[6] 'The development of Mathematical Series in India after Bhāskara II', *Bulletin of the National Institute of Sciences of India* (*BNISI*), **21**, 320–43, 1963.

[7] *See* M. Cantor's *Vorlesungen Über Geschichte der Mathematik*, 2nd ed., **3** (c/o *JASB*, **15**, Part II, 1, 1949).

[8] *TRAS*, of Great Britain and Ireland, **3**, Part III, 513, 516, 519.

[9] *BNISI*, **21**, 320–43, 1963; *JBBRAS*, **20** (n.s.), 66–69, 1944.

[10] *Chambers's Encyclopaedia*, **10**, 227, 1935; Cajori, F., *A History of Mathematics*, 2nd revised and enlarged ed., 206.

[11] *TRAS*, of Great Britain and Ireland, **3**, Part III, 522, 1834.

[12] *Dṛggaṇita* of Parameśvara, ed. by K. V. Sarma, *Vishvesharanand Vedic Research Institute*, Hoshiarpur, 1963.

[13] Rāja Rāja Varma, *History of Sanskrit Literature in Kerala*, **1**, 529.

[14] Ullur, Parameśvara Iyer, *Kerala Sahitya Caritra*, **2**, 564.

[15] See *A Bibliography of Sanskrit Works on Astronomy and Mathematics* by S. N. Sen, A. K. Bag and S. R. Sarma, 167 (under PARAMEŚVARA).

[16] MSS. of *Trippunitura Sanskrit College Library and the Adyar Library*, ch. 2 (c/o *JBBRAS*, **20** (n.s.), 77, 1944).

[17] See *JBBRAS*, **20** (n.s.), 81, 1944.

Paper 30

K. V. Sarma (1972)

Kerala Astronomers

Excerpt from:
K. V. Sarma, *A History of the Kerala School of Hindu Astronomy (in Perspective)*, Vishveshvaranand Institute, Hoshiarpur, 1972, Chapter IV, pp. 41–81.

Note

A History of the Kerala School of Hindu Astronomy (in perspective) is the first definitive book giving a comprehensive description of the contribution of Kerala to astronomy and mathematics. The book, among other things, contains details of the lives and works of about 80 astronomers and mathematicians belonging to the Kerala School. It has also identified 752 works belonging to the Kerala school. (Wikipedia)

K. V. Sarma

For a biographical sketch, see p. 79.

Source: Internet Archive

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CHAPTER IV

KERALA ASTRONOMERS

1. Introductory

The approach of the Kerala astronomers to the study of that discipline and the broad trends of their writings, as set out in the previous chapters, should form the natural background for an apprisal of the literature produced in the land on astronomy and, for the sake of fullness, also on the allied discipline of astrology. Such an apprisal is attempted in the two chapters that follow. In the present chapter, the major writers are taken note of and in the next chapter is presented a fully documented Inventory of the available literature on these disciplines. It may be noted that the said Inventory carries references to the manuscripts, editions and studies relating to as many as about 600 works on astronomy, astrology and mathematics and details of about 100 authors and authorities.

2. Earlier documentation

Several efforts have been made before to take stock of Hindu astronomical and allied literature, including that of Kerala, though these attempts have hardly succeeded in giving any adequate idea of the extent of the literature produced in Kerala. 'A brief bibliography of Hindu mathematics' by G.R. Kaye[1] records but a single work from Kerala, being the commentary of Paramādīśvara (sic.) on the *Āryabhaṭīya*. The *Gaṇakataraṅgiṇī or Lives of Hindu astronomers* by Sudhakara Dvivedi mentions just four Kerala authors and two works.[2] 'A consolidated list of Hindu mathematical works' by K. Balagangadharan[3] includes just 18 Kerala authors and 42 works. K.K. Raja's well-documented paper on 'Astronomy and mathematics in Kerala : An account of the literature',[4]

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1. *JI. and Proc. of the Asiatic Soc. of Bengal*, (NS) 7 (1911) 679-86.

2. Rev. edn. by Padmakara Dvivedi, Banares, 1933. The Keralite authorities referred to herein are Parameśvara, Kṛṣṇācārya, Govinda and Vidyāmādhava, among authors, and *Bhaṭadīpikā* and *Daśādhyāyī*, among works.

3. *Mathematics Student*, 15 (1947) 55-70.

4. *Adyar Library Bulletin*, 27 (1963) 118-67.

takes into its purview only some of the more prominent authors in these two disciplines, numbering about 20, and their works, numbering about 80. S.N. Sen's *Bibliography of Sanskrit works on astronomy and mathematics*,[1] carries the documentation of just 13 Kerala authors and 47 works. The major histories of Kerala literature, *viz.*, *Kerala Sahitya Charitram* (in 5 vols.) by Ulloor S. Parameswara Aiyar,[2] *Keraliya Samskrita Sahitya Charitram*, (in 6 vols.)[3] and *Kerala Sahitya Charitram* : *Charcchayum Pūraṇavum*, (in 2 vols.),[4] both by Vaṭakkumkūr Rajaraja Varma, *Malayala Bhasha Charitram* by P. Govinda Pillai,[5] and *Kerala Bhasha Sahitya Charitram* (in 7 vols.), by R. Narayana Panikkar,[6] especially the first three, notice a much larger quantom of the literature, though they, too, fail to do adequate justice to the subject. In fact, more than half of the literature on Joytiṣa produced in Kerala does not find a mention in these volumes. Moreover, the accounts given in these histories are not documented and are, mostly, distributed in their different volumes. Neither do these histories, except that of Ulloor, carry any index to help the reader in locating a work or an author dealt with therein. Under the circumstances as stated above, the wealth of material presented, in a chronological manner, in the following pages may be expected to provide a far more comprehensive account of the literary output of Kerala in the discipline of Jyotiṣa. It has been thought not necessary to repeat here the details which are available in the earlier accounts, but it was felt essential to set out all new information not noticed in therein. Therefore, in the following pages, the stress will be on new information, identifications and correlations, although, for the sake of completeness and continuity, a brief mention will be made also about currently known authors and works, for further details about which the earlier writings referred to in the footnotes might be consulted with advantage. For the documentation of the manuscripts of the works and for extracts therefrom, the annotated Inventory in the next chapter might be referred to.

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1. Part I : *Manuscripts, Texts, Translations and Studies*, National Institute of Sciences of India, New Delhi-1, 1966.

2. Pub., University of Kerala, Trivandrum, 1953-57.

3. Pub., Vol. I, Kamalalaya Book Depot, Trivandrum, M. E. 1113 (A. D. 1938), Vols. II-VI, Mangalodyam, Trichur, M. E. 1122-40 (A.D. 1947-65).

4. Pub., Mangalodayam, Trichur, 1967-69.

5. Reprint, National Book Stall, Kottayam, 1955.

6. Pub., Vidyavilasini Book Depot, Trivandrum, n.d.-1951.

3. Vararuci I (4th cent. A.D.)

Vararuci is the father figure in the astronomical tradition of Kerala. He is supposed to have lived in the first half of the 4th century, this date having been arrived at on the basis of the dates of birth and death of his eldest son Meḷattoḷ Agnihotri, which are given, according to tradition, in the Kali chronograms *yajñasthānam surakṣyam* (12,70,701) and *purudhīḥ samāśryaḥ* (12,57,921) which fall, respectively, in A.D. 343 and 378. The manuscript tradition of the land ascribes to Vararuci the authorship of the 248 *Candra-vākyas* ('moon-sentences'), popularly called *Vararuci-vākyas*, beginning with *gīr naḥ śreyaḥ*, and also the promulgation of the *kaṭapayādi* notation of depicting numbers which has been used in the composition of the said *Vakyas*.[1]

4. Vararuci II

In horary astrology, the work *Kerala-dvādaśabhāvavākyāni*, in 12 chapters, is ascribed to Vararuci; the manuscripts of the work carry expressive colophons such as *iti Keralīya-Vararuci-vākye* and *Kerala Vararuci-prokta*. Two anonymous works, both entitled *Kerala-dvādaśabhāva-phalam*, one in verse and the other in prose, are allied to Kerala-Vararuci's work mentioned above. Another work of Vararuci is the *Vārarucika*, an extenstive treatise on general astrology in 12 chapters. Still another work called *Vararuci-kerala* called also *Jātakarahasya* and *Kerala-rahasya* on horoscopy is, also, attributed to him. The Vararuci, who is the author of the above works on astrology might be identical, but it is not possible to assert that he is the same as the author of the *Candra-vākyas*.

5. Haridatta (*c.* 650-700)

Haridatta, as noted earlier, has the distinction of promulgating,

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1. On Vararuci, see Raja, pp. 120-22; Ul. vol. I. pp. 77-81, 165; Va. vol. I. pp. 90-95. There is another astronomer named Vararuci, the author of the well-known *Vākyakaraṇa* (ed. T. S. K. Sastri and K. V. Sarma, Madras, 1962), which is the source-book of the *Vākya-pañcāṅga*, popular in South India, especially the Tamil region. But this Vararuci, whether the name is real or apocryphal, belonged to the Tamil region, as is clear from the introductory verses of that work. It has also been shown that this is a 13th century work, having been composed between 1282 and 1306, as can be inferred from the *śodhya-dina* ('deductive day') made use of in that work as a 'zero-day' for computations (*vide op. cit.*, Intro., p. ix). For some Keralite legends on Vararuci, see Koṭṭārathil Śaṅkuṇṇi, 'Parayiperru pantiru kulam' (Mal.), ('The twelve families born of a woman of the Paraya caste'), *Aitihyamālā*, (6th edn., Trichur, 1961), pp. 60-90.

In 683, at Tirunāvāy in Kerala, a new system of astronomical computation called *Parahita*, which presented distinct advantages over the prevailing Āryabhaṭan system.[1] One of the important corrections which he introduced was called the *Śakābda-saṁskāra* or *Bhaṭa-saṁskāra*.[2] His *Grahacāranibandhana*, of which and of whose summary critical editions are available,[3] is the working manual of the system. His comprehensive treatise on astronomy entitled *Mahā-mārganibandhana*, referred to in the *Grahacāranibandhana* (III. 44), is yet to be recovered. The well-known verse *vāgbhāvonāt* etc., enunciating his *Śakābda-saṁskāra*, which does not occur in the shorter manual, must be from this treatise ;[4] so also Nīlakaṇṭha Somayāji's quotation in his *Grahaṇādi-grantha* (ms., p. 59) which reads : *atra Haridattaś cāpy āha—'caturdaśabhāgāntaś calanty eva grahās tataḥ'*. The *Parahita* system of computation, which grew highly popular, has played a leading role in the propagation and practice of astronomy in Kerala.[5]

6. Govindasvāmin (*c.* 800-850)

In Govindasvāmin, teacher of Śaṅkaranārāyaṇa (see below) who was the court astronomer of King Ravi Varma Kerala, we have one of the ablest exponents of Bhāskara I and of the Āryabhaṭan system. His elaborate *Bhāṣya* on the *Mahābhāskarīya*[6] contains new ideas and mathematical elaborations which remain to be fully recognised and expounded in terms of modern mathematics.[7] An original work of his on astronomy and mathematics, which is quoted under the title *Govindakṛti* by later writers like Śaṅkaranārāyaṇa, Nīlakaṇṭha

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1. See above, pp. 7-8.

2. On this see above, p. 9.

3. Cr. ed. with Intro., K.V. Sarma, K.S.R. Institute, Madras, 1954.

4. For this verse and its exposition, see above, p. 9.

5. On Haridatta, see: K.V. Sarma, Intro. to his edn. of *Grahacāranibandhana*, *op. cit.*, and 'The original 'text of Parahita-gaṇita', (Mal), *Mathrubhumi Weekly* (Kozhikode), 10.10.1954 ; Raja, 123-26.

6. Cr., ed, T. S. Kuppanna Sastri, Madras, 1957.

7. For an exposition of some of these, see R. C. Gupta, 'Second order of interpolation in Indian mathematics', *Indian Jl. of Hist. of Sc.*, 4 (1969) 86-98 ; 'Fractional parts of Āryabhaṭa's sines and certain rules found in Govindasvāmin's *Bhāṣya* on the *Mahābhāskarīya*', *ibid.*, 6 (1971) 51-59.

Somayāji and Nārāyaṇa,[1] is yet to be recovered. So also is his work on astrology which is referred to by Nīlakaṇṭha as *Govindapaddhati*.[2] However, manuscripts of his short commentary called *Prakaṭārtha* or *Sampradāyapradīpikā* on *Parāśarahorā* are known but it is yet to be issued in print.[3]

7. Śaṅkaranārāyaṇa (*c.* 825-900)

Ravi Varma, the 9th century ruler of the Cera dynasty of Kerala had as his chief court astronomer Śaṅkaranārāyaṇa disciple of Govindasvāmin. Śaṅkaranārāyaṇa was a native of Kollapurī on the Arabian coast. The only known work of his is a commentary on the *Laghubhāskarīya* which he wrote in A. D. 869.[4] Besides being highly elucidatory with regard to the subject treated therein, the work throws light on the keen interest evinced by the royalty in the promotion of the study of astronomy, the presence of an astronomical observatory at the capital city Mahodayapuram (modern Koṭungalloor near Cochin), the construction of buildings with due reference to mathematical principles, arrangements made in the city for announcing the times of the day—which all point to the conditions that greatly favoured the flowering of astronomical studies in that part of India.[5]

8. Udayadivākara (11th cent.)

Udayadivākara, surnamed Jyotiṣikabhaṭṭa, author of the elaborate commentary called *Sundarī* on the *Laghubhāskarīya*, which takes as its epoch 1073,[6] probably hailed from Kerala. In the *Sundarī* he quotes as many as twenty verses from an unnamed work of a

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1. For instance, see *Śaṅkaranārāyaṇa* in his com. on *Laghubhāskarīya*, *TSS*, No. 162, pp. 19, 55, 95 ; Nīlakaṇṭha, *Grahaṇādigrantha*, ms., pp. 35, 42, 44.

2 See his *Grahaṇādigrantha*, ms., p. 44.

3. On Govindasvāmin, see : T.S.K. Sastri, Intro , to his edn. of *Mahābhāskarīya*, *op. cit.*, pp. xlvi-l ; Raja, 127-28 ; Ul. II.96-97. The identification of our author with the 13th cent. Govinda Bhaṭṭatiri, au. of *Muhūrtaratna*, made by Ulloor (I.169, II. 96, 97) is not correct. Again, Ulloor (I.169) refers to a com. on the *Āryabhaṭīya* by our author, but no such com. is known.

4. Ed. *Trivandrum Skt. Ser.*, (*TSS*), No. 162, (Trivandrum, 1949).

5. On Śaṅkaranārāyaṇa, see, Intro. to the above edn., pp. x-xv ; Raja, 128-29 ; Ul. I. 165-67 ; Va. IV, 221-23. Va.'s statement (IV. 222) that *Bṛhatkarmanibandhana* is a work of this author is not correct. *Bṛhatkarmanibandhana* is another name for the *Mahābhāskarīya* of Bhāskara I.

6 Śaka 935, as given in his com. on *Laghu.*, II. 29.

mathematician Jayadeva and comments on them.[1] His comments on Jayadeva are quoted also in the *Līlāvatīvyākhyā (Kriyākramakarī)* by Nārāyaṇa (Madras Ms. R. 2754, pp. 182, 183-84, 185). Possibly Udayadivākara has commented on Jayadeva's work in full.

9. Acyuta I

An extensive work on analytic astrology, called severally as *Devakerala*,[2] *Keralajyotiṣa*, *Candranāḍī* and *Candrakalānāḍī*, introduces itself as to have been compiled from the teachings of Acyuta, a brahman from Kerala. Acyuta propitiated, by penance, Bṛhaspati, who taught him the *Jātakaskandha*. In a similar manner, Acyuta propitiated Śukra and Parameśvara also and got from them their teachings as well, which he propagated.[3] Neither the date of this Acyuta nor any personal details about him is known. The *Devakerala* which represents his teachings is, obviously, a compilation by his disciples, among whom Veṅkaṭeśa of the Kāśyapagotra is specifically mentioned as the redactor of the *Uttarabhāga* of the work.[4]

10. Keralācārya (? 12th cent.)

According to current tradition, Keralācārya hailed from Trichur in Central Kerala and belonged to the Kāśyapagotra. It is also said that he was a minister of King Rāma Varmā who ruled in Kerala from 1108-32. He wrote on Jyotiṣa, Āyurveda and Tantra. His works on

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1. For a study on these verses, see K. S. Shukla, 'Ācārya Jayadeva, the mathematician', *Gaṇita*, (Bharat Ganita Parishad, Lucknow), 5 (1954) 1-20.

2. Ed. by T. S. Ranganatha Josyar, in 3 parts, *Madras Gov. Or. Ser.*, Madras, 1952-56.

3. *Cf.* : *Kerale viṣaye kaścid Acyuto nāma bhūsuraḥ /*
Bṛhaspatim samuddiśya sa cakre tapa uttamam //

prasādasumukhas tasmai prāha devapurohitaḥ /
tapasā tava tuṣṭo 'smi vatsa kim te karomy aham // ...

yan mayānukṛtam pūrvam Jātakaskandham uttamam /
dviṣahasreṇa saṅkṣipya tvatkṛte procyate 'dhunā // ...

Acyuto 'pi punaḥ prītas tapasārādhya Bhārgavam /
tasmāt sahasrasaṅkhyākam prāptavān Bhārgavam matam // ...

śiṣyān adhyāpayāmāsa śāstram tad Devakeralam / ...

4. *Cf.* : *abhivandya mataṅgākhyam śrīmad-veṅkaṭanāyakam /*
Kāśyapo Veṅkaṭeśo 'ham phalam vakṣyāmi bhūbhujām //
(Intro. verse)

Jyotiṣa included *Kerala-saṁhitā*, *Kerala-nāḍīgrantha*, *Keralīyapraśna-mārga* and *Keralapraśnasāra*. However, as observed earlier, these works must, obviously, be non-Keralite treatises based on Keralite teachings attributed to Keralācārya.[1]

11. Vyāghrapāda

Aṅkaṇaśāstra,[2] called also *Grahasamaya* and *Navagrahasamaya*, has, obviously, been composed by a devotee of god Śiva enshrined at Vaikkom (Skt. Puṇḍarīkapura) in Central Kerala and attributed to sage Vyāghrapāda, the establisher of that place. It is an interesting treatise in analytical astrology, intended as a supplement and corrector to general horoscopic astrology and contains 389 verses, divided into four chapters. Herein, each of the twelve *lagna-rāśis* (ascending zodiacal segments) are divided into nine *aṅkaṇa-s* ('court-yards'), each *aṅkaṇa* being presided over by a planet having individual characteristics. Minute prognostications are made on the basis of the effects of these planets.[3]

12. Kṛṣṇa (c. 1200)

Kṛṣṇa, called also Kṛṣṇācārya, is the author of *Cintājñāna*, a comprehensive work on astrology in thirtytwo chapters. No personal details about him are known but his being quoted frequently as an authority by later writers, the earliest of them being Govinda Bhaṭṭatiri of Talakkuḷam (1237-95), is a pointer to his popularity and his date, which might be taken to be about 1200 or before. He may be identical with Kṛṣṇa, the author of a commentary on the *Horā* of Varahamihira, of which a manuscript has come to light. Kṛṣṇa's *Cintājñāna* has a popular commentary called *Caturasundarī* by Viṣṇu.[4]

13. Kṛṣṇa-śiṣya (*c.* 1200)

Kṛṣṇa had in one of his disciples, who does not disclose his name or give any other personal details, an ardent admirer and propagator of his teachings. This disciple has composed two works, one a metrical commentary on the *Horā* and the other a short treatise in verse on astrological query, entitled *Praśnaphalaprāptikāla-nirṇaya*. In both these works, he mentions his teacher in high terms

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1. See above, pp. 38-40.
2. Ed. by K. Raghavan Pillai, *TSS*, No. 222, Trivandrum, 1968.
3. On this work, see Intro. to the edn. cited above and Ul. Sup. I. 221-22.
4. On Kṛṣṇa, see Ul. I. 169-70.

and, in the latter, goes to the extent of saying that the minds of those who have imbibed the teachings of Kṛṣṇācārya would remain unassailed so far as prognastication is concerned.[1]

14. Sūryadeva Yajvan (1191-*c.* 1250)

Sūryadeva Yajvan of the Nidhruva gotra and nephew of another Sūryadeva was a versatile commentator and recognised authority. He gives his date of birth as '*viśveśa*' (1113) *śāka* in the colophon to his commentary on the *Tripraśnādhyāya* of the *Laghumānasa* of Muñjāla. His elaborate commentaries on the *Laghumānasa* and the *Āryabhaṭīya* are available, but that on the *Mahābhāskarīya-bhāṣya* of Govindasvāmin is known only from his reference to it. On astrology, he commented on the *Mahāyātrā* of Varāhamihira, and the *Jātaka-karmapaddhati* of Śrīpati, both of which are popular.[2]

15. Vidyāmādhava

Vidyāmādhava, whose commentary on the *Kirātārjunīya*, is perhaps, the best exposition of that *mahākāvya*, was equally at home in Jyotiṣa. His *Muhūrtadarśana*, called also *Muhūrtamādhavīya* and *Vidyāmādhavīya*,[3] is an extensive work in 15 chapters, discussing and prescribing auspicious times for all types of social and religious functions. He was a member of the Tuḷu brahman family of Nīlamana, his village being Guṇavatī (Guṇavanta), near Gokarṇa, on the Arabian coast. He was the son of Nārāyaṇa, an eminent scholar attached to the court of King Mallappa. The *Muhūrtadaraśna* has been extremely popular in Kerala and has six Keralite commentaries on it, two in Sanskrit and the others in Malayalam.[4]

16. Viṣṇu of Nīlamana

Viṣṇu was the son of Vidyāmādhava and had the distinction of commenting upon his father's *Muhūrtadarśana*. This commentary,

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1. See the verses extracted in the Inventory, below, under the respective works.

2. On Sūryadeva, see Raja, 131-32, Sen, p. 124 ; Ul. II. 97-98 ; Ul. Sup. I. 219 ; Va. I. 390-92.

3. Ed. by Shama Sastry, *Vidyamādhavīyam* with Viṣṇuśarma's *Muhūrtadīpaka*, *Bibl. Sanskrita*, Nos. 63, 67, 70, Or. Res. Inst., Mysore, 1923, 1925, 1926.

4. On Vidyāmādhava, see, Ul. II. 95 ; Ul. Sup. I.540 ; Va. I. 441-44.

called *Dīpikā*,[1] is highly elucidatory and informative and has been as much popular in the land as its text.[2]

17. Govinda Bhaṭṭatiri of Talakkuḷam (1237-95)

Govinda Bhaṭṭatiri of Talakkuḷattūr family belonged to the village of Ālattūr in S. Malabar and is renowned as the progenitor of the famous Pāzhūr Kaṇiyār family of astrologers A line of astronomical tradition which he started continued for more than 700 years.[3] The dates of his birth and death are given by tradition by the Kali chronograms *rakṣed govindam arkaḥ* (15,84,362) and *kālindīpriyatuṣṭaḥ* (16,12,891), which fall, respectively, in A. D. 1237 and 1295. His elaborate commentary *Daśādhyāyī* on the *Horā*, called so on account of its commenting only chs. I to X of the work, deals, in substance, with the entire *Horā*.[4] His *Muhūrtaratna* has been very popular. He is said to have written also a *Muhūrtapadavī*, which formed the basis for several later works of that name (Ul. II. 110).[5]

18. Tāmaranallūr (14th cent.)

A member of the Tāmaranallūr family has produced an authoritative work entitled *Muhūrtavidhi*, called also *Muhūrtam Bhāṣā*. The work is popularly known as *Tāmaranallūr Bhāṣā*, after its author. It deals with the fixation of auspicious times for functions and is composed

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1. Ed. with text, *op. cit.*

2. The colophon to the commentary on ch. XI reads :

Muhūrtādarśanādarśavyākhyāne Viṣṇunā kṛte /
vyācaṣṭaikādaśādhyāyam Vidyāmādhavanandanaḥ //

On the basis of this Ulloor (II. 95) distinguishes between Viṣṇu and Vidyāmādhava's son and states that while the disciple Viṣṇu is the main commentator, ch. XI was commented by Vidyāmādhava's son. But, for the identity of the two, see colophon to the com. on ch. V :

ittham Vidyāmādhavīye Muhūrtādarśe 'Vidyāmādhavasyātmajena' /
vyākhyāto 'bhūd gumphite 'Viṣṇunāmnā' vīryāvīryam darśanam pañcamo 'yam //

3. See above, pp. 5-6.

4. Pub. under the title *Daśādhyāyī*, Venkatesvara Press, Bombay, Sam. 1969 (A.D. 1912) ; Vidyakalpataru Press, Palghat, (1905), (in grantha script).

5. On Govinda Bhaṭṭatiri, see : Raja, 132-33 ; Ul. I. 167-69, II. 110 ; Va. I. 353-59, II 499-500. For the legends about him see, Koṭṭārathil Śaṅkuṇṇi, 'Talakkuḷattūr Bhaṭṭatiriyum Pāzhūr paṭippurayum' (Mal.), *Aitihyamālā*, I, (6th edn., Trichur, 1961), 99-108.

in early *bhāṣā-miśra* (Skt.-Mal.) verses. The author, who does not reveal his personal name, makes mention of his patron Ceyyūr (?Cellūr) Nārāyaṇan Nampūtiri and might have hailed from Periñcellūr in Malabar.[1]

19. Nityaprakāśa Yati (14th-15th cent.)

Nityaprakāśa Yati or °Bhaṭṭāraka, about whom the only personal information known is the name of his preceptor, *viz.*, Ānandaprakāśa, has commented on the *Horā*, both in Sanskrit and in Malayalam. Both the commentaries are elaborate and are called *Prakāśikā*. The Sanskrit commentary is called also *Viṣamākṣaravivṛti*. The language of the Malayalam commentary, which is characterised by mediaeval traits,[2] would suggest for it a date in the 14th-15th century.

20. Kumāra Gaṇaka (*c.* 14th-15th cent.)

Kumāra Gaṇaka is the author of *Raṇadīpikā*,[3] which he composed at the instance of Deva Śarmā, younger brother of Govinda, a Kerala prince, probably of the principality of Cempakaśśeri (modern Ampalappuzha). The work deals with politics, royal conduct and war, on the background of natural and horary astrology. It is divided into eight chapters, called *viveka-s*, devoted, respectively, to *Naya*, *Yātrā*, *Jayājaya*, *Kāla*, *Sūlacakrādi*, *Pañcasvara*, *Mṛgavīrya* and *Bhūbala*. The author writes very readable poetry and reveals a liberal and progressive bent of mind.[4]

21. Rudra I (*c.* 1325-1400)

Parameśvara Vaṭaśreṇi (1360-1455), of *Dṛggaṇita* fame, has mentioned in several of his works, Rudra, the eminent astronomical authority at whose feet he learnt the discipline. This Rudra's has, however, not been identified as the author of any work.[5]

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1. On Tāmaranallūr, see Ul. I. 430-1 ; Ul. Sup. I 146 ; Va. III 396. Two manuscripts of the work, Ker. 4147-A and PM 4113-B, call themselves *Muhūrtapadavī Bhāṣā*. It remains to be seen whether it is related to the *Muhūrtapadavī* of Govinda Bhaṭṭatiri of Talakkuḷam.

2. *E. g.*, see the lines : *yāvad gataḥ śītakaro* etc. (*Horā* 24. 6) : *enpatu praśnam peṇṇina pozhutu praśnarāśikku ettanayām rāśiyile candran ninṟān enṟu inta candran ninṟa rāśi piṭittu akkaṇṇakkeṇṇi avvirāśiyile candṟan nilkaṟpiṟantān enṟu colluka* / (*Des. Cata. of Skt. Mss. in H H. the Maharaja's Palace Library, Trivandrum*, Ms. No. 995, (vol. IV, p. 1315).

3. Ed. *TSS*, No. 95, Trivandrum, 1923.

4. On Kumāra Gaṇaka, see Ul. II. 112-13 ; Va. II. 735-39

5. On Rudra, see Raja, 138.

22. Mādhava of Saṅgamagrāma (*c.* 1340-1425) : Mādhava I

Mādhava, referred to by later astronomers as *Golavid* ('Master of spherics'), was an astute mathematician who belonged to Saṅgamagrāma, identified with Irinjālakkuḍa, near Cochin. According to an old astronomical document, he belonged to the sub-caste of Kerala brāhmans called Emprān-s and the name of his house was Ilaññippaḷḷi.[1] In his *Veṇvāroha*,[2] he evolved a facile procedure to read out the true positions of the Moon every 36 minutes.[3] He uses, in this work, a date in A. D. 1400 as the epoch, which gives a clue to his date. Among his known works are *Lagnaprakaraṇa* and a table of moon-mnemonics correct to the seconds. His *Mahājyānayanaprakāra* and *Madhyamānayanaprakāra*, for which short commentaries are available, contain novel theorems and computational methods evolved by him and used by later writers.

An important work of Mādhava, which may be identified here, is his *Agaṇita*. The astronomical document mentioned above states that Mādhava is the author also of an *Agaṇitapañcāṅga*. An anonymous *Agaṇitagrahacāra* which has been quoted by *Karaṇapaddhati* (IV. 14, 16, 18) and is available in manuscript form mentions the *śodhyābdas* ('deductive years') for the computation of Mars, Mercury, Jupiter, Venus, Saturn and Moon's Higher Apsis as Śaka 1320, 1318, 1340, 1158, 1301 and 1276, corresponding to A.D. 1398, 1396, 1418, 1236, 1379 and 1354.[4] Since the principle underlying the *sodhyābdas* is to fix them in such a way that the largest possible number of

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1. *Cf.* the words: *Mādhavan Veṇvārohādīnām karttā. ... Mādhavan Ilaññippaḷḷi Emprān*, in a Kerala manuscript, No. 9886, preserved in the Oriental Institute, Baroda, and ed. by K. V. Sarma in the paper 'Some direct lines of astronomical tradition in Kerala', Pt. Charudeva Shastri Felicitation Vol., Delhi, 1972.

2. Cr. ed. by K. V. Sarma, with the commentary of Acyuta Piṣaraṭi, Sanskrit College, Trippunithura, 1956.

3. For a note on other works of this genre, see above, pp. 32-33.

4. *Cf.* Ms. T. 280, *Cata. of the Skt. Mss. in the Curator's Office, Trivandrum*, Vol. IV, p. 1305 :

śakābdāt 'naraloko'nād rāghavair dhītsunā kujaḥ /
'divyaloko'-nanīlāgrais tattvajñair bhājite budhaḥ //

'navaloko'-nasārāṅgair gajair āpte gurur bhavet /
'hemapuṇyo'-naśākābdāt sāravair govarair bhṛguḥ //

'yajñaloko'-nanītaṁśair dharmair āpte śanir bhavet /
'tīrthapriyo'na-śākābdād gandhajñair bhājite tamaḥ //

years will be cut off from the number of the current year, thereby providing maximum ease in calculation, the *śodhyābdas* selected would be as near as possible to the date of the composition of a work, which date would, naturally, be just ahead of the largest *śodhya*. In the case of the *Aganitagrahacāra* in question, the *śodhyābdas*, with A. D. 1418 as the latest among them, agrees, surprisingly, with the date of Mādhava. This goes to confirm that in the present *Aganitagrahacāra* we have a hitherto unidentified work of Mādhava, *viz.* his *Aganitapancāṅga* which is mentioned in the said astronomical document. Again, it seems quite possible that Mādhava had composed a comprehensive treatise on astronomy and mathematics, which yet remains to be identified and which may be supposed to contain the numerous single and groups of verses enunciating computational procedures, theorems and formulae which are quoted as Mādhava's by later writers. Possibly, Mādhava wrote also a work named *Golavāda*[1] which gained for him the appellation *Golavid* by later scholars.[2]

23. Parameśvara of Vaṭaśreṇi (*c.* 1360-1455) : Parameśvara I

Parameśvara, one of the foremost astronomers of Kerala who revised the *Parahita* system of computation through his *Dṛggaṇita* in 1430, has made significant contributions to Hindu mathematics and astronomy. He was a Ṛgvedin of the Āśvalāyana-sūtra and the Bhṛgu-gotra. He hailed from the village of Aśvatthagrāma (Mal. Ālattūr) and his house Vaṭaśreṇi (Mal. Vaṭaśśeri) was situated on the confluence of river Nīlā (Mal. Bhāratappuzha) with the Arabian Sea, where, on the sandy expanse, he carried on investigations for fiftyfive years. He also observed a large number of eclipses, of which he has recorded the details in his *Siddhāntadīpikā*.[3] His grandfather was a disciple of

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1. A matter of melencholy interest might de mentioned here, at least for record. The presence of a work entitled *Golavāda* written in continuation of Mādhava's *Veṇvāroha* in a manuscript preserved in the Eḷankunnappuzha Naṭakkal Kovilakam, near Ernakulam, raised in the present writer the hope that that could, possibly, be a thitherto unknown work of Mādhava. When, however, he proceeded to the said Kovilakam and, with a good deal of persuasion, caused the wooden box containing the family collection of manuscripts to be opened, he found that all the manuscripts in that box, including that of *Golavāda*, had been completely eaten away by white ants. For further details, see his note on the subject in the *Mathrubhumi W*, (Kozhikode), 17.6.1956, p. 4.

2. On Mādhava, see : K. V. Sarma, Intro. to the edn. of *Veṇvāroha*, *op. cit.*; 'Saṅgamagrāma Mādhava', (Mal.), *Mathrubhumi W*, 4. 11. 1956, pp. 17-18, 59 ; Raja. 133-36 ; UI. II. 91-100 ; Va. I. 387-88, III. 388-90.

3. On Parameśvara's extensive astronomical observations, see above, pp. 2-4.

Govinda Bhaṭṭatiri of Talakkuḷam and he himself was a pupil of Rudra I, Nārāyaṇa son of Parameśvara and Mādhava of Saṅgamagrāma.

Parameśvara was a prolific writer, author of about 30 works, including original treatises and commentaries, both on astronomy and astrology. Among his original writings on astronomy might be mentioned the *Dṛggaṇita* (1430),[1] three works on spherics, being the *Goladīpikās* I-III (1443),[2] three works on improved computation and rationale of eclipses, *viz.*, *Grahaṇāṣṭaka*,[3] *Grahaṇamaṇḍana*[4] and *Grahaṇanyāyadīpikā*,[5] a text on the computation of the Moon-shadow, entitled *Candracchāyāgaṇita* and a rationale on the computation of mnemonic tables, *Vākyakaraṇa*.[6] He has commented on the *Āryabhaṭīya*,[7] *Mahābhāskarīya*,[8] *Mahābhāskarīya-bhāṣya*,[9] *Laghubhāskarīya*,[10] *Sūryasiddhānta*, *Laghumānasa*[11] *Līlāvatī*, *Goladīpikā* I[12] and *Vyatīpātāṣṭaka*. Some of his commentaries, like, for instance, those on the *Āryabhaṭīya* and *Mahābhāskarīya-bhāṣya*, are extremely valuable for the historian of Hindu astronomy, since they contain the enunciation of some of his new findings, theories and interpretations. Two of his works on astronomy, *viz.*, *Vākyadīpikā* and *Bhādīpikā*, are yet to be recovered. His writings on astrology include *Ācārasaṅgraha* in two versions, one of them ending with *Vivāhānukūlyam*, *Jātakapaddhati*, and *Ṣaḍvargaphalam* and commentaries on the *Jātakakarmapaddhati* of

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1. Cr. ed., K.V. Sarma, V.V.R.I., Hoshiarpur, 1963.

2. I: Cr. ed., K.V. Sarma, Adyar Library, Madras, 1957; II; Ed. T. Ganapati Sastri, *TSS*, No. 49, Trivandrum, 1916. The traditional date of composition of one the *Goladīpikās* (which one it is not possible to say) is given by the chronogram *dīyatām* (M.E. 618) equivalent to A.D. 1443, is given by Achyutattu Vasudevan Moossatu in his collection of chronograms entitled *Maṟakkātta nalukal* ('Unforgettable days'), Kunnamkulam, M.E. 1130 (A.D. 1955).

3. Cr. ed., K.V. Sarma, K S R. Inst., Madras, 1959.

4. Cr. ed., K.V. Sarma, V.V.R.I., Hoshiarpur, 1965.

5. Cr. ed., K.V. Sarma, V.V.R.I., Hoshiarpur, 1966.

6. This *Vākyakaraṇa*, in about 75 verses, is different from the *Vākyakaraṇa* attributed to Vararuci III.

7. Ed., H. Kern, Leiden, 1874; Udaya Narain Singh, Etawah, 1906.

8. Pub., *Anandasrama Skt. Ser.*, No. 126, Poona, 1945.

9. Cr. ed., T.S. Kuppanna Sastri, Gov. Or. Mss. Lib. Madras, 1957.

10. Pub., *Anandasrama Skt. Ser.*, No, 128, Poona, 1946.

11. Pub., *Anandasrama Skt. Ser.*, No, 123, Poona, 1944.

12. Cr. ed., K.V. Sarma, Adyar Library, Madras, 1957.

Śrīpati, *Praśnaṣaṭpañcāśikā* of Pṛthuyaśas, an anonymous *Muhūrtāṣṭāka* and the *Muhūrtaratna* of Govinda Bhaṭṭatiri.[1]

24. Dāmodara of Vaṭaśreṇi (*c.* 1410-1510) : Dāmodara I

Of Dāmodara, son of Parameśvara of Vaṭaśreṇi, no fullfledged work is known, but his pupil Nīlakaṇṭha Somayāji refers to him as an erudite astronomer and quotes from his writings. Thus, in his *Bhāṣya* on *Āryabhaṭīya* (*Kālakriyā* 17-21), Nīlakaṇṭha says : *tac coktam asmad Ācāryaiḥ : sarvatra viṣkambhadalam śrutau vā vyāsārdhake syāt viparītakarṇaḥ |* (Edn., *TSS*, No. 110, p. 47). Further down, in the same context, Nīlakaṇṭha quotes a longer piece with the introductory statement : *nibaddham ca tat tadaiva asmadgurubhiḥ pañcabhir upajātibhiḥ* :

arkasphuṭenānayanam prakuryāt
svamadhyamasyātra vitungabhānoḥ |
bhujāguṇam koṭiguṇam ca kṛtvā
mṛgādikendre 'ntyaphylākhyakoṭyoḥ ||

bhedaḥ kulīrādigate tu yogas
tadvargayuktād bhujavargato yat |
padam viparyāsakṛtaḥ sa karṇas
trijyākṛtes tadvihṛtas tu karṇaḥ ||

tenāhatām uccavihīnabhānor
jīvām bhajed vyāsadalena labdham |
svocce kṣipec cāpi tam ādyapāde
cakrārdhataḥ śuddham api dvitīye ||

cakrārdhayuktam tu tṛtīyapāde
saṁśodhitam maṇḍalatás caturthe |
evamkṛtaḥ sūkṣmataras tu madhyaḥ
pūrvam padam yāvad ihādhikam syāt |

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1. On Parameśvara, see : K.V. Sarma, Intro. to his edns. of Parameśvara's works, *op. cit.*; '*Dṛggaṇita* of Parameśvara', (Mal.), *Mathrubhumi W*, 7.10.1956, pp. 29-30; 'Fiftyone years' penance' (Mal.), *Mathrubhumi W*, 7.10.1956, pp. 29-30 ; Raja, 136-43 ; Sen, pp. 166-70 ; Ul. II. 100-5 ; Ul Sup., I. 215-19 ; Va. I. 378-83.

antyāt phalāt koṭiguṇaś caturthe
tv ārabhyate yady adhikātra koṭiḥ |
sarvatra viṣkambhādalam śrutau vā
vyāsārdhake syād viparītakarṇaḥ ||
(*Ibid.*, p. 48)

Elsewhere, too, Nilakaṇṭha quotes Dāmodara : *E.g.*, *prakārāntareṇa 'candrabāhuphala' ityādinā śrīmad-Dāmodarāhvayāsmadgurumukhodgatena ślokenoktā |* (*Grahaṇādigrantha*, ms., p. 61). Later writers also mention Dāmodara. Dāmodara might have composed certain works which are yet to be identified, and from which the above-mentioned passages should have been quoted.[1]

25. Ravi Nampūtiri (*c.* 1425-1500)

Nilakaṇṭha Somayāji (1444-1545) mentions in the colophon to his *Āryabhaṭīya-bhāṣya, Gaṇitapāda*, Ravi as his teacher on Vedānta. He pays his respects to Ravi also in the beginning of his *Siddhāntadarpaṇa*. Ravi was also an erudite scholar in Jyotiṣa and has been identified as the author of *Ācāradīpikā*, an elaborate metrical commentary on the *Muhūrtadīpikā*.[2]

26. Nilakaṇṭha Somayāji (1444-1545) : **Nilakaṇṭha I**

Nilakaṇṭha Somayāji, the centenarian astronomer of Kuṇḍapura (Mal. Tṛkkaṇṭiyūr) in South Malabar, was, perhaps, as eminent as his grand-teacher, Parameśvara of Vaṭaśreṇi. In a detailed colophon to his *Bhāṣya* on the *Āryabhaṭīya, Gaṇitapāda*, he gives full details about himself. Elsewhere also, in his works, he records particulars about his person. He was a Numpūtiri of the Garga-gotra and hailed from the family of Keḷallūr (Mal. Kerala-nal-ūr, Skt. Kerala-sadgrāma). He was the son of Jātavedas and had a younger brother named Śaṅkara. His date of birth is given in the chronogram given in his *Siddhāntadarpaṇa-vyākhyā*, *viz.*, *tyajāmyajñatām tarkaiḥ* (16,60,181) which falls in Dec. 1444. He is referred to as a 'living' authority by Mādhava of Īñcakkāzhvā in his *Praśnasāra*, composed in 1542-43. Both he and his brother Śaṅkara were patronised by Kauṣītaki Āḍhya

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1. On Dāmodara, see Ul. Sup. I. 210-11 ; Va. I. 388. Va. identifies (I. 388) the present author with Dāmodara, au. of *Muhūrtābharaṇa*, but this is not possible, for the former belonged to the Bhārgava-gotra while the latter was of the Bhāradvāja-gotra ; *cf.* the Intro. verse to *Muhūrtābharaṇa* (Ms. Kerala T. 71) : *teṣām viduṣām madhye Bhāradvājagotrajātā ye |*

2. On Ravi, see K.V. Sarma, Intro. to the edn. of *Golasāra*, p. xv ; Ul. II. 113-14.

Netranārāyaṇa (Āzhvāñceri Tamprākkal), the hereditory religious head of the Nampūtiris. Nīlakaṇṭha spent his student days at the house of Parameśvara of Vaṭaśreṇi and received occasional instruction from him, his regular teacher being Parameśvara's son, Dāmodara. Nīlakaṇṭha had also another teacher by name Ravi, who is identified as the author of *Ācāradarśana*.

Nīlakaṇṭha was a versatile scholar, though his writings, which are prolific and erudite, are all on astronomy. His *Tantrasaṅgraha* (A.D. 1500)[1] is a comprehensive treatise on astronomy. His *Grahaṇanirṇaya* and *Candracchāyāgaṇita*, the latter with his own commentary, deal with revised procedures on the respective topics. In his *Golasāra*[2] he gives a résumé of some of his views on certain astronomical topics and in his *Siddhāntadarpaṇa*,[3] he sets out the astronomical constants, as determined by him, the rationale of which he expounds in his commentary on that work. In some elaborate tracts on eclipses and certain other topics (*Grahaṇādigrantha*) he expounds the tradition and rationale of eclipse computation, methods for determining corrections (*saṃskāra-s*) etc. His *Sundararājapraśnottara* forms his answers to certain astronomical problems posed by a contemporary astronomer from the adjoining Tamil region who has commented on the *Vākyakaraṇa* ascribed to Vararuci.[4] Nīlakaṇṭha sets out much original contribution through his extensive *Bhāṣya* on the *Āryabhaṭīya*[5] which is considered to be his masterpiece. But, by far the most instructive work of Nīlakaṇṭha is the *Grahaparīkṣākrama*,[6] where, in about 200 verses, he sets out the procedures for the observation of the planets, sometimes with instruments, and for their computation using the data

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1. Ed. with the commentary of Śaṅkara Vāriyar, *TSS*, No. 185 (Trivandrum, 1958).

2. Cr. ed., K.V. Sarma, V.V.R.I., (Hoshiarpur, 1970).

3. Cr. ed., K.V. Sarma, Adyar Library, (Adyar Library, Madras, 1955).

4. Cr. ed., T.S.K. Sastri and K.V. Sarma, K.S.R. Inst., Madras, 1962.

5. Ed. *TSS*, Nos. 101, 110, 185, (Trivandrum, 1930, 1931, 1957).

6. This work has been quoted in full in his *Āryabhaṭīya-bhāṣya*, *Gola*. 48, (*TSS*, No. 185, pp. 132-49). An adapted edition of this has been issued by Puliyur Purushottaman Namputiri, (Bombay, 1950). A perfect ms. of the work is available with K.V. Sarma.

obtained from the observations. A commentary in Malayalam expounding the said procedures is also known.[1]

27. Śaṅkara of Keḷallūr (*c.* 1475-1575) : Śaṅkara I

Nīlakaṇṭha Somayāji's younger brother Śaṅkara, an astronomer in his own right, was, like his brother, patronised by the Azhvāñceri Tamprākkaḷ, at whose house he was teaching astronomy as mentioned in Nīlakaṇṭha's *Āryabhaṭīya-bhāṣya, Gaṇita.* 26, (edn., p. 156). The said *Bhāṣya* had been composed for the sake of Śaṅkara, who was also entrusted with its propagation.[2]

28. Citrabhānu (*c.* 1475-1550)

Citrabhānu Nampūtiri, author of *Karaṇāmṛta,* an advanced manual on astronomical computation in four chapters, was a pupil of Nīlakaṇṭha Somayāji, whom he mentions as 'Gārgya' in the beginning of his work. He belonged to the Gautama-gotra and hailed from the village of Covvaram (Skt. Śivapuram) near Trichur. The date of composition of *Karaṇāmṛta* which is indicated in that work by the Kali chronogram *buddhyonmathyoddhṛtam yatnāt* (16,91,513) (A.D. 1530) gives the clue to its author's date.[3]

29. Citrabhānu-Śiṣya (*c.* 1500-75)

Bhāvacintāvali, in three chapters, is a work on astrology by an anonymous pupil of Citrabhānu. That the pupil has written on astrology would suggest that the teacher Citrabhānu, besides being an astronomer, was also an authority on astrology.[4]

30. Nārāyaṇa I (*c.* 1500-75)

Nārāyaṇa, disciple of another Nārāyaṇa and Citrabhānu, ardent admirer of Nīlakaṇṭha Somayāji and highly devoted to Subrahmaṇya,

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1. On Nīlakaṇṭha, see : K.V. Sarma, 'Gārgya Kerala Nīlakaṇṭha Somayājin', *Jl. of Or. Res.*, 26 (1956-57) 21-39 ; Intro. to the edns. of *Siddhāntadarpaṇa* and *Golasāra, op. cit.*, ; Raja, 143-52 ; Sen, 155-57 ; Ul. II. 117-20 ; Ul. Sup. I. 235-37 ; Va. I. 3 3-87.

2. On Śaṅkara, see K V. Sarma, Intro. to *Golasāra, op. cit.*, p. xiii.

3. On Citrabhānu, see Raja, 153-55 ; Ul. II. 273-75 ; Ul. Sup. I. 305-6.

4. On Citrabhānu-Śiṣya, see Raja, 154 (where the work is wrongly called *Bhāvacintāmaṇi*) ; Ul. II.275.

an esteemed associate of Nīlakaṇṭha, was an original thinker and highly informed commentator. He wrote, in A.D. 1529, an instructive commentary called *Laghuvivṛti* on *Pañcabodha IV*.[1] His *Uparāga-kriyākrama* in five chapters is a detailed exposition of eclipse computation. He also wrote two commentaries on the *Līlāvatī*, one short and the other nearly five times as long, both called *Kriyākramakarī* and *Karmadīpikā* (called also *Karmapradīpa* and *Karmapradīpikā* according to the exigencies of the metre of the verses containing these names).[2] The longer *Kriyākramakarī* is especially valuable to the historian of Kerala astronomy and mathematics for the profuse references it contains to earlier authors and authorities, some of which are now lost, and for the theories and procedures enunciated in that commentary.[3]

31. Saṅkara Vāriyar (*c.* 1500-60) : Saṅkara II

The author of *Laghuvivṛti* (A. D. 1556), which is an erudite commentary on the *Tantrasaṅgraha*, was a disciple of Nīlakaṇṭha Somayāji and protege of (Netra) Nārāyaṇa (Āzhvāñceri Tamprākkaḷ), both of whom are mentioned in the beginning of that commentary, is identified with Śaṅkara Vāriyar of Tṛkkuṭaveli family. Though the name of the author is not mentioned in the commentary itself, some manuscripts of that commentary (Kerala Univ. 8351, 8906, C. 524) carry the following post-colophonic statement, in Malayalam, by the scribe : *ī vyākhyānam Tṛkkuṭaveli-c-Caṅkaravāriyar oṭukkattu camaccatu / Āzhvāñcerikku veṇṭiṭṭu sukhame śikṣiccu camacu ennu Paraṅṅoṭṭu paṟaññu keṭṭu* / : 'This commentary was composed last by Tṛkkuṭaveli Śaṅkara Vāriyar. It is stated to have been said by Paraṅṅoṭṭu that it was composed with great care for the sake of Āzhvāñceri'. The person referred to here as Paraṅṅoṭṭu is very likely to be Paraṅṅoṭṭu

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1. *Cf.* the chronograpm *prajñālokoddhṛtam yas* (Kali day 16,91,302) in the verse at the end of this work :

prajñālokoddhṛtam yasmāt gūḍham artham vivṛṇvatī /
vyākhyaivam Pañcabodhasya ciram tiṣṭhatu bhūtale //

2. Some scholars distinguish between the authors of the two commentaries on the *Līlāvatī* and suggest that Śaṅkara Vāriyar of Tṛkkuṭaveli might have written the longer *Kriyākramakarī* (*cf.* Raja, 155) ; again, some miss to recognize '*two*' *Kriyākramakarī-s* (Ulloor II, 121). The matter has been examined in detail and the authorship of Nārāyaṇa of both the commentaries is being set out by the present writer, elsewhere.

3. On Nārāyaṇa, see Ul. II. 121 ; Ul. Sup. I. 222, 227-28.

Jyeṣṭhadeva, an younger contemporary of Nīlakaṇṭha, and, therefore, the statement is quite likely to be authentic.

Hitherto, Śaṅkara Vāriyar has been credited with the authorship only of the above-said commentary. However, three more of his works can now be identified. Thus, in the beginning of the present commentary, he makes mention of a larger commentary of his on *Tantrasaṅgraha*:

Nārāyaṇam jagadanugrahajāgarūkam
Śrī-Nīlakaṇṭham api sarvavidam praṇamya |
yat Tantrasaṅgraha-gatam grahatantrajātam
tasyā'parām' ca Vivṛtim vilikhāmi Laghvīm ||

Again, in the final colophon at the end of the work, he states that the present commentary is an adaptation of a larger commentary of his named *Kriyākalāpa* ('Detailed demonstration') on the *Tantrasaṅgraha*. *Cf.*:

iti Tantrasaṅgrahasya Kriyākalāpam krameṇa saṅgṛhya |
racite tadvyākhyāne pūrṇo 'bhūd aṣṭamo 'dhyāyaḥ ||

A unique manuscript of this longer commentary has been recorded in the Inventory below. It can also be shown that the anonymous work entitled *Karaṇasāra*, in four chapters, whose authorship has been left open (*cf.*, Ul. II. 121-22) and a Malayalam commentary on it which is, at present, attributed to Śaṅkaran Nampūtiri of Mahiṣamaṅgalam (Ul. II. 272, 474), are really the works of Śaṅkara Vāriyar.[1]

32. Jyeṣṭhadeva (*c.* 1500-1610)

Jyeṣṭhadeva, this name being, most probably, the Sanskritised form of his personal name in the local language, has the distinction of

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1. Evidences for these identifications are being set out by the present writer elsewhere. Some scholars (*cf.* Raja, 155) ascribe to Śaṅkara Vāriyar the elaborate commentary *Kriyākramakarī* on the *Līlāvatī* on the basis of the similarity of some expressions in one of the introductory verses of that work with those in his commentary on the *Tantrasaṅgraha* However, there are clear evidences, which, as indicated above are being set out elsewhere, to show that the *Kriyākramakarī* is a work of Nārāyaṇa and not of Śaṅkara Vāriyar.

being the author of the popular *Yuktibhāṣā*[1] or *Gaṇitanyāyasaṅgraha*,[2] which forms an elaborate and systematic exposition of the rationale of mathematics in its Pt. I and of astronomy in its Pt. II. The Sanskrit version of this work, known as *Gaṇitayuktibhāṣā*, is also, in all probability, his work. An old astronomical document informs that he was the pupil of Dāmodara of Vaṭaśreṇi and was a member of the Paraṅṅoṭṭu family[3] of the Ālattūr village in South Malabar.[4] Nīlakaṇṭha Somayāji, whose *Tantrasaṅgraha* he mentions towards the beginning of his work, was his respected elder. He himself was the teacher of Acyuta Piśāraṭi who mentions him in reverential terms at the close of his *Uparāgakriyākrama* (A.D. 1592). M. Whish records a tradition that the author of the *Yuktibhāṣā* was the author also of a *Dṛkkaraṇa*.[5] The *Dṛkkaraṇa* in question, which is now available in a single manuscript, is a comprehensive metrical treatise in Malayalam on astronomy. It does not give anywhere the name of its author, but gives in its last verse, its date of composition in the words *kolambe barhisūnau* (M.E. 783=A.D. 1608). In view of this date and the mention of the tradition recorded by Whish, it is quite possible that this is a work of Jyeṣṭhadeva.[6]

33. Jyeṣṭhadeva-Śiṣya (*c.* 1550-1625)

A disciple of Jyeṣṭhadeva, who studied *Tantrasaṅgraha* under him, has composed a metrical commentary on that work; this

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1. Pt. I. ed with elaborate notes by Ramavarma Maru Thampuran and A. R. Akhileswara Aiyar, (Mangalodayam, Trichur, 1948). See also, A. Venkataraman, 'Some interesting proofs from Yukti-bhāṣā,' *Mathematics Student*, 16 (1968) 1-7.

2. An extremely unreliable edition of the whole work has been issued under this title by the Govt. Or. Mss. Library, Madras, 1954.

3. An old Malayalam Ms. No. 9386, preserved in the Or. Inst., Baroda, reproduced by K.V. Sarma, in his paper 'Direct lines of astronomical tradition in Kerala', Pt. Charudeva Shastri Felicitation Volume, Delhi, 1972.

4. See Pārayil Rāman Nampūtiri, (ed.) *Nampūtirimār*, Trichur, M.E. 1093, (A.D. 1918), p. 65.

5. M whish, 'On the Hindu quadrature of the circle' etc., *Trans. Royal As. Soc.*, 3 (1835) 523.

6. On Jyeṣṭhadeva, see K.V. Sarma, 'Jyeṣṭhadeva and his identification as the author of *Yuktibhāṣā*', *Adyar Lib. Bul.*, 22 (1958) 35-40. On *Yuktibhāṣā*, see Raja, 156-58; Ul. III. 439-40; Ul. Sup. I. 612-16.

commentary extends, unfortunately, only to the first four of the eight chapters of that work.[1]

34. Māttūr Nampūtiri-s : Puruṣottama I and Subrahmaṇya I (*c.*1475-1550)

Two *Muhūrtapadavī*-s, out of the seven texts that go under that name, have been widely known as the works of two brothers belonging to the Māttūr (Skt. Mahāvastu) family in the village of Pāññāḷ (Skt. Pāñcālagrāma), near Chelakkara in Cochin. These two works, comprising, respectively, of 36 and 43 verses, condense in themselves, independently, the prescription of auspicious times for all the major social and religious functions and indication of the baneful times which are to be avoided. Their popularity, as attested by their numerous manuscripts as available and the several commentaries on them is matched only by the obscurity that surrounded the names of their authors. It has now been possible to identify both these writers. A unique manuscript of the first of these two works, preserved in the India Office, London, (Catal. No. 8070), carries a colophon giving the name of its author as Puruṣottama :

V(B)ṛhadāvṛtigehasambhaven dvijarājñā Puruṣottamena /
gurunāthakṛpābalāt kṛtam yat tad idam sadviduṣām mude 'stu nityam ||

With regard to the second of the two works, a hitherto unnoticed commentary thereon available in the Āzhvāñceri Mana (List No. 81) etc., mentions in its introductory verse the name of the author of the text as Subrahmaṇya. *Cf.* :

'vṛddhimdheyād hitāyā'sau savitā vas trayīmayaḥ
yadraśmisambhramād bhānti jyotīṁṣy etāni santatam /
śrīsvarṇabhūmir vidvadbhir sevyā, padyair iyam kṛtā
Muhūrtapadavī yena Subrahmaṇyam praṇaumi tam ||

It is also of interest to note that the first words of the said verse, *viz.*, *Vṛddhimdheyāddhitāya* (16,91,994), gives the date of composition

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1. *Cf.* the concluding verse :

ity eṣa Parakroḍavāsadvijavara-samīrito yo 'rthaḥ /
sa tu Tantrasaṅgrahasya prokto 'dhyāye caturthe 'bhūt ||

The view expressed by some scholars (*cf.* Ul. II. 120, Va. II. 498 and Intro., to the edn. of *Tanrasaṅgraha*, *TSS* 188) that the author of the commentary was a 'Brāhman of the Parakroḍa village' is not warranted by this verse.

of the work as A D. 1531, enabling the date of the author to be fixed correctly.[1]

35. Nārāyaṇa of Kāṇvavastu (*c.* 15th cent.) : Nārāyaṇa II

Nārāyaṇan Nampūtiri hailing from the village of Kāṇvavastu, which is differently identified with Tṛkkaṇṇapuram and Kaṇṇanparampu in Malabar, was the son of Keśava.[2] Nārāyaṇa's *Muhūrtadīpaka* is a comprehensive treatise in about 400 verses on the selection of auspicious times for social and religious functions and has been cited as an authority by Śaṅkara of Mahiṣamaṅgalam (see below).[3]

36. Rudra Vāriyar (*c.* 1475-1550) : Rudra II

Rudra (Mal. Uzhuttira) Vāriyar belonged to the Deśamaṅgalam Vāriyam, in South Malabar, which, during the middle ages, was a renowned centre of Sanskrit studies and a rich repository of manuscripts.[4] Rudra wrote in A.D. 1527 his elaborate commentary called *Naukā* or *Vivaraṇa* on Varāhamihira's *Horā*.[5] His other known work is the *Aṣṭamaṅgalapraśna* on a type of astrological query which is very popular in Kerala.[6]

37. Śaṅkara of Mahiṣamaṅgalam (1494-1570) : Śaṅkara III

In the matter of the popularisation of studies on Jyotiṣa among the masses in Kerala, Śaṅkaran Nampūtiri of Mazhamaṅgalam

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1. On Māttūr Nampūtiri-s, see K. Sankara Menon, Intro. to his edn. of *Muhūrtapadavī II* with *Bālaśaṅkaram*, *Śri Vanchi Setu Lakshmi Series*, No. 5, (Trivandrum, 1926); Ul. II.110-11 ; Va. II. 601-12.

2. Ulloor (II. 107) suggests that Nārāyaṇa might have been one of the teachers of Parameśvara of Vaṭaśreṇi. But this is not possible, for the name of the father of the present Nārāyaṇa is Keśava (*cf.* : *kaścid dvijo guṇakhyāta-'Keśavākhyadvij-ātmajaḥ' / Kāṇvavastv-ākhvayagrāmajanmā Nārāyaṇāhvayaḥ //*), while the name of the father of Parameśvara's teacher Nārāyaṇa was Parameśvara (*cf.* Nilakaṇṭha Somayāji's statement on the subject : *Parameśvaras tu Rudra-'Parameśvarātmaja-Nārāyaṇa'-Mādhavādibhyo golavidbhyo gaṇitagola-yuktīr api bālya eva samyag gṛhītvā* etc. (*Āryabhaṭīya-bhāṣya*, *Gola.* 48 ; edn., Pt. III, p. 154).

3. On Nārāyaṇa, see Ul. II. 107 ; Va. II. 612-13.

4. For an account of this repository of Mss., see K.V. Sarma, 'Deśamaṅgalam Mss. collection' (in Mal.), *Mathrubhumi Weekly*, (Kozhikode), 8.2.1957.

5. Ed. *TSS*, No 91, Trivandrum, 1926; 2nd edn., Trivandrum, 1958.

6. On Rudra, see Ul. II. 114-17 ; Ul. Sup. I. 531-34.

(Skt. Mahiṣamaṅgalam) family had an important part to play. Śaṅkara hailed from the Perumanam village near Trichur but spent most of his active life at Chengannoor with his teacher Parameśvaran Potti of Vāzha-māveli house. In his work, *Rūpānayanapaddhati*, on grammar, Śaṅkara gives the date of his birth in the chronogram *jātoyam haṁsatulye*('hani) (Kali day 16,78,168), which falls in A.D. 1494.

Śaṅkara wrote a large number of works both on astronomy and on astrology, mostly in simple Malayalam poetry and easy prose. These works include *Gaṇitasāra*, *Candraganitakrama* and *Ayanacalanādi-gaṇita* in astronomy, *Jātakakrama* in horoscopy and *Praśnamālā* with *Bhāṣā* in astrological query. A work which he composed in two versions is the *Kāladīpaka*, called in general parlance *Ceriya Kāladīpakam* (Short Kāladīpaka) and *Valiya Kāladīpakam* ('Long Kāladīpaka'). He has a *Jātakasāra* in Sanskrit and another with the same title in Malayalam. He has commented on *Pañcabodha* (II) and *Pañcabodha* (IV).[1] His commentaries on *Laghubhāskarīya*, *Muhūrtapadavī*, *Pañcabodha* IV and his own *Kāladīpaka* II, all called *Bālaśaṅkaram*, have greatly helped the popularisation of these works among the people. On *Pañcabodha* IV, besides the *Bālaśaṅkaram*, he wrote another shorter (?metrical) elucidation entitled *Pañcabo-*

1. A good amount of confusion has occurred in earlier writings in the identification of Śaṅkara's works. Thus, the *Jātakasāra*, noticed by Ulloor (II. 474) is, really, by a namesake of our author. Our Śaṅkara has actually composed two works under the title *Jātakasāra*, one in Sanskrit and the other in Malayalam; Ulloor does not mention the former and describes the latter as an anonymous work (Ul. III. 241-42), failing to take note of the apparent Mahiṣamaṅgalm characteristics in it. Similarly, Ulloor does not take note of Śaṅkara's *Praśnamālā* and describes Śaṅkara's own *Bhāṣā* on it as an anonymous work (Ul. III. 242). Again, Ulloor describes Śaṅkara's *Bālaśaṅkaram*, called also *Pañcabodhakriyākrama* on *Pañcabodha* IV as the *Pañcabodha* of Śaṅkara, while Śaṅkara's commentary on *Pañcabodha* (II) is not noticed. Certain other works, *viz*, *Bhāṣāsaṅgraha*, *Praśnasāra* and *Karaṇasāra* have also been ascribed to our author but they do not appear to be his works. Of these, *Bhāṣāsaṅgraha* is a commentary by Vāsudeva on *Muhūrtapadavī* V and the *Praśnasāra* and *Karaṇasāra* are the works of different authors named Śaṅkara. A detailed study on the identification of Śaṅkara's works is being presented by the present writer elsewhere.

dhārthadarpaṇa. Saṅkara is also reputed to have composed a 'Register of muhūrtas' for a 1000 years.[1]

38. Mādhava of Iñcakkāzhvā (*c.* 1500-75) : Mādhava II

Mādhava was a member of the Nampūtriri family of Iñcakkāzhvā in Rāmamaṅgalam in the Mūvāṭṭupuzha Taluk (Dt. Kottayam). He composed his *Praśnasāra* in A.D. 1543, in 16 chapters, based on earlier works like the *Horā*, but incorporating in it numerous local practices. The work is historically significant for it provides the names of several contemporary astronomical and astrological authorities like Nīlakaṇṭha Somayāji of Keḷallūr, Parameśvaran Poṭṭi of Vāzha-māveli, Attimaṭṭam, Vākkaṭ, Paḷḷimaṭṭam, Mecceri, Koyikkara and Araṇappuram.[2]

39. Acyuta Piśāraṭi (*c.* 1550-1621) : Acyuta II

Acyuta Piśāraṭi of Tṛkkaṇṭiyūr (Skt. Kuṇḍapura) in South Malabar, was a versatile scholar and original thinker on astronomy. It was he who enunciated, for the first time, in Indian astronomy, the correction called 'Reduction to the the ecliptic', in his work *Sputanirṇaya* (before A.D. 1593) and set out its rationale, elaborately, in his work *Rāśigolasphuṭānīti*.[3] As pointed out earlier, (see above pp. 12-14), this correction was first introduced in Western astronomy by Tycho Brahe, at about the same time. Acyuta was a protege of King Ravi Varma of Veṭṭattunāḍ (Skt. Prakāśaviṣaya) and was the teacher of the renowned poet and grammarian Melputtūr Nārāyaṇa Bhaṭṭa, who, in the *carama-śloka* which he composed at the death of Acyuta, expressed the date of the latter's demise in the chronogram, *vidyātamā svar asarpat* (17,24,514), which falls in A.D.1621. Acyuta has composed about a dozen works on Jyotiṣa, including *Karaṇottama*,[4]

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1. On Śaṅkara, see Koṭṭārattil Śaṅkuṇṇi, 'Muzhamaṅgalattu Nampūri', *Aitihyamālā* IV, (4th edn., Trichur, 1958), pp. 89-93; Raja, 155-56; Ul. II. 269-72, 469-75; Va. II, 613-22; Ulloor, *Vijñānadīpikā*, (Trivandrum), IV, pp. 187-95.

2. On Mādhava see Ul. II. 475-77. R. Narayana Panikkar (II. 234-35) wrongly describes Mādhava's *Praśnasāra* as a commentary on the well-known *Praśnamārga*. It may be noted that the *Praśnamārga* was composed in 1650, more than hundred years after the composition of *Praśnasāra* (1543).

3. Ed. by K.V. Sarma, Adyar Library, Madras, 1955.

4. Ed. by K. Raghavan Pillai, *TSS*, No. 213, Trivandrum, 1964.

on astronomical computation, in five chapters, *Uparāgakriyākrama*,[1] and *Uparāgaviṁśati*[2] on eclipse computation, *Chāyāṣṭaka* on shadow computation and *Jātakābharaṇa* (?) and *Horāsāroccaya* on horoscopy. He has commented on the *Veṇvāroha* of Saṅgamagrāma Mādhava,[3] *Sūryasiddhānta* and on his own *Sphuṭanirṇaya* and *Karaṇottama*.[4]

40. Nīlakaṇṭha (16th-17th cent.) : Nīlakaṇṭha II

Nīlakaṇṭha II, about whom nothing more is known, is the author of a comprehensive treatise on arithmetic, entitled *Kaṇakkusāram*, couched in *maṇipravāḷam* verses in Malayalam and a commentary thereon. The author claims to have based his work on Sanskrit texts like the *Līlāvatī* and old Malayalam texts like *Kaṇakkatikāram*. It is noteworthy that, besides the general mathematical procedures, this work deals also with practices relating to local grain transactions, housebuilding, weighing of gold and silver, land tenure, masonry, ground measurement etc.[5]

41. Nārāyaṇa III

Laghudarśinī, a short work on astrology, is the work of a Nārāyaṇa, about whom nothing more than his name is known at present.

42. Dāmodara II of Maṅgalaśreṇi (*c.* 1575-1675)

In Dāmodaran Nampūtiri of the Maṅgalaśśeri house in Kaṇṇātipparampu in the Chirakkal taluk in N. Malabar, we have a reputed authority in astrology, of the times. His disciples included Iṭakramañceri Nampūtiri, author of *Bhadradīpa* and the anonymous author of *Jyotiṣasaṅgraha* (II), noticed below. Vaṭakkumkūr mentions

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1. This work, in four chapters, contains 130 verses, and not 20 as stated by Raja, 161.

2. Ed. by Rama Varma Maru Thampuran, *Ravivarma Granthāvali*, (Quarterly of the Skt. College, Trippunithura), 2 (1954) i-iii, 16 pp.

3. Ed. by K.V. Sarma, with text, Skt. College, Trippunithura, 1956.

4. On Acyuta, see, S. Venkitasubramonia Iyer, 'Acyuta Piṣāraṭi and his works', *Jl. Or. Res.* 22 (1952-53) 40-46 : K.V. Sarma, Intro. to his edns. of *Rāśigolasphuṭānīti* and *Veṇvāroha*, *op. cit.* ; Raja, 158-62 ; Ul. II. 319-26 ; Ul. Sup. I. 315 ; Va. II. 748-57.

5. On Nīlakaṇṭha, see Ul. II. 478-79 ; Ul. Sup. I. 226-27 ; Va. IV. 276.

two of his works, *viz.*, *Praśnarīti* (!) and *Līlāvatī-vyākyā*, which, he adds, have been popular. Manuscripts of these works have, however, yet to be identified.[1]

43. Iṭakramañceri Nampūtiri (*c.* 1625-1700)

He is author of *Bhadradīpa-gaṇita* (A.D. 1665) but prefers to remain anonymous and refers to himself only by the name of his family, *viz.*, Iṭakramañceri (or Eṭakramañceri). He mentions his native place as Perumānūr in Malabar and his teacher as Dāmodaran Nampūtiri[2] of the Maṅgalaśśeri family. In eleven chapters, couched in easy Sanskrit-Malayalam verses, he provides the layman with the elements of astronomical computation. A shorter work entitled *Bhūgolañāyam*, composed in the same style and devoted to a description of the earth as suspended in the atmosphere, its main landmarks, the construction of the armillary sphere etc., has much in common with Ch. V of the *Bhadradīpa* and is likely to be another composition of our author.[3]

44. Maṅgalaśreṇivipra-Śiṣya (17th cent.)

An author, who does not give out his name and describes himself as a pupil of a Nampūtiri of the Maṅgalaśśeri family, has composed, in Malayalam, a *Jyotiṣasaṅgraha* (II). The Maṅgalaśśeri Nampūtiri in question is very likely to be the same as the Dāmodaran Nampūtiri, author of *Praśnarīti* I and teacher of Iṭakramañceri Nampūtiri, noticed above.

45. Panakkāṭṭu or Iṭakkāṭṭu Nampūtiri (*c.* 1625-1725)

The *Praśnamārga*, the most popular and authoritative work on *praśna* in Kerala, was composed in about A.D. 1650 (*cf. kolambe 'muraha'-saṅkhye*, M.E. 825) by a Nampūtiri of the Panakkāṭṭu family. Since his family was situated in a place called Iṭakkāṭ (or Eṭakkāṭ) (Skt. Madhyāṭavī or Madhyāraṇya), in North Malabar, he was known by that name as well. K. Mahadeva Sastri gives the name of the

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1. On Dāmodara, see Va. III. 378-79.

2. Va. III. 379 takes the word 'Dāmodara' occurring in the Intro., verse 2 of the *Bhadradīpagaṇita* as the author's name on the basis of the wrong reading 'Dāmodaro', instead of that of his teacher, as per the correct reading 'Dāmodaram'.

3. On this author, see Ul. III. 232-34.

author as Parameśvara, on what grounds it is not known.[1] The author gives the name of his parents as Mahādeva and Śrī. In 32 chapters, the work presents a detailed exposition of the different topics and types of *praśna*. The author has also supplied a lucid exposition called *Durgamārthapradarśinī* on the work. It has several commentaries in Malayalam also, including some produced during modern times. Panakkāṭṭu Nampūtiri is said to have composed in 1676 a work on astronomy, which is not available now.[2]

46. Iṭakkāṭṭu (or Eṭakkāṭṭu) Kukkaṇiyāḷ (*c.* 1675-1750)

The personal name of this author seems to be Śaṅkaran Kaṇiyār. He belonged to the house of Kaṇiyāṅkaṇṭi in Eṭakkāṭ in North Malabar and studied astrology under Panakkāṭṭu Nampūtiri. In later life, he came under the influence of a Yogi and spent his life in meditation. His *Praśnarīti*[3] in 20 chapters is very popular among astrologers of the Kaṇiyār caste. In its first twelve chapters, it is a free metrical rendering in Malayalam of his teacher's *Praśnamārga* and deals, in the remaining eight chapters, with other allied and miscellaneous matters.[4]

47. Rāma-śiṣya (17th cent.)

Manuscripts of a free explanatory rendering, in Malayalam verses, of the *Laghujātaka* or *Laghuhorā* of Varāhamihira, called *Laghuhorāvivaraṇam* have been identified. The author does not reveal his name but mentions his teacher Rāma and the date of composition of the work, *viz.*, A.D. 1640. The author says in the beginning of the work that he had already composed a *Horāsāra* in Malayalam, possibly a free explanatory rendering, like the present work, of the *Horā*.

48. Puruṣottama II (*c.* 1650-1725)

A unique manuscript of the *Uparāgapariccheda* of a *Pañcabodha-śataka* (*Pañcabodha* V) is known. In 47 verses, this section of the work sets out the computation of solar and lunar eclipses, using its own

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1. See his *Des. Cata. of Skt. Mss. in the Curator's Office Library, Trivandrum*, vol. IV, (Trivaudrum, 1939), p. 1539.

2. On Panakkāṭṭu Nampūtiri, see Ul. III. 87-89 ; Va. III. 400-5.

3. Ed. serially in the *Kavanodayam* (Mal. monthly, Kozhikode).

4. On this author see Ul. III. 236-37 ; Ul. Sup. I. 536 ; Va. III.405-6.

revised multipliers, divisors and other constants. For the calculation of the precession of the equinox, it mentions the epoch *prauḍhaśrī-guṇasevya* (17,53,242). This date works out to A.D. 1699 and gives a clue to the date of the author.[1]

49. Putumana Somayāji (*c.* 1660-1740)

The *Karaṇapaddhati*,[2] in ten chapters, is a comprehensive treatise on astronomy by an anonymous Somayāji of the Putumana or Putuvana (Skt. Nūtanagṛha or Nūtanavipina) family of Śivapuram (Trichur). The availability of manuscripts of the work in Tamil and Telugu scripts indicate its popularity in those regions as well. The work has been commented in Malayalam,[3] Sanskrit and Tamil. Its date of composition is said to be given in the concluding verse of the work by the chronogram *gaṇitam etad samyak* (17,65,653) which falls in A.D. 1732.[4] While *Karaṇapaddhati* is the best known work of the author in Jyotiṣa, it has been shown that he has written certain other works also in that discipline.[5] In *Nyāyaratna*, he deals, in eight chapters, with certain aspects of *Sphuṭa*, *Vikṣepa*, *Chāyā*, *Viparītacchāyā*, *Grahaṇa*, *Śṛṅgonnati*, *Mauḍhya* and *Vyatīpāta*. The object of writing the work is expressed in its concluding verse thus :

nyāyapradarśanāyaivam gaṇiteṣvalpacetasām |
gaṇitābdhau nimajjyaitat Nyāyaratnam mayoditam ||

'This *Nyāyaratna* has been composed by me by delving into the depths of astronomy for demonstrating astronomical rationale to the dullwitted.'

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1. On Puruṣottama, see Ul. Sup. I. 224-25.

2. Ed. *TSS*, No. 126, (Trivandrum, 1937) ; with two old Malayalam commentaries, *Madras Govt. Or. Ser.*, No 98, (Madras, 1956) ; with a modern commentary by P.K. Koru, (Astro Press, Cherp, Kerala, 1953).

3. See fn. above.

4. Some scholars like Ulloor (II. 107-8) and Vaṭakkumkūr (I. 529-30) have held that this work was written in A D. 1430. For a refutation of this view and for conclusive evidences for a much later date, see K. V. Sarma, 'The date of Putumana Somayāji', (Mal.), *Mathrubhūmi W*, (Kozhikode), 5. 2. 1956. It has also been argued that the said chronogram, which according to the tradition recorded by Whish is the date of composition of the work, may not be so. However, in view of date indicated by the evidences set out in the said paper, there is no incongruity in believing the tradition.

5. On these, see K. V. Sarma, 'The hitherto unknown works of Putumana Somayāji' (Mal.), *Mathrubhūmi W*, *ib.* 29.1.1956.

The *Nyāyaratna* occurs in two versions, which differ slightly by the presence or absence of certain verses and in the arrangement thereof. The *Veṇvārohāṣṭaka* of Putumana Somayāji is a manual for the accurate determination of the moon at short intervals. His *Pañcabodha* III[1] is a practical manual for the computation of *Vyatīpāta, Grahaṇa, Chāyā, Śṛṅgonnati* and *Mauḍhya. Grahaṇagaṇita* IV commences with the characteristic introductory verse of Putumana Somayāji, *viz., madīya-hṛdayākāśe* etc. and so is likely to be his work. So also *Grahaṇāṣṭaka* II, which, though a self-contained work, forms a part of the said *Grahaṇa-gaṇita*. K. Rama Varma Raja makes mention of a tradition which attributes to our author a work called *Mānasagaṇitam.*[2] This is likely to be the Malayalam commentary on *(Laghu)mānasa* which has now been identified and whose introductory verse contains words reminiscent of Putumana Somayāji's expressions. In horoscopy, he wrote the highly popular *Jātakādeśa(mārga)*[3] which is known in two recensions, both of which might have been written by him.[4]

50. Vāsudevasvāmi : Vāsudeva I

Vāsudevasvāmi, who describes himself as a devotee of Viṣṇu, is the author of a treatise entitled *Kāladīpa*, dealing, mainly, with natural astrology. It is preserved in a unique manuscript procured from the Punnattūrkoṭṭa Mana, Koṭṭappaṭi, in Malabar. It is a comprehensive treatise dealing with a wide range of topics, which the author enumerates towards the beginning of his work.[5]

51. Śyāmalavāraṇarāja

The *Kāladīpa* of Vāsudevasvāmi has an elaborate commentary which gives the name of its author as Śyāmalavāraṇarāja. The

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1. Included in the *Jyotiśśāstrasubodhinī*. Pt. II, by Punnaaśśeri Nīlakaṇṭha Śarmā, (Triśśivaperur, M. E. 1104) and in *Pañcabodham Bhāṣā*, ed. by Kanippayyoor Sankaran Namputirippad, Kunnamkulam, M.E. 1107.

2. 'The brahmins of Kerala', *Jl. of the Royal As. Soc.*, 1910, p. 635.

3. Pub., Panchangam Press, Kunnamkulam, 5th edn., M. E. 1125; A.R.P. Press, Kunnamkulam, M.E. 1105.

4. On Putumana Somayāji, see K. V. Sarma, 'Putumana Somayāji', Proc. All-India Or. Conf., 18th Session, Annamalainagar, 1956, pp. 562-64; Intro. to *Karaṇapaddhati, op. cit.*, (Madras). pp. xxi-xxxi : Raja, 162-65 ; Ul. II. 107-9 ; Ul. Sup. I. 212-15 ; Va. I. 529-31 ; A. K. Bag ; 'Trigonometrical series in the *Karaṇapaddhati*, etc.', *Indian Jl. of Hist. of Sc.*, 1 (1966) 98-106.

5. On *Kāladīpa*, see Va. III. 392-94.

name is obviously the Sanskritisation of the author's Malayalam name. Since he describes himself as a 'rājā' and a manuscript of the work has been found only in in the palace of Punattūrkoṭṭa, it is likely that the author is a member of that household.[1]

52. Dāmodara of Bhāradvāja-gotra : Dāmodara III

He was a Nampūtiri who hailed from a village east of Tṛppaṟaññoṭ in Malabar and was the brother of Keśava, who was also his teacher. Two earlier scholars of his family were Yajña and Yajña's son Dāmodara. The *Muhūrtābharaṇa* of our author is an exhaustive text in nine chapters, devoted, respectively to : definitions, merits of neutral times, demerits of baneful times, merits of auspicious occasions, comparative potencies, auspicious times for : (a) religious functions beginning with *niṣeka* to marriage, (b) agriculture, consecration etc., (c) royal installations and (d) miscellaneous functions. A special distinction of the *Muhūrtābharaṇa* is that it provides very readable poetry from a deft-handed poet.[2]

53. Kṛṣṇa II

The *Āryabhaṭīya* has a commentary in Malayalam by an author whom its only known manuscript (India Office 6273) labels as Kṛṣṇa. He is different from Kṛṣṇadāsa, another commentator (see below), as shown by the difference in the beginning of the two commentaries.

54. Keraliya-dvija

An author who calls himself merely as Keraliya-dvija and does not give out his real name has written a good Malayalam commentary on the *Karaṇottama* of Acyuta Piṣaraṭi. No personal detail is known about this author.

55. Govinda-śiṣya

There is an anonymous author who describes himself as a devotee of Lord Śiva at Vaikkam (Skt. Vyāghra-grāma) in Central Kerala and as a pupil of Govinda. He has composed an elucidatory and highly popular commentary called *Bālabodhinī* or *Bālaprabodhinī* on the

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1. On *Kāladīpavyākhyā*, see Va. III 393.

2. On Dāmodara, see Ul. II. 105-6 ; Ul. Sup. I. 210-11 ; Va. I. 383. Va's equating our Dāmodara of the 'Bhāradvāja gotra' with Dāmodara of Vaṭaśreṇi, who belongs to the 'Bhārgava gotra' is, obviously, not correct. So also his suggestion that Keśava was the father and not the brother of Dāmodara as found in the original manuscript of the work.

Jātakapaddhati of Parameśvara of Vaṭaśreṇi. His teacher Govinda is identified, generally, as Govinda Piṣārati of the Kailāsapuram house near Vaikkom.[1]

56. Veṇāḍ-brāhmaṇa

The *Jātakodaya* is a succinct work on astrology in 103 verses, whose author does not give out his name, but styles himself as a brāhman of Veṇāḍ.[2] Veṇāḍ, as is well known, is the name of the southern part of the erstwhile State of Travancore in Kerala.

57. Azhvāñceri Tamprākkaḷ (*c.* 1725-1800)

The painstaking enterprise and zeal for propagating, among the Nampūtiris, studies on Jyotiṣa, evinced by a member of the Āzhvāñceri Tamprākkaḷ's family is apparent from his series of adapted texts and extensive commentaries thereon, covering the entire field of astronomy and astrology in eleven books. In the penultimate volume of the said works, *viz.*, *Phalasārasamuccaya-Bhāṣā*, while setting out his objectives, he makes a mention of his earlier works : "In the Malayalam area, those who are not eligible to learn Jyotiṣa study it. And, brāhmaṇas ask them about *muhūrta, praśna* etc. Since neither of these is proper, I, with a view to change this state of affairs through making the brāhmaṇas learn Jyotiśśāstra, summarised the most essential matters thereof in the *Jyotiśśāstrasaṅgraha* and, also prepared a commentary in Sanskrit-Malayalam on this *Saṅgraha*. Later, in order to enable the understanding of the working of the matters set out in the *Saṅgraha*, a *Saṅgrahasādhanakriyā* was written and also a Malayalam commentary thereon. Then I compiled the *Jātakasārasaṅgraha* and its Malayalam commentary. Then, in order to be of help in the drawing of horoscopes, I wrote the *Jātakanītimārga* and a Malayalam commentary on it. Later, with a view to enable an understanding of the results of astrological aspects, I compiled the work entitled *Phalasārasamuccaya*. Now, I am writing the present work, being its Malayalam commentary." The *Jyotiśśāstrasaṅgraha* is a résumé on the essentials of practical astronomy and astrology (text, 650 granthas ; *Bhāṣā* 1700 gr.) is in two parts : I. *Siddhāntaskandha* or *Gaṇitaskandha*, on astronomy and II. *Horāskandha*, comprising of Muhūrta, Jātaka and Praśna. The *Saṅgrahasādhanakriyā* (text, 150 gr. ; *Bhāṣā*, 600 gr.) is intended to indicate the working of the matters set out in the *Jyotiśśāstrasaṅgraha*.

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1. On this work see, Ul.III. 89-90 ; Ul. Sup. I. 464 ; Va. III. 382-83.
2. The actual expression is : *dvijena...dvi* (?) *Veṇāṭikadeśajanmanā*.

The *Jātakasārasaṅgraha* (text, 230 gr.) in six sections and the *Jatakānītimārga* (text, 200 gr. ; *Bhāṣā*, 1300 gr.) in fourteen sections are complementary texts on horoscopy. The *Phalasārasamuccaya*, the author's *magnum opus*, (text, 1300 gr.; *Bhāṣā*, 21,000 gr.), is an extensive work in three sections, devoted, respectively, to Jātaka, Praśna and Muhūrta. The Tamprākkaḷ wrote also a detailed commentary called *Tamprākkaḷ Bhāṣā* in Malayalam on *Muhūrtapadavī* II.[1]

The author does not give out his personal name anywhere in his works, but gives his family name, *viz.*, Āzhvāñceri Tamprākkaḷ.[2] Some of the manuscripts of his works are dated. The earliest of these dates, *viz.*, A.D. 1756, which is possibly, the date of composition of the work, occurs in a manuscript of *Gaṇitasārasaṅgraha* Section of *Jyotiśśāstra-saṅgraha*, Kerala C. 2183-I. This provides a clue to the author's date, which might be put as *c.* 1725-1800.

58. Vāsudeva of Vaḷḷimana : Vāsudeva II

Vāsudeva hailed from Kaṇṇamaṅgalam and wrote a metrical commentary on a hitherto unrecognised *Muhūrtapadavī* (V) beginning with the words *cakrārdham mṛtivaidhṛtam ca gulikaḥ*. Two versions of this commentary, which is called *Bhāṣāsaṅgraha*, are known, one in 178 verses and the other in 185 verses ;[3] to all appearances, Vāsudeva himself has effected this revision.[4]

59. Tuppan Nampūtiri of Iṭavaṭṭikkāṭ (*c.* 1725-1800)

Tuppan Nampūtiri was a member of the Iṭavaṭṭikkāṭ family of

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1. Pub. Keralakalpadruman Press, Trichur.

2. One of the ms. *Jyotiśśāstrasaṅgraha-Bhāṣā* (Kerala C. 2197-A) has the name of Kaypañceri Tamprākkaḷ mentioned in the post-colophonic statement, though, without being 'specifically' stated as the author.

3. *Cf.* the concluding verses of the two versions :

   i. *'dāsakṛt'* (178) *ślokasaṅkhyā ca Vāsudevena nirmitā /*
   Vaḷḷimanṟu gṛham, deśam collinṟū Kaṇṇamaṅgalam // and

   ii. *'mandāyuḥ'* (185) *ślokasaṅkhyā ca Vāsudevena nirmitā /*
   Vaḷḷimanṟu gṛham, deśam collinṟū Kaṇṇamaṅgalam //

4. Ulloor, inconsiderate of the content of this work and the colophon (as above) at the end, wrongly takes (Ul. II. 272, 473) this work as an independent treatise and ascribes it to Śaṅkara of Mahiṣamangalam on the basis of a superficial similarity of its introductory verses with those in the works of Śaṅkara.

Pāṅkode in Kunnathunad taluk, Dt. Kottayam, and was patronised by Maharaja Rāma Varmā, surnamed *Dharmarājā*, of Travancore. He was also a poet of some merit. His work on Jyotiṣa is *Muhūrtapadavī* VI, in 40½ verses.[1]

60. Nārāyaṇa of Iṭavaṭṭikkāṭ (*c.* 1728-1800) : Nārāyaṇa IV

Nārāyaṇan Nampūtiri of Iṭavaṭṭikkāṭ family was the younger brother of Tuppan Nampūtiri and was, like his brother, patronised by *Dharmarājā* of Travancore. He too was a poet of merit, both in Sanskrit and Malayalam and has left some exquisite writings. He wrote in Jyotiṣa *Muhūrtapadavī* VII, in 36 verses.[2]

61. Parameśvara II

The *Praśnaṣaṭpañcāśikā* of Pṛthuyaśas and the *Horā* of Varāhamihira have concise commentaries called *Pārameśvarī* by a Parameśvara who has to be distinguished from his namesakes. In the commentary on the *Horā*, which is also called *Horābhiprāyanirṇaya*,[3] Parameśvara often adds his own explanatory verses in elucidation of the text.

62. Parameśvara, pupil of Śaṅkara : Parameśvara III

The *Horā* of Varāhamihira has a commentary called *Jātaka-candrikā* or *Viśvārthadīpinī* by a Parameśvara who describes himself as a disciple of a Śaṅkara, but about whom nothing more is known.[4]

63. Bhāradvāja-dvija (*c.* 1750-1800)

An astronomer who calls himself merely as *Bhāradvāja* has composed two important works, *viz.*, (1) *Gaṇitayuktayaḥ* which sets out the rationale of several mathematical and astronomical procedures and (2) *Karaṇadarpaṇa*, an advanced manual for astronomical computation. The former work cites frequently a 'Bhāṣyakāra' who is, obviously, Nilakaṇṭha Somayāji, Jyeṣṭhadeva, the author of *Yuktibhāṣa*,

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1. On Tuppan Nampūtiri, see Ul. III. 470-76.
2. On Nārāyaṇa, see Ul. III. 470-76.
3. Ed. by K. Raghavan Pillai, *TSS*, Trivandrum, 1963.
4. On this author, see Va. III. 394.

and *Nyāyaratna*, most probably, the work of that name by Putumana Somayāji. The *Karaṇadarpaṇa* has been commented in Malayalam.

64. Nārāyaṇa of Perumanam: Nārāyaṇa V

Nārāyaṇan Nampūtiri of Perumanam village, near Trichur, is the author of an astronomical treatise entitled *Tantrasāra*. There is also a Malayalam commentary available on the work.[1]

65. Kṛṣṇadāsa (Koccukṛṣṇan Āśān) (1756-1812)

Koccu-kṛṣṇan Āśān was born in the family of Neṭumpayil in the Tiruvalla taluk of Kerala, as the son of an erudite astrologer named Rāman Āśān. He studied Jyotiṣa under his father as also under Śūlapāṇi Vāriyar of Kozhikode. He came of a long line of astronomers and astrologers and had several disciples who continued that tradition.[2] He was a great devotee of God Kṛṣṇa and was a poet with several works to his credit. His works on Jyotiṣa, all intended for the novice, include *Pañcabodha* VIII, in Malayalam verse, *Bhāṣājātakapaddhati*, being a free rendering-cum-commentary of the popular *Jātakapaddhati* of Parameśvara of Vaṭaśreṇi, incorporating several matters not dealt with in the original.[3] *Kaṇakkuśāstram* presenting mathematical procedures in Malayalam verse and a *Bhāṣā-Golayukti* which he mentions in his *Bhāṣājātaka-paddhati* as a work which he intended to write but which yet remains to be traced.

A hitherto unknown work of Kṛṣṇadāsa is a commentary in Malayalam prose on the *Āryabhaṭīya*. The commentary is elucidative and quotes several authorities including Bhāskara I, *Laghubhāskarīya*, Saṅgamagrāma Mādhava, Parameśvara of Vaṭaśreṇi, *Karaṇapaddhati* and a *Prakāśikā* which remains to be identified.[4] A Malayalam

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1. On this, see Raja, 153.

2. *Vide* above, pp. 5-6.

3. Ed. by K. Sankara Menon, *Bhāṣājātakapaddhati*, *Sri Vanchi Setu Lakshmi Series*, No. 3, Trivandrum, 1926.

4. The relevent quotation occurs in the com. on *Gītikā* 5 and gives the number of Kalpa days up to the beginning of Kali: *Prakāśika enna granthattil varttamānakaliyugattil ninnu munpu atītaṅṅaḷāya kalpadivasaṅṅaḷ 'śarāśviṣaṭkhā-driśarādrivedakṛteṣuyugmasvara-sammitāḥ syāt'* (7,25,44,75,70,625) *ennu paṭhikkappeṭṭiṭṭumuṇṭu.*

quotation from Parameśvara[1] poses a problem, for all known works of Parameśvara are in Sanskrit. Possibly, Parameśvara has written also in Malayalam or the passage in question is only a view of Parameśvara expressed in Malayalam.[2]

66. Śaṅkara of Muktisthala (17th cent.) : Śaṅkara IV

This author was a Nampūtiri who hailed from Mūkkola (Skt. Muktisthala) in North Malabar. He refers, in his works, to Nārāyaṇa, his teacher on Jyotiṣa, and to his patron, the Zamorin of Kozhikode. In his *Mantrasāra*, he says that the name of his house was 'Rājakula' (Mal. ?) and that he imbibed his knowledge in *mantraśāstra* from Dāmodara, son of Nīlakaṇṭha of the Bhāradvāja-gotra, who belonged to Kuṇḍapura (Mal. Tṛkkaṇṭiyūr).[3] He wrote, on natural astrology, the *Sāmudrasāra*, called also *Ārūḍhapraśna* and *Lāñchana-śāstra*, a comprehensive work which dealt with omens, palmistry, astrological query etc. He is the author also of another work, in Malayalam *maṇipravāḷa* verses, tentatively entitled *Āyuḥpraśna*, but dealing, besides *Āyus*, with *Aṣṭamaṅgala*, *trisphuṭa*, *viṣṭi* etc.[4]

67. Śaṅkara V

A Śaṅkara, about whom little personal details are available, is the author of two works, entitled *Jātakasāra* (III) and *Praśnasāra* (I).[5] Both are in Malayalam prose interspersed with Sanskrit verse and have four chapters each. The former deals with the main topics of practical astrology starting from the drawing of a horoscope, and the latter with astrological queries relating to age, marriage, children and profit and loss. He is, probably, the author also of *Praśnānuṣṭhana-paddhati*, which occurs in two versions, I and II.

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1. *Cf.* : *'iviṭe yojanakarṇam candrante madhya-yojanakarṇam ākunnu' enniññane Parameśvarācāryar paraññiṭṭumuṇṭu.*

2. On Kṛṣṇadāsa, see Ul. III. 519-27 ; Ul. Sup. I. 647-48 ; Va. III. 384-85.

3. For extracts, see Va. II. 719.

4. On Śaṅkara, see Ul. III. 86-87 ; Va. II. 718-23.

5. Ulloor does not notice this *Jātakasāra*, which is different from its namesake by Śaṅkara of 'Mahiṣamaṅgalam'. Of *Praśnasāra* II, Ulloor (II. 473) takes Mahiṣamaṅgalam Śaṅkara as the author. This is unlikely, for this work does not bear any of the characteristic *mudrās* of the Mahiṣamaṅgalam author.

68. Bhūtanāthapura-Somayāji

An anonymous Somayāji of Bhūtanāthapura, who was a disciple of Vaidyanātha has composed a comprehensive work on astrological query, entitled *Praśnasāra* (III).

69. Śaṅkara VI

Śaṅkara, about whom nothing more is known than his name, has composed a commentary in Malayalam to the *Praśnasāra* III of Bhūtanāthapura-Somayāji.

70. Śrīkumāra, son of Nīlakaṇṭha

Śrīkumāran Nampūtiri, son of Nīlakaṇṭha and pupil of Nārāyaṇa, is the author of *Praśnāmṛta*, a succinct work on astrological query. The author is good at versification and writes in a pleasing language. He mentions his village as 'Dvipakānana' (? Mal. Ānakkāṭ).

71. Nārāyaṇan Iḷayatu of Maccāṭ (1765-1843) : Nārāyaṇa VI

Nārāyaṇan Iḷayatu was a member of the Chāntampiḷḷi family in Maccāṭ near Cochin. He was a gifted poet, author of several poems in Sanskrit and Malayalam and a resourceful astrologer whose predictions have developed into legends.[1] He was patronised by the royal house of Cochin, especially by Śaktan Thampurāṇ. In Jyotiṣa, he wrote the *Jyotiṣabhāṣāvalī*, called also *Maccāṭṭu-Bhāṣā*, being an introduction to astrology, and the *Jātakādeśaratna*,[2] based on the *Jātadādeśamārga* of Putumana Somayāji.[3]

72. Parameśvara of Puradahanapura (*c.* 1775-1839) : Parameśvara IV

Parameśvara, disciple of Agnidatta, was a member of the Nampūtiri family of Purayannūr (Skt. Puradahanapura), in the Vaḷḷuvanāṭ taluk of South Malabar. He was a poet of merit and is the author of two long poems in Malayalam. His significant contribution

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1. Several of these feats are described by Puthezhathu Rama Menon in his book *Śaktan Thampurān*, (Mangalodayam, Trichur).

2. Manuscripts of this work are yet to be recovered, but the work is referred to by Puruṣottama III, the grandpupil of our author, in his *Praśnāyana*, III.19, VIII 88, XV. 41.

3. On Nārāyaṇa, see Kuṭṭamaśśeri Nārāyaṇa Pishāroṭi, 'The works of Maccāṭṭu Iḷayatu' (Mal.), *Sahitya Parishat Traimāsikam* (Ernakulam) 9 (M.E. 1119) 130-31, Ul. III 527-29; Ul. Sup. I. 618-49; Va. IV. 216-21.

to Jyotiṣa is his commentary *Varadīpikā*, composed, in A.D. 1815 (*cf.* *'naḷadhī'-saṅkhya-kolambe*, M.E. 990), on the *Muhūrtapadavī* II of Māttūr Puruṣottaman Nampūtiri. This commentary is highly elucidative, profusely documented and extremely elaborate, in 2000 granthas for a text of 35 verses. One of the important works quoted by Parameśvara is the *Prācīna-Muhūrtapadavī* (*Muhūrtapadavī* I), supposed to have been written by Govinda Bhaṭṭatiri of Talakkuḷam.[1]

73. Śrīkaṇṭha Vāriyar of Veḷḷārakkāḍ

The popular *Jātakapaddhati* of Parameśvara of Vaṭaśreṇi has a lucid commentary in Malayalam which has been made available in print in a slightly adapted form by Kanippayyoor Sankaran Namputirippad.[2] The edition contains a verse, carried over from the original manuscript, to the effect that it was written (*likhitavān*) by Śrīkaṇṭha, a 'devadāsa' of Śvetagrāmāṭavi (Mal. Veḷḷārakkāḍ). It is not definite, however, whether Śrīkaṇṭha was only the scribe or the real author.

74. Ghaṭigopa (*c.* 1800-60)

Ghaṭīgopa, which is, presumably, not his real name but only the Sanskritisation of his personal or popular name,[3] calls himself a disciple of Parameśvara and a devotee of God Padmanābha, the presiding deity of Trivandrum. His contribution to astronomy is in the form of two commentaries, one in Sanskrit and the other in Malayalam, on the *Āryabhaṭīya*. The commentary in Malayalam occurs in two

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1. On Parameśvara, see Ul. III. 476-77 ; Ul. Sup. I. 627 ; Va. II. 605-12.

2. Pub. under the title *Balapiṇḍavum āyurdāyavum*, Kunnamkulam, M.E. 1102 (A.D. 1927).

3. The South Indian village accountant is called *Maṇiyakkāran* ; this term, which can be rendered, literally, as 'clock-man' could give rise to the Sanskritisation 'Ghaṭīgopa'. To hazard another specific suggestion : Prince Goda Varma Koyittampurān (1810-60) a member of the scholarly family of Kilimanoor and a resident of Trivandrum, had the distinction of constructing, all by himself, a clock, and, thereby, getting as present a golden bracelet from the reigning ruler of Travancore (Ul. IV. 57 ; RNP V. 21). Following this achievement, he gained the popular appellation 'Maṇikkāran Thampurān' ('Prince of the clock'). This prince was also a good astronomer. Can it be that he composed the commentaries on the *Āryabhaṭīya* under the Sanskritised version of his name 'Ghaṭīgopa' ? In that case, the date arrived at on the basis of internal evidence (see below) and the author's being a devotee of God Padmanābha, would fit in well.

distinct versions, the longer one being nearly one and a half times in extent as the shorter. In the shorter version, under *Kalakriyā* 4, Ghaṭigopa gives the rationale for the 248 *candra-vākyas*. An indication of his date is given by his mention of Putumana Somayāji's (1675-1750) *Pañcabodha*, *Chāyākhaṇḍa* 20, in the said shorter version of his commentary. Ghaṭigopa might, for this reason, be palced after 1800.[1]

75. Goda Varmā, Vidvān Iḷaya Tampurān (1800-51)

Among the renowned scholars and patrons of literature produced by the scholarly royal house of Koṭungallūr, near Cochin, Goda Varmā, better known as Vidvān Iḷaya Tampurān, occupies a place in the front rank. He was a versatile scholar and has written profusely, both in Sanskrit and in Malayalam. In astronomy he has produced erudite commentaries in Sanskrit on the *Gaṇitādhyāya (Bhāskarīya-gaṇita)* and the *Golādhyāya* of the *Siddhāntaśiromaṇi* of Bhāskara II.[2]

76. Śaṅkara Varmā of Kaṭattanāṭ (1800-38)

Prince Śaṅkara Varmā, known also as Appu Tampuran, belonged to the royal house of Kaṭattanāṭ in North Malabar. He was an astute astronomer and his *Sadratnamālā*,[3] in six chapters, is a compendium of the Kerala school of mathematics and astronomy. The date of composition of the work is indicated in the concluding verse of the work by the chronogram *lokāmbe siddhisevye* (17,97,313), which falls in A. D. 1823. The author himself has written an elaborate commentary on the work, which, however, extends only up to verse 32 of the last chapter.[4]

77. Subrahmaṇya Śāstrī (1829-88)

Subrahmaṇya Śāstrī was a versatile scholar who hailed from the village of Nalleppalli in Chittur taluk (Cochin). His erudition extended to different disciplines, including Jyotiṣa, Mantraśāstra, Viṣavaidya,

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1. On Ghaṭigopa, see Ul. III. 234-35 ; Ul. Sup. II. 535.

2. On Goda Varmā, see Ul. IV.31-44 ; Ul. Sup. II. 19-23 ; Va. IV. 315-44.

3. Ed. with auto-commentary, in *Kavanodayam*, (Mal. monthly, Kozhikode), vol. 16 (1898),

4. On Śaṅkara Varmā, see Raja, 165-66 ; Sen, 191 ; Ul. III. 499-501 ; Ul. Sup. I. 635-37 ; Va. IV. 384-93.

Music and Dramaturgy, and has, to his credit, several works in Sanskrit, Malayalam and Tamil. His contribution to Jyotiṣa is his *Agaṇita* (A.D. 1851) which enunciates procedures to compute the planets for a thousand years.[1]

78. Subrahmaṇya of Kunnattu Mana (1835-1903) : Subrahmaṇya II

Subrahmaṇyan Tirumunpu was a member of Kunnattu Mana of Bhūtanāthapura (Payyanur) in North Malabar. He wrote a commentary called *Bhāvaprakāśaka* on the *Muhūrtadarśana Vidyāmādhava*. This commentary is very extensive, though it has not grown popular.[2]

79. Puruṣottaman Mūssatu (*c.* 1850-1900) : Puruṣottama III

Puruṣottaman Mūssatu of Kūttaleṭattu Mana is the author of *Praśnāyana*,[3] a comprehensive work in 1018 verses, distributed in fifteen sections called *ayana*'s, composed in A.D. 1881.[4] The author mentions, at the beginning of his work, his teachers as Bhāskara, Śaṅkara, Vahni and Nārāyaṇa and, at the close, he indicates the inspiration behind its composition :

> *Madhyāraṇyadvijāsyād vigalita iha yaḥ Praśnamārgo 'marādau'*
> *kolambe, chātraśiṣyo 'sya ca viracitavān Jātakādeśaratnam |*
> *tacchiṣyasyānumatyā Sthalipuranilayasy 'Āyanam' 'Praśna'to 'sau*
> *Śṛprā-deśādhivāsī vyaracayam aham apy 'Uttamaḥ Pūruṣāṇām' ||*

According to this statement, Madhyāraṇyadvija (Iṭakkāṭ or Panakkāṭ Nampūtiri) composed his *Praśnamārga* in the Kollam year *marādi* (825=A.D. 1650)[5] and a grand-pupil of his (identified with Nārāyaṇan Iḷayatu of Maccāṭ, 1765-1842) composed the *Jātakādeśaratna*. At the instance of a pupil of his from Sthalipura (Mal. Taḷipparampu), his

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1. On Subrahmaṇya Sastri, see Ul. IV. 292 ; Ul. Sup. II. 122-23 ; Va. IV 607; RNP IV. 481-82.

2. On this author, see Ul. Sup. I. 540 ; Va. I. 444.

3. Ed. by K. Raghavan Pillai, *TSS*, No. 223, Trivandrum, 1968.

4. *Cf.*, *Praśnāyana* X. 14, *yātā 'rājadhavāḥ' samāḥ* and com. thereon *idānīm, 'rājadhavā'* (4982) *ity akṣārasankhyā, samāḥ kalybdā, atītāḥ* / (Edn., *op. cit.*, p. 176). Kali 4982=A.D. 1881.

5. On this, see above, p. 66 § 45.

grand-pupil[1] Puruṣottama, who hailed from Śrprādeśa (Mal. Trprayāer ?) composed the *Parśnāyana*. The work has a succinct commentary included in the edition of the work, which, too, to all appearances, has been composed by the author himself.[2]

80. Rāma Varma Koyittampurān (1853-1910)

Rāma Varmā of the royal house of Grāmam was a scion of the principality of Parappanāṭ which took refuge in Travancore when Tipu Sultan of Mysore invaded Malabar in 1807. He was a versatile scholar and author of several literary works. He studied Jyotiṣa under Prince Ampurājā at the Chirakkal Palace in North Malabar and composed, in that discipline, the *Jyotiṣapradīpa*, an instructive introduction to astronomy.[3]

81. The later phase

Among later scholars who continued the tradition of Jyotiṣa and who, by their expositions and interpretations, helped to sustain and promote astronomical and astrological studies in the land, a mention might be made of four important names: (1) Rāma Vāriyar of Kaikkulangara (1833-97), author of *Sāmudrikaśāstra* and *Gauliśāstra* and commentator of *Horā* and *Praśnamārga*;[4] (2) A. R. Rajaraja Varma Koyittampurān (1853-1918) who wrote two studies entitled, respectively, *Karaṇa-pariṣkaraṇa* and *Pañcāṅgaśuddhipaddhati*[5] and *Jyotiṣaprakāśana*,[6] being

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1. Elsewhere also in this work, Puruṣottama refers to his teacher's teacher as the author of *Jātakādeśaratna*. *Cf.*, *madguror-śrīguror vaktrāj Jātakādeśam utthitam* (III. 19) and *Śrīman-madgurunāthasadgurumukhāj jātaḥ sphuṭam Jātakādeśas tena* (VIII. 88). The contraction of *Jātakādeśaratna* to *Jātakādeśa* in the above references is, obviously, due to metrical exigencies.

2. On Puruṣottama, see Ul. Sup. I. 225.

3. On Rāma Varma, see Ul. IV.544-50.

4. On Rāma Vāriyar, see the biography included in P. V. Krishna Variyar, *Maṇmaraññā sāhityakāranmār* (Mal.); Ul. IV. 267-76; Ul Sup. II. 107-16; Va. IV. 587-607.

5. Pub. Trivandrum.

6. Ms. No. 719, *Des. cata. of Skt. Mss. in the Curator's Office Library*, vol. V, Trivandrum, 1939, pp. 1489-91.

an exposition of the *Goladhyaya* of Bhaskara II ;[1] (3) Vasunni Mussatu of Vellanasseri (1855-1914), author of an erudite commentary in Malayalam on *Pañcabodha* ;[2] and (4) Punnasseri Nampi Nilakantha Sarma (1858-1935) who compiled a very instructive manual on astronomy under the title *Jyotisśastrasubodhini* with an extensive elucidation[3] and wrote a *Pañcabodhakriya-Bhasa*, besides commenting, elaborately, on the *Camatkaracintamani* and the *Prasnamarga*, both published through the Bharata Vilasam Press, Trichur.[4]

82. Modern times

It is significant that, even during modern times, compendiums on astronomy and manuals of computation in Sanskrit continue to be composed by traditional exponents of the discipline.[5] And, what is more interesting is that some of the Western advances, corrections and methods have been duly taken note of and appropriately incorporated in the Sanskrit texts and elucidated in the Malayalam expositions, in these works.[6]

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1. On Rajaraja Varma, see Ul. V.791-827 ; Ul. Sup. II. 436-49 ; Va. VI. 27-65.

2. On Vasunni Mussatu, see Ul. IV. 698-703 ; Ul. Sup. II. 373-76.

3. Pub. in 2 parts, Trissivaperur, M.E. 1097, 1104 (A.D. 1922, 1929).

4. On Nilakantha Sarma, see Ul. V. 964-70 ; Ul. Sup. II. 500 ; Va. V. 192-204.

5. See, for instance, *Ganitanirnaya* by P. S. Purushottaman Namputiri, (2nd edn., Quilon, M.E. 1127—A.D 1952) ; *Ganitaprakasika* by K.V.A. Rama Potuval, (Kannoor, 1950) and *Suddha-Drggaṇitam* by V. P. Kunhikkanna Poduval, (Payyannur, 1956). All these works are in Sanskrit and are provided with Malayalam expositions by their authors themselves.

6. See, for instance, the innovations adopted in the *Ganitanirnaya* of Purushottaman Namputiri, (*op. cit.*), such as fourteen of the Western corrections for the moon, a revision of the traditional astronomical constants in the light of the modern value for the mean motion of the Sun, sine-tables prepared on the basis of the ellipse instead of the circle and revised mnemonics for computations.

Paper 31

R. C. Gupta (1974)

Addition and Subtraction Theorems for the Sine and the Cosine in Medieval India

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R. C. Gupta

Radha Charan Gupta (born 1935) is an Indian historian of mathematics. Gupta graduated from the University of Lucknow, securing Master's degree in 1957. He earned his Ph.D. in history of mathematics from Ranchi University in 1971 under the supervision of T. A. Sarasvati Amma. Then he served as a lecturer at Lucknow Christian College (from 1957 to 1958) and in 1958 he joined Birla Institute of Technology, Mesra. In 1982 he was awarded a full professorship. He retired in 1995 as the Emeritus Professor of the history of mathematics and logic.

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Addition and Subtraction Theorems for the Sine and the Cosine in Medieval India

The paper deals with the rules of finding the sines and the cosines of the sum and difference of two angles when those of the two angles are known separately. The rules, as found in the important medieval Indian works, are equivalent to the correct modern mathematical results. Indians of the said period also knew several proofs of the formulas. These proofs are based on simple algebraic and geometrical reasoning, including proportionality of sides of similar triangles and the Ptolemy's theorem. The enunciations and derivations of the formulas presented in the paper are taken from the works of the famous authors of the period, namely, Bhāskara II (AD 1150), Nīlakaṇṭha Somayājī (1500), Jyeṣṭhadeva (sixteenth century), Munīśvara (1st half of the seventeenth century) and Kamalākara (2nd half of the seventeenth century).

1 Introduction

According to Carl B. Boyer[1], the introduction of the sine function represents the chief contribution of the *Siddhāntas* (Indian astronomical works) to the history of mathematics. The Indian Sine (usually written with a capital *S* to distinguish it from the modern sine) of any arc in a circle is defined as the length of half the chord of double the arc. Thus the (Indian) Sine of any arc is equal to $R \sin A$, where R is the radius (norm or *Sinus totus*) of the circle of reference and $\sin A$ is the modern sine of the angle, A, subtended at the centre by the arc. Likewise, the (Indian) Cosine function is equivalent to $R \cos A$ and similarly for the Versed Sine and its complement. The *Āryabhaṭīya* of Āryabhaṭa I (born 476 AD) is the earliest extant historical work of the dated type in which the Indian trigonometry is definitely used.

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The modern forms of the Addition and the Subtraction Theorems for the sine and the cosine functions are:

$$\sin(A+B) = (\sin A)\cdot(\cos B) + (\cos A)\cdot(\sin B) \tag{1}$$
$$\sin(A-B) = (\sin A)\cdot(\cos B) - (\cos A)\cdot(\sin B) \tag{2}$$
$$\cos(A+B) = (\cos A)\cdot(\cos B) - (\sin A)\cdot(\sin B) \tag{3}$$
$$\cos(A-B) = (\cos A)\cdot(\cos B) + (\sin A)\cdot(\sin B) \tag{4}$$

The present paper concerns the equivalent forms of the above four Theorems for the corresponding Indian trigonometric functions. Statements as well as derivations of these formulas, as found in important Indian works, are described in it.

2 Statement of the Theorems

Bhāskara II (AD 1150) in his *Jyotpatti*, which is given at the end of the *Golādhyāya* part of his famous astronomical work called *Siddānta-śiromaṇi,* states[2]

चापयोरिष्टयोर्दोर्ज्ये मिथः कोटिज्यकाहते ।
त्रिज्याभक्ते तयोरैक्यं स्याच्चापैक्यस्य दोर्ज्यका ॥ २१ ॥
चापान्तरस्य जीवा स्यात् तयोरन्तरसंमिता ॥ २१$\frac{1}{2}$ ॥

The Sines of the two given arcs are crossly multiplied by (their) Cosines and (the products are) divided by the radius. Their (that is, of the quotients obtained) sum is the Sine of the sum of the arcs; their difference is the Sine of the difference of the arcs.

$$R\sin(A\pm B) = \frac{(R\sin A)\cdot(R\cos B)}{R} \pm \frac{(R\cos A)\cdot(R\sin B)}{R} \tag{5}$$

Thus we get the Addition Theorem (called the *Samāsa-Bhāvanā* by Bhāskara II) and the Subtraction Theorem (called the *Antara-Bhāvanā*) for the Sine.

We have some reason (see below and also Sect. 3) to believe that Bhāskara II was aware of the corresponding Theorems for the Cosine. According to the *Marīci* commentary (=*MC*) by Munīśvara (1638) on the *Jyotpatti*, a reason for Bhāskara's omission *(upekṣā)* of the Cosine formulas was that the following alternately shorter procedure, after having obtained $R\sin(A\pm B)$, was known[3]

$$R\cos(A\pm B) = \sqrt{R^2 - \{R\sin(A\pm B)\}^2} \tag{6}$$

Kamalākara (1658) also mentions (or quotes *MC*) in his commentary on his own *Siddhānta-tattva-viveka* (=*STV*) that the *Ācārya* (Bhāskara) has not followed or given the Cosine Theorems because of the exactly same reason as stated in the *MC*.[4]

In the late Āryabhaṭa School the Addition-Subtraction Theorem for the Sine was known as the *Jīveparaspara-Nyāya* and is attributed to the famous Mādhava of

Saṅgamagrāma (*circa* 1340–1425) who is also referred as *Golavid* (Master of spherics).[5] The *Tantrasaṅgraha* (=*TS*), composed by Nīlakaṇṭha Somayājī (AD 1500), gives Mādhava's rule in Chap. 2 as[6]

> जीवे परस्परनिजेतरमौर्विकाभ्यामभ्यस्य विस्तृतिदलेन विभज्यमाने ।
> अन्योन्ययोगविरहानुगुणे भवेतां यद्वा स्वलम्बकृतिभेदपदीकृते द्वे ॥ १६ ॥
>
> The Sines (of two arcs) reciprocally multiplied by the Cosines and divided by the radius, when added to and subtracted from each other, become the Sines of the sum and difference of the arcs (respectively). Or (we get the same results when the mutual addition and subtraction is performed with) the two (positive) square-roots of the (two) differences of their own (that is, of the two Sines themselves) and *lamba* squares.[7]

So that the first part of the rule gives the formula (5), while the second part contains the alternate formula

$$R\sin(A \pm B) = \sqrt{(R\sin A)^2 - (lamba)^2} \pm \sqrt{(R\sin B)^2 - (lamba)^2} \tag{7}$$

Nīlakaṇṭha in the *Āryabhaṭīya-bhāṣya* (=*NAB*) and Śaṅkara Vāriar (AD 1556) in his commentary (=*TSC*)[8] on the above rule explains that the *lamba* involved is to be calculated from the relation

$$lamba = \frac{(R\sin A)\cdot(R\sin B)}{R} \tag{8}$$

Thus it will be noticed that the form (7) is mathematically equivalent to the formula (5).

An important point to note is that the *TSC* (pp. 22–23) makes it clear that the word *jīve* (which we have translated as Sines) can be also taken to mean Cosines. But in such a case the phrase *nijetaramaurvikās* ('the other chords') should be taken to mean, as the *TSC* points out, the corresponding Sines. In other words we can get the same Addition and Subtraction Theorems for the Sine, if we interchange "sin" and "cos" with each other in the right-hand sides of (5). Following this interpretation the form (8) can also be expressed as

$$R\sin(A \pm B) = \sqrt{(R\cos A)^2 - (lamba)^2} \pm \sqrt{(R\cos B)^2 - (lamba)^2} \tag{9}$$

where the *lamba* will now be given by

$$lamba = \frac{(R\cos A)\cdot(R\cos B)}{R} \tag{10}$$

This interpretation also leads to the same Addition and Subtraction Theorems for the Sine.

For a geometrical interpretation of the quantity *lamba*, see Sect. 4.

Almost the same Sanskrit text of Mādhava's rule is also found quoted in the *NAB* (part I, p. 58) where it is explicitly mentioned to be *'Mādhava-nirmitaṃ padyam'*,

that is a stanza composed by Mādhava. In this connection the *NAB* (part I, p. 60) also mentions the variant readings:

विस्तृतिगुणेन and विस्तृतिदलेन

For the Addition and Subtraction Theorems of the Cosine function, we may quote the *Siddhānta-sārvabhauma* (= *SSB*, 1646 AD), II, 57 which says[9]

यदंशज्ययोर्घातहीनाऽधिका च स्वकोटिज्ययोराहतिस्त्रिजकाप्ता ।
तदंशैक्यविश्लेषहीनाभ्रनन्दांशयोर्ज्ये स्तः ॥ ५७ ॥

The product of the Sines of the degrees (of two arcs) subtracted from or added to the product of their Cosines, and (the results) divided by the radius, become the Sines of the sum or difference of the degrees diminished from the ninety degrees.

That is,

$$R\sin(90^\circ - \overline{A \pm B}) = \frac{(R\cos A \cdot R\cos B \mp R\sin A \cdot R\sin B)}{R} \tag{11}$$

which are equivalent to the Theorems (3) and (4).

The *STV* (AD 1658), III, 68–69 (p. III) puts all the four Theorems side by side in the following words clearly.

मिथः कोटिज्यकानिघ्न्यौ त्रिज्याप्ते चापयोर्ज्यके ।
तयोर्योगान्तरे स्यातां चापयोगान्तरज्यके ॥ ६८ ॥

दोर्ज्ययोः कोटिमौर्व्योश्च घातौ त्रिज्योद्धृतौ तयोः ।
वियोगयोगौ जीवे स्तश्चापैक्यान्तरकोटिजे ॥ ६९ ॥

Multiply the Sines of the two arcs crossly by the Cosines and divide (separately) by the radius. Their (that is, of the two quotients obtained) sum and difference are the Sines of the sum and difference of the arcs (respectively). The products of the Sines and the Cosines are (each) divided by the radius. Their (that is, of the two quotients just obtained) difference and sum are (respectively) the Cosines of the sum and difference of the arcs.

That is,

$$R\sin(A \pm B) = \frac{(R\sin A)\cdot(R\cos B)}{R} \pm \frac{(R\sin B)\cdot(R\cos A)}{R}$$

$$R\cos(A \pm B) = \frac{(R\cos A)\cdot(R\cos B)}{R} \mp \frac{(R\sin A)\cdot(R\sin B)}{R}$$

Immediately after the statement of the above Theorems, the author, Kamalākara, in the next two verses (*STV*, III, 70–71), says

एवमानयनं चक्रे पूर्वं स्वीयशिरोमणौ ।
भावनाभ्यामतिस्पष्टं सम्यगार्योऽपि भास्करः ॥ ७० ॥
तस्य चानयनस्यार्यैः सिद्धान्तज्ञैः पुरोदिता ।
वासना बहुभिः स्वस्वबुद्धिवैचित्र्यतः स्फुटा ॥ ७१ ॥

> Such a computation, which is quite evident from the two *bhāvanās* (see the next Section), was given earlier also by the highly respected Bhāskara in his (*Siddhānta-śiromaṇi*).† And many accurate proofs of that computation have been given previously by the respected astronomers according to the manifoldness of their intelligence.

Below we outline the various derivations as found in some Indian works and which indicate the ways through which Indians understood the rationales of the Theorems.

3 Method Based on the Theory of Indeterminate Analysis

The second-degree indeterminate equation

$$Nx^2 + k = y^2 \tag{12}$$

is called *varga-prakṛti* (square-nature) in the Sanskrit works. In connection with its solution the following two Lemmas, referred to as Brahmagupta's (AD 628) Lemmas by Datta and Singh,[10] have been quite popular in Indian mathematics since the early days.

Lemma I: If $x_1,\ y_1$ is a solution of (12) and $x_2,\ y_2$ that of

$$Nx^2 + g = y^2 \tag{13}$$

then (*Samāsa-bhāvanā*)

$$x = x_1y_2 + y_1x_2, \quad y = y_1y_2 + Nx_1x_2$$

is a solution of the equation

$$Nx^2 + kg = y^2 \tag{14}$$

Lemma II: (*Antara-bhāvanā*)

$$x = x_1y_2 - y_1x_2, \quad y = y_1y_2 - Nx_1x_2$$

is also a solution of (14).

An elaborate discussion of the subject including references to Sanskrit works, translations, proofs, and terminology is available and need not to be reproduced here.[11] Dr. Shukla's paper on Jayadeva (not later than 1073) is an additional noteworthy publication in this connection.[12]

Now, as indicated by the terminology used by Bhāskara II and clearly explained by his great mathematical commentator, Munīśvara, it is evident that Bhāskara II

†Alternately, the first verse may be translated thus: 'This was very clearly computed earlier by the respected Bhāskara also through the *bhāvanās* in his *Siddhānta-śiromaṇi*'.

arrived at the truth of the Addition and Subtraction Theorems for the Sine (and Cosine) possibly by applying the above Lemmas as follows.

As explained in the *MC* (pp. 150–151) on *Jyotpatti* 21–25, if we compare the Eq. (12) with the relation

$$-(R\sin Q)^2 + R^2 = (R\cos Q)^2 \tag{15}$$

we see that we can take

$$\begin{aligned} &guṇaka(\text{multiplier}) \quad N = -1\\ &kṣepaka(\text{interpolator}) \quad k = R^2\end{aligned}$$

$$\begin{aligned} x &= R\sin Q\\ y &= R\cos Q\end{aligned}$$

Hence, on identifying

$$\begin{aligned} k &= g = R^2\\ x_1 &= R\sin A,\ y_1 = R\cos A\\ x_2 &= R\sin B,\ y_2 = R\cos B\end{aligned}$$

we at once see, from Lemma I, that

$$x = (R\sin A)\cdot(R\cos B) + (R\cos A)\cdot(R\sin B) \tag{16}$$

and

$$y = (R\cos A)\cdot(R\cos B) - (R\sin A)\cdot(R\sin B) \tag{17}$$

is a solution of the equation

$$-x^2 + R^4 = y^2$$

that is, $\left(\frac{x}{R}\right)$ and $\left(\frac{y}{R}\right)$ will be a solution of (12), since

$$-\left(\frac{x}{R}\right)^2 + R^2 = \left(\frac{y}{R}\right)^2 \tag{18}$$

Thus comparing (15) and (18), we see, from (16) and (17), that

$$\frac{(R\sin A)\cdot(R\cos B) + (R\cos A)\cdot(R\sin B)}{R}$$

and

$$\frac{(R\cos A)\cdot(R\cos B) - (R\sin A)\cdot(R\sin B)}{R}$$

will represent some sort of additive solutions for the Sine and Cosine functions, respectively. The above were taken to represent $R\sin(A+B)$ and $R\cos(A+B)$, respectively. From mathematical point of view there is a lacuna in such an identification without further justification.[13]

Similarly, by using Lemma II. the expansions of $R\sin(A-B)$ and $R\cos(A-B)$ were identified.

Such a derivation undoubtedly supports the view that Bhāskara must have been aware of the Addition and Subtraction Theorems for the Cosine, although he did not state them.

The *SSB*, II, 58–59 (pp. 144–145). whose author is same as that of *MC*, also gives the same derivation of the Theorems (also see *STVC*, pp. 112–113).

4 Geometrical Derivation as Given in the *NAB*

While explaining Mādhava's Sanskrit stanza (*Jīve-paraspara* etc.) and the implied rules, which we have already mentioned above, the *NAB* (part I, p. 59) says:

> तत्राद्यपादत्रयात्मकमेकं वाक्यम्, चरमः पादो वाक्यान्तरमिति विभागः। तत्राद्ये वाक्ये त्रैराशिकेन तदानयनं प्रदर्श्यते। अन्यस्मिन् भुजाकोटिकर्णद्वारा वर्गमूलपरिकल्पनया

The first three lines (of the stanza) form one rule (or method). The last line represents another rule. This is the break-up. We demonstrate the derivation of the first rule by (applying) the Rule of Three (that is, the proportionality of sides in the similar triangles). The other (rule) will follow from the relation between the base, upright and hypotenuse (or Sine, Cosine and radius) by extracting the square-root.

The geometrical demonstration given in the *NAB* (part I. pp. 58–61) may be substantially outlined as follows:

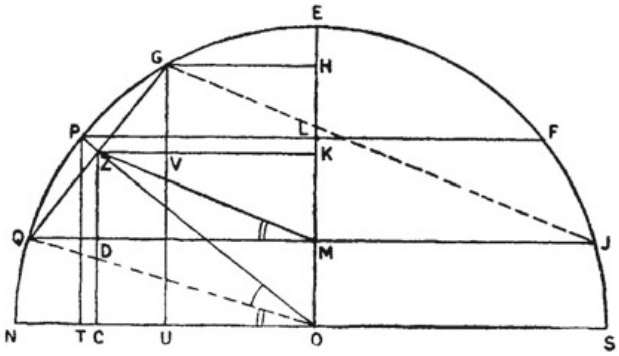

Fig. 1 Geometrical demonstration of *jīvepaspara-nyāya* as outlined by Nīlakaṇṭha

In Fig. 1 (East direction is upwards).

$$\text{arc } EP = A$$
$$\text{arc } PQ = \text{arc } PG = B \text{ (being less than } A).$$

So that

$$\text{arc } EQ = A + B$$
$$\text{arc } EG = A - B.$$

Here OP and QG intersect at Z and

$$PL = R\sin A = TO$$
$$OL = R\cos A$$
$$QZ = R\sin B = ZG$$
$$OZ = R\cos B = OP - PZ$$
$$QM = R\sin(A + B), \text{ which is required to be found out.}$$

In order to find QM, we determine its two portions QD and DM, made by the line ZC (drawn westwards from Z), separately and add them. Now to find the southern portion DM (which is equal to KZ) we have, from the similar right triangles OZK and OPL,

$$\frac{ZK}{OZ} = \frac{PL}{OP}$$

or

$$\frac{DM}{(R\cos B)} = \frac{(R\sin A)}{R}$$

giving

$$DM = \frac{(R\sin A)\cdot(R\cos B)}{R}. \tag{19}$$

Again, to find the northern portion DQ, we have, from the similar right triangles DQZ and OLP

$$\frac{DQ}{ZQ} = \frac{OL}{OP}$$

or

$$\frac{DQ}{(R\sin B)} = \frac{R\cos A}{R}$$

giving

$$DQ = \frac{(R\cos A)\cdot(R\sin B)}{R}. \tag{20}$$

By adding (19) and (20) we get QM which represents $R\sin(A+B)$. Thus is proved the Addition Theorem for the Sine.

For proving the Subtraction Theorem, drop perpendicular GU from G on ON. It divides ZK, which is equal to DM given by (19), into two portions VZ and VK. The northern portion VZ is equal to DQ given by (20) because the hypotenuse ZG is equal to the hypotenuse ZQ. Hence the southern portion

$$VK = ZK - DQ$$

or

$$GH = DM - DQ.$$

That is,

$$R\sin(A-B) = \frac{(R\sin A)\cdot(R\cos B)}{R} - \frac{(R\cos A)\cdot(R\sin B)}{R}$$

the required Subtraction Theorem.

Again, since

$$\begin{aligned}\frac{(R\sin A)\cdot(R\cos B)}{R} &= \frac{(R\sin A)\cdot\{\sqrt{R^2-(R\sin B)^2}\}}{R}\\ &= \sqrt{(R\sin A)^2 - \left\{\frac{(R\sin A)\cdot(R\sin B)}{R}\right\}^2}\\ &= \sqrt{(R\sin A)^2 - (lamba)^2}\end{aligned}$$

we can easily get the form (7) from the form (5) mathematically (see *NAB*, part I, pp. 86–87).

The *NAB* (part I, pp. 87–88) has also given some further geometrical interpretations and computations which we now indicate. In Fig. 1, $R\sin(A+B)$, that is, QM, is the base of the triangle ZQM. The second (or smaller) Sine, $R\sin B$, that is, QZ is the left side. The greater Sine, $R\sin A$, is the right side ZM (How?).

The foot of the perpendicular *(lamba)*, D, divides the base into two segments (*ābādhās*) DQ and DM which have been already found out. So that the *lamba*, given by (8), can be easily identified with the length ZD, the altitude of the triangle ZQM (this follows from $ZD^2 = ZQ^2 - DQ^2$). Then, from (see *NAB* part I, p. 88)

$$\sqrt{DM^2 + ZD^2} = ZM$$

we get, using (8) and (19),

$$ZM = R\sin A.$$

5 A Proof Based on Ptolemy's Theorem

Jyeṣṭhadeva (circa 1500–1610)[14] wrote *Yuktibhāṣā* (=*YB*) in Malayalam. Part I of the work presents an elaborate and systematic exposition of the rationale of the mathematical formulas.[15]

YB, (pp. 206–208 and 212–213) explains Mādhava's rules concerning the Addition and Subtraction Theorems for the Sine more or less on the same lines as given in the *NAB*. However, the *YB* (pp. 237–238) also indicates a proof of the Addition Theorem for the Sine by applying the so-called Ptolemy's theorem, namely:

'In a cyclic quadrilateral the sum of the products of the opposite sides is equal to the product of the diagonals'.

Of course, before indicating this use of the Ptolemy's theorem, the *YB* (pp. 228–236) has given a proof of it. According to Kaye,[16] a proof of the Ptolemy's theorem was also given by a commentator (Pṛthūdaka?, ninth century) of Brahmagupta (AD 628), the famous Indian mathematician who knew the correct expressions (which immediately yield the Ptolemy's theorem on multiplication) for the diagonals of a cyclic quadrilateral.[17]

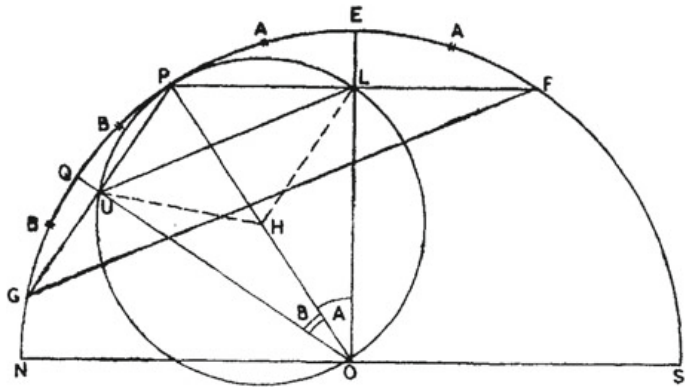

Fig. 2 Geometrical demonstration for *jīvepaspara-nyāya* as outlined in *Yuktibhāṣā*

The proof indicated in the *YB* and as explained by its editors (pp. 237–239) may be outlined as follows:
In Fig. 2

$$\text{arc } PE = A$$
$$\text{arc } QP = \text{arc } QG = B.$$

The radius OQ intersects PG in U. Thus PL and OL are the Sine and the Cosine of A and PU and OU those of B. From the cyclic quadrilateral $LPUO$, we have, by applying the rule of *bhujā-pratibhujā* etc. (that is, the Ptolemy's theorem),

$$PL.\, OU + OL \cdot\ PU = LU \cdot\ OP$$

or

$$(R\sin A)\cdot(R\cos B)+(R\cos A)\cdot(R\sin B)=LU\cdot R \tag{21}$$

The relation (21) will establish the Addition theorem for the Sine provided we are able to identify that LU represents the Sine of $(A+B)$. For seeing this, it may be noted that LU is the full chord of the arc $(LP+PU)$ in the circle which circumscribes the quadrilateral in question and whose radius is $\frac{R}{2}$ (as the centre of this smaller circle will be at H. The middle points of the radius OP equal to R). Thus

$$\begin{aligned} LU &= 2\left(\frac{R}{2}\right)\sin\left\{\frac{(2A+2B)}{2}\right\} \\ &= R\sin(A+B) \end{aligned}$$

We can also prove this by observing that LU is parallel to and half of the side FG in the triangle PFG. But FG itself is the full chord of the arc GPF in the bigger circle, so that

$$FG=2R\sin\left\{\frac{(2A+2B)}{2}\right\}.$$

6 A Geometrical Proof Quoted in the *MC* (1638)

The *MC* (pp. 154–155) contains a geometrical proof, ascribed to others (*kecid*), which is only slightly different from that found in the *NAB* (see Sect. 4). It may be outlined as follows:

Firstly, the *MC* asks us to draw a figure similar to Fig. 1 which may be referred now. In the triangle ZQM, the base QM is the desired Sine of the combined arc $(A+B)$. The smaller side QZ is $R\sin B$. The distance between Z and M, that is, the larger (lateral) side ZM is equal to $R\sin A$ evidently (*pratyakṣa-pramāṇāvagatā?*) In order to know the base QM, its two segments QD and DM should be found out.

Now the *MC* finds QD exactly in the same manner as *NAB* (see the derivation of the relation (20)). Similarly, from the similar right triangles DMZ and OZQ, we have

$$\frac{DM}{MZ}=\frac{OZ}{OQ}$$

or

$$\frac{DM}{(R\sin A)}=\frac{(R\cos B)}{R}$$

which gives the bigger segment DM and hence their sum $(QD+DM)$ proves the Addition Theorem for the Sine.

After this, the *MC* also indicates the method for proving the Subtraction Theorem for the Sine.

We note that, in proving the Addition Theorem above, the *MC* does not give any theoretical details to demonstrate that the length ZM is equal to $R \sin A$. One way of proving this could be by noting that ZM is parallel to and half of the side GJ in the triangle QGJ; and GJ is itself the full chord of the arc GEJ which is easily seen to be equal to $2A$, so that $GJ = 2R \sin A$.

Alternately, we can see that a circle, of radius $\frac{R}{2}$, drawn on OQ as the diameter will pass through the points Q, Z, M and O and ZM will be a full chord (subtending angle $2A$ at the centre) of this smaller circle. So that we have

$$ZM = 2\left(\frac{R}{2}\right)\sin A.$$

Once the flank sides of the triangle ZQM are thus identified, the perpendicular ZD could also be obtained directly by using a well-known geometrical rule equivalent to[18]

$$\text{perp.} = \frac{\text{product of flank sides}}{\text{twice the circum-radius}}$$

giving

$$\begin{aligned} ZD &= \frac{ZQ \cdot ZM}{2 \cdot \left(\frac{R}{2}\right)} \\ &= \frac{(R \sin B) \cdot (R \sin A)}{R}. \end{aligned}$$

Thus, knowing ZQ, ZM and ZD, we can easily get the segments QD and DM and hence the required length QM. This provides an alternate and independent rationale of the Addition Theorem for the Sine in the form (7).

7 Proofs Found in *STVC*

We have already mentioned the observation of *STV*, III, 71 that several proofs of these Theorems were given by the previous writers. One set of derivations as given in the *STVC* (pp. 125–129) may be briefly outlined as follows:

In Fig. 3 arcs EP and EQ are equal to A and B, respectively. Other constructions are obvious from the figure. It can be easily seen that

$$\begin{aligned} PK &= PL + MQ = R \sin A + R \sin B \\ QK &= OM - OL = R \cos B - R \cos A \end{aligned}$$

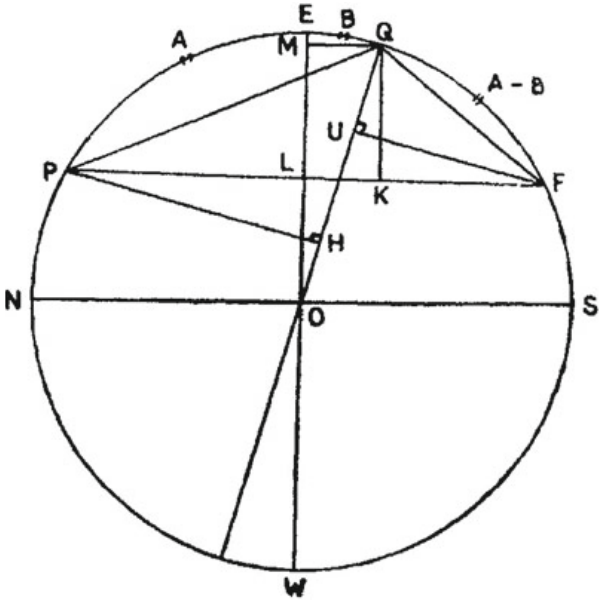

Fig. 3 Proof given in *Siddhānta-tattvaviveka* of Kamalākara

Therefore,

$$\begin{aligned} PQ^2 &= (R\sin A + R\sin B)^2 + (R\cos B - R\cos A)^2 \\ &= 2R^2 + 2R\sin A \cdot R\sin B - 2R\cos A \cdot R\cos B \end{aligned} \tag{22}$$

But PQ is the full chord of the arc $(A + B)$, so that

$$\frac{PQ}{2} = R\sin\left\{\frac{(A+B)}{2}\right\} \tag{23}$$

Now from a rule given in the *Jyotpatti*, 10 (p. 282), which the *STVC* (p. 126) quotes, we have

$$R\sin\frac{(A+B)}{2} = \sqrt{\left(\frac{R}{2}\right)\cdot R\ \text{vers}\ (A+B)} \tag{24}$$

That is,

$$\begin{aligned} R\ \text{vers}\ (A+B) &= \left(\frac{2}{R}\right)\cdot\left\{\frac{R\sin(A+B)}{2}\right\}^2 \\ &= \left(\frac{1}{2R}\right)\cdot PQ^2, \qquad \text{by (23)} \end{aligned}$$

Using (22), we easily get

$$R \text{ vers } (A+B) = R + \frac{(R\sin A \cdot R\sin B - R\cos A \cdot R\cos B)}{R}$$

from which the required expression for $R\cos(A+B)$ follows, since

$$R\cos\,(A+B) = R - R \text{ vers } (A+B).$$

Here it may be pointed out that the *STVC* (p. 126) also states that PQ, which we have found above from the triangle PQK, is also the hypotenuse for the right-angled triangle PQH (PH being perpendicular to the radius OQ). Incidentally, this gives an alternate procedure for proving the Addition Theorem for the Cosine. For, we have

$$PK^2 + QK^2 = PQ^2 = PH^2 + QH^2$$

or

$$(R\sin A + R\sin B)^2 + (R\cos B - R\cos A)^2 = \{R\sin(A+B)\}^2 + \{R - R\cos(A+B)\}^2$$

or

$$2R^2 + 2R\sin A \cdot R\sin B - 2R\cos A \cdot R\cos B = 2R^2 - 2R\cdot\ R\cos(A+B)$$

giving the required expansion of $R\cos(A+B)$.

Anyway, after getting the expression for $R\cos(A+B)$, the *STVC* (pp. 127–128) derives the corresponding expression for $R\sin(A+B)$ by using the relation

$$\{R\sin(A+B)\}^2 = R^2 - \{R\cos(A+B)\}^2.$$

Again, in the same figure the arc QF represents $(A-B)$. Also we have

$$FQ^2 = FK^2 + QK^2 = (PL - QM)^2 + (OM\ - OL)^2$$

or

$$\left\{2R\sin\frac{(A-B)}{2}\right\}^2 = (R\sin A - R\sin B)^2 + (R\cos B - R\cos A)^2 \qquad (25)$$

If we proceed as we did in the case of proving $R\cos(A+B)$ above, we easily get the desired expression for $R\cos(A-B)$. Alternately we get the same expansion by starting with the relation

$$FK^2 + QK^2 = FU^2 + QU^2$$

and proceeding as before.

Finally, the corresponding Subtraction Theorem for the Sine can be derived from that for the Cosine.

It is interesting to note that an equivalent of the identity (25) already occurs in the *jyotpatti*, 13, (p. 282). Thus Bhāskara II's familiarity with the relation (25) and that implied in (24) (where $A - B$ should be used for $A + B$) was enough to derive the Subtraction Theorems by this method (if he wanted to do so).

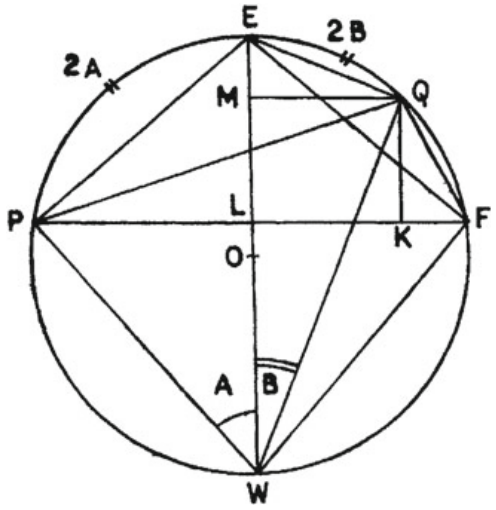

Fig. 4 Another proof given in *Siddhānta-tattvaviveka*

Another proof given in the *STVC* (pp. 130–135) may be briefly outlined as follows: In Fig. 4

$$\text{arc } EP = \text{arc } EF = 2A$$
$$\text{arc } EQ = 2B$$

It is important to note that the *STVC* says that the radius of the circle drawn is $\frac{R}{2}$ where R is the *Sinus totus*, so that the full chords EP, EQ etc., will themselves behave as the Sines. That is, we have

$$EP = 2\left(\frac{R}{2}\right)\sin A = R\sin A$$
$$EQ = R\sin B$$
$$PW = R\cos A$$
$$QW = R\cos B$$

etc., and, of course

$$EW = R$$

Now, by the methods of finding the altitude and segments of the base in a triangle, we have

$$\begin{aligned}
\text{segment } EM &= \frac{(R\sin B)^2}{R} \\
\text{segment } WM &= \frac{(R\cos B)^2}{R} \\
\text{perpendicular } QM &= \frac{(R\sin B)\cdot(R\cos B)}{R} \\
\text{segment } EL &= \frac{(R\sin A)^2}{R} \\
\text{segment } WL &= \frac{(R\cos A)^2}{R} \\
\text{perp } PL = \text{perp } LF &= \frac{(R\sin A)\cdot(R\cos A)}{R}.
\end{aligned}$$

(Of course, all these results also follow from similar right triangles in the figure.) Now we have

$$\begin{aligned}
PQ^2 &= PK^2 + QK^2 \\
&= (PL + QM)^2 + (EL - EM)^2
\end{aligned}$$

On substituting from the above expressions, simplifying, and taking the square-roots we easily get the required expression for $R\sin(A+B)$ represented by PQ.

Again we have

$$\begin{aligned}
FQ^2 &= FK^2 + KQ^2 \\
&= (PL - QM)^2 + (EL - EM)^2
\end{aligned}$$

Thus, following the same procedure, we get the required expression for $R\sin(A-B)$ represented by QF.

However, before closing this article, it may not be out of place to mention that by using the Ptolemy's theorem in Fig. 4, we get the expressions for PQ and QF almost in one step. For, Ptolemy's theorem applied to the quadrilateral $EPWQ$ yields

$$EP\cdot QW + PW\cdot EQ = PQ\cdot EW$$

which gives the desired PQ; and Ptolemy's theorem applied to the quadrilateral $EQFW$ yields

$$EQ\cdot FW + QF\cdot EW = EF\cdot QW$$

which gives the desired QF.

Acknowledgements I am grateful to Prof. M. B. Gopalakrishnan for reading the relevant passages from the *YB* for me and to Dr. T. A. Sarasvati Amma for some discussion on the subject.

References and Notes

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2. The *Siddhānta-śiromaṇi* (with the author's own commentary called *Vāsanābhāṣya* edited by Bapudeva Sastri, Chowkhamba Sanskrit Series Office, Benares (Varanasi), 1929, p. 283. (Kashi Sanskrit Series No. 72). The *Jyotpatti*, consisting of 25 Sanskrit verses, given towards the end of the above edition, may be regarded either as Chapter XIV of Or as an Appendix to the *Golādhyāya* part of the work.
3. *Golādhyāya* (with *Vāsanābhāṣya* and *Marīci*) edited by D. V. Apte. Anandarama Sanskrit Series No. 122. Part I, Poona, 1943, p. 152.
   The *MC* on *jyotpatti* is found here in Chap 5 itself, instead of at the end of the work. All page references to *MC* are as per this edition.
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12. Shukla, K.S., "Ācārya Jayadeva, the mathematician," *Gaṇita* 5, No. 1 (June 1954), pp. 1–20.
13. If we compare Eq. (12) with the identity
$$\tan^2 Q + 1 = \sec^2 Q,$$
we see that (tan A, sec A) and (tan B, sec B) are solutions of the equation
$$x^2 + 1 = y^2,$$
whose *samāsa-bhāvanā* solution. therefore, will be given by
$$x = \tan A \cdot \sec B + \sec A \cdot \tan B$$
$$y = \sec A \cdot \sec B + \tan A \cdot \tan B.$$
But here x and y do not represent tan $(A + B)$ and sec $(A + B)$.
14. Sarma, K.V., *op. cit.,* pp. 59–60.
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Paper 32

A. K. Bag (1976)

Mādhava's Sine and Cosine Series

Indian Journal of History of Science, Vol. 11 no. 1 (1976), pp. 54–57

A. K. Bag

For a biographical sketch of A. K. Bag, see p. 289.

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MĀDHAVA'S SINE AND COSINE SERIES

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Indian scholar Mādhava (1350-1410) gave a table of almost accurate values of half-sine chords for twenty-four arcs drawn at equal intervals in a quarter of a given circle. The paper discusses the basis of arriving at such accurate values and has shown that Mādhava established the following sine and cosine series before Newton (1642-1727), De Moivre (1707-38) and Euler (1748), and used these relations to compute his table. These are :

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$

The rational of the relations has also been discussed in a nutshell in the paper.

The originality of Mādhava[1] (1350-1410) in astronomy, particularly in the application of refined mathematics in it in medieval period, is now being slowly recognised. He has given values of 24 half-sine chords for twenty-four arcs in a quarter of a circle drawn at equal intervals of 225′, viz. 225′, 450′, 675′, ... 5400′. The corresponding twenty-four sine values given by him are as follows.[2]

224′ 50″ 22‴	448′ 42″ 58‴	670′ 40″ 16‴
889′ 45″ 15‴	1105′ 1″ 39‴	1315′ 34″ 7‴
1520′ 28″ 35‴	1718′ 52″ 24‴	1909′ 54″ 35‴
2092′ 46″ 3‴	2266′ 39″ 50‴	2430′ 51″ 15‴
2584′ 38″ 6‴	2727′ 20″ 52‴	2858′ 22″ 55‴
2977′ 10″ 34‴	3083′ 13″ 17‴	3176′ 3″ 50‴
3255′ 18″ 22‴	3320′ 36″ 30‴	3371′ 41″ 29‴
3408′ 20″ 11‴	3430′ 23″ 11‴	3437′ 44″ 48‴

These values are correct to more or less eight to nine places of decimals. How Mādhava arrived at such accurate values of sine table, has been discussed in the paper.

The following passage found in the *Tantra-saṃgraha*[3] (1501 A.D.) has left distinct hints that the results contained in the lines were of Mādhava. The verses run as follows :

nihatya cāpa vargeṇa cāpam tattatphalāni ca/
haret samūlayugvargaistrijyāvargahataiḥ kramāt//
cāpam phalāni cādhodhonyasyoparyupari tyajet/
jīvāptyai, saṅgraho 'syaiva vidvān-ityādinakṛtaḥ//
nihatya cāpavargena rūpam tattatphalāni' ca/
hared vimulayugvargaistrijyāvargahataiḥ kramāt//
kintu vyāsadalenaiva dvighnenādyam vibhājyatām/
phalānyadhodhaḥ kramaśo nyasyoparyupari tyajet//
śarāptyai, saṅgraho 'syaiva stenastrītyādinā kṛtaḥ/

English Translation : Multiply the arc by the square of itself (multiplication being repeated any number of times) and divide the result by the product of the square of even numbers increased by that number and square of the radius (the multiplication being repeated same number of times). The arc and the results obtained from above are placed one below the other and are subtracted systematically one from its above. These together give the *jīvā* ($r \sin\theta$) collected here as found in the expression beginning with *vidvān* etc. Multiply the unit (i.e. radius) by the square of the arc (multiplication being repeated any number of times) and divide the result by the product of square of even number decreased by that number and square of the radius (multiplication being repeated same number of times). Place the results one below the other and subtract one from its above. These together give the *śara* ($r - r\cos\theta$), collected here as found in the expression beginning with *stena*.

If t_n and t'_n be the n-th expression for *jīvā* and *śara*, then for a small arc s, and radius r,

$$t_n = \frac{s^{2n} \cdot s}{(2^2+2)(4^2+4)\ldots[(2n)^2+2n]r^{2n}} \qquad (n = 1, 2, 3, \ldots).$$

The successive terms $t_1, t_2, t_3 \ldots$ are,

$$t_1 = \frac{s^3}{3!\,r^2},\ t_2 = \frac{s^5}{5!r^4},\ t_3 = \frac{s^7}{7!r^6},\ t_4 = \frac{s^9}{9!r^8},\ \ldots.$$

Then according to the rule,

$$\begin{aligned} jīvā &= (s-t_1)+(t_2-t_3)+(t_4-t_5)+\ldots \\ &= s - \frac{s^3}{3!r^2} + \frac{s^5}{5!\,r^4} - \frac{s^7}{7!\,r^6} + \frac{s^9}{9!\,r^8} - \frac{s^{11}}{11!r^{10}} + \ldots \qquad \ldots (1) \end{aligned}$$

Again, $$t_n' = \frac{s^{2n} \cdot r}{(2^2-2)(4^2-4)\ldots[(2n^2)-2n]r^{2n}} \quad (n = 1, 2, 3, \ldots)$$

The successive terms $t_1', t_2', t_3', \ldots$ are :

$$t_1' = \frac{s^2}{2!\,r}\ ,\quad t_2' = \frac{s^4}{4!\,r^3}, \ldots\quad t_6' = \frac{s^{12}}{12!\,r^{11}} \ldots$$

As per rule, *śara* $= (r - t_1') + (t_2' - t_3') + \ldots$ ...

$$= r - \frac{s^2}{2!r} + \frac{s^4}{4!r^3} - \frac{s^6}{6!\,r^5} + \frac{s^8}{8!\,r^7} - \frac{s^{10}}{10!\,r^9} + \frac{s^{12}}{12!\,r^{11}} - \quad \ldots \quad (2)$$

when $s = r\,\theta$, the eqns (1) and (2) reduce to

$$\left.\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \ldots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots \end{aligned}\right\} \quad \ldots \quad (3)$$

Fortunately the passages beginning with *vidvān* and *stena* referred to in the above verses have been preserved in both *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha (1443-1545) and *Karaṇapaddhati*.[5] In the former it has been clearly stated that the values of the first five terms t_5, t_4, t_3, t_2, t_1 of the eqn (1) and of t_6', t_5', t_4', t_3', t_2', and t_1' of eqn (2) were given by Mādhava (*evāha mādhavaḥ*) when $s = 5400'$ and $r = 3437'\ 44''\ 48'''$. The values are : *vidvān* $(= 44''' = t_5)$, *tunna bala* $(= 33''\ 6''' = t_4)$, *kavi-śanicaya* $(= 16'\ 5''\ 41''' = t_3)$, *sarvārthaśīlasthiro* $(= 273'\ 57''\ 47''' = t_2)$, *nirvirdhāṅganarendrarung* $(= 2220'\ 39''\ 40''' = t_1)$ and *stena* $(= 6''' = t_6')$, *stripiśuna* $(= 5''\ 12''' = t_5')$, *sugandhinaganud* $(= 3'\ 9''\ 37''' = t_4')$, *bhadrāṅgabhavyāsana* $(= 71'\ 43''\ 24''' = t_3')$, *mīnāṅganarasiṃha* $(= 872'\ 3''\ 5''' = t_2')$, *unadhanakṛtbhūreva* $(= 4241'\ 9''\ 0''' = t_1')$.

These values when substituted in eqn (1) containing terms from t_1 to t_5, *jīvā* comes out to be $3437'\ 44''\ 48'''$, the 24th sine value given in the table of Mādhava (here $s = 5400'$). Similarly if s is replaced gradually by $225'$, $450'$, $675'$ ... Mādhava's sine table is obtained. Proceeding in a similar way and substituting values in eqn(2), the cosine table is obtained. This evidently shows that Mādhava, followed by the authors of *Tantrasaṃgraha* and *Karaṇapaddhati*, used the eqns (1) and (2) for the computation of the sine and cosine tables.

How Mādhava arrived at the equations (1) and (2) is not yet definitely known. The *Tantrasaṃgraha* (ch. 2, verse 12½) of Nīlakaṇṭha and *Karaṇapaddhati* (ch. 6, verse 19) have both given that for small arc, *jīvā* $= s - \frac{s^3}{3!\,r^2}$ (approximately).

The *Yuktibhāṣā*[6] has given the complete rational of the eqns (1) and (2). Its author Jyeṣṭhadeva(c. 1500-1600) in an effort to find an expression for the difference between any arc and its sine chord, divided the circumference of the quarter of a circle into

n equal divisions and considered the first and second sine differences. He then found the sum of the first n sine differences and cosine differences by considering all sine chords to be equal to corresponding arc and the small unit of the circumference to be equal to one unit, which evidently gives $jīvā = s - \frac{s^3}{3!r^2}$ and $śara = \frac{s^2 r}{2!r^2}$.

Since sine values are not actually equal to its arc length, further correction was applied *ad-infinitum* to each of the terms of the values obtained for *jīvā* and *śara*, which ultimately gives rise to the eqns (1) and (2). It would not be quite unlikely to presume that the rational was first established by Mādhava before Jyeṣṭhadeva could make use of it.

In Western mathematics Newton(1642-1727) is often given credit for the expansion of sine and cosine series No. (3). The result was established later algebraically on a solid foundation by De Moivre (1707-38) and Euler (1748)[7]. It is clear from the discussion that the Indian scholar Mādhava (1350-1410) used and possibly established the series (1), (2) and (3) of course in finite form before Newton, De Moivre and Euler, and laid the foundation of his sine table.

I express my gratitude to Sri S. N. Sen for his kind interest in the paper.

Notes and References

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[4] Trivandrum Sanskrit Series No. 101 (*Gaṇitapāda*), p. 113; *Yuktibhāṣā*, p. 145.

[5] Ch. 6, verses 14-15 (for edition vide Trivandrum Sanskrit Series, No. 126); ch. 6, verse 7 refers to the circle of radius- $\frac{21600 \times 10^{10}}{31,41,59,26,536} = 3437' 44'' 48'''$, used for the computation of sine and co-sine table.

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Paper 33

R. C. Gupta (1977)

Parameśvara's Rule for the Circum-radius of a Cyclic Quadrilateral

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R. C. Gupta

For a biographical sketch of R. C. Gupta, see p. 341.

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Parameśvara's Rule for the Circum-radius of a Cyclic Quadrilateral

The expression for the circum-radius of a cyclic quadrilateral in terms of its sides, usually attributed to L'Huilier in 1782, was known in India to Parameśvara (circa 1430). The present paper contains the original Sanskrit text of the rule, its English translation, and a discussion of its derivation as given by Sańkara Vāriar in his *Kriyākramakarī* (sixteenth century) along with relevant historical remarks.

चक्रीय चतुर्भुज के भुजमानों से उसके परिगत वृत्त की त्रिज्या के ज्ञान एवं प्रकाशन (१७८२ ई.) का श्रेय एक यूरोपीय लेखक को दिया गया है। लेकिन तत्संबंधी नियम भारत में परमेश्वर (लगभग १४३० ई.) को पहले से ही ज्ञात था। निम्न लेख में उस नियम का मूल संस्कृत पाठ, आंग्ल अनुवाद, तथा शंकर कृत क्रियाक्रमकरी (१६ वीं शताब्दी) में वर्णित उपपत्ति का विवेचन सुसंगत ऐतिहासिक टिप्पणियों के साथ दिये हैं।

Let $ABCD$ be a cyclic quadrilateral with sides AB, BC, CD, and DA; equal to a, b, c, and d, respectively. Smith [1958, II, 287] stated that S. A. J. L'Huilier discovered and published in 1782 a formula which reduces

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}} \tag{1}$$

where R is the radius of the circle circumscribing the quadrilateral and s is its semi-perimeter. The formula (1) was already known about 350 years earlier in India and is given verbally by Parameśvara (circa 1360–1455) in his commentary (before 1432) on the *Līlāvatī* (circa 1150) of Bhāskara II [Saraswathi 1969, 69; Sarma 1972, 19].

The purpose of the present paper is to bring to the notice of scholars the Sanskrit verse and the Indian derivation of the rules as found in another commentary,

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K. Ramasubramanian (ed.), *Gaṇitānanda*,
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called *Kriyākramakarī* (sixteenth century), on the *Līlāvatī* recently published [Sarma (editor) 1975].

The composition of the *Kriyākramakarī* (=*KKK*) commentary was started by Saṅkara Vāriar (c. 1500–1560) and, after his death, finished by Mahiṣamaṅgalam Nārāyaṇa (1540–1610). The rule and its rationale are found in the portion (c. 1534) which was written by Saṅkara [Sarma (ed.) 1975, xxii].

The original Sanskrit text of the rule as found in the *KKK* (p. 363), and which is almost the same as that given by Parameśvara, is

दोष्णां द्वयोर्द्वयोर्घातयुतीनां तिसृणां वधे ।
एकैकोनेतरत्र्यैक्यचतुष्केण विभाजिते ॥
लब्धमूलेन यद् वृत्तं विष्कम्भार्धेन निर्मितम् ।
सर्वं चतुर्भुजं क्षेत्रं तस्मिन्नेवावतिष्ठते ॥

This may be translated almost literally thus:

> The three sums of the products of the sides taken two at a time are to be multiplied together and divided by the tetrad formed by diminishing one (of the sides) at a time from the sum of the other three. If a circle is drawn with the square-root of the quotient (just obtained) as semi-diameter, the whole quadrilateral figure will be located therein.

That is,

$$R = \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}} \qquad (2)$$

which is equivalent to (1). After explaining the rule, the *KKK* gives (pp. 364–365) its rationale *(upapatti)*, using the following three results.

Lemma I The product of the flank sides of any triangle divided by the diameter of its circumscribed circle is equal to the altitude of the triangle.

Lemma II The area of the cyclic quadrilateral is given by

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)} \qquad (3)$$

Lemma III (cf. Ptolemy's theorem): Let $ABCD'$ be the quadrilateral formed from $ABCD$ by interchanging the sides AD and CD, that is, by taking $AD' = CD = c$ and $CD' = AD = d$. If x, y, z denote the three diagonals AC, BD, and BD', respectively, then $yz = ab + cd$, $zx = bc + da$, $xy = ca + bd$.

Lemma I was known to Indians for about a thousand years before the date of the KKK. In the equivalent form circum-radius $= \frac{\text{(product of flank sides)}}{\text{(twice the altitude)}}$, ... it is implied in a rule given by Brahmagupta (AD 628) in his *Brāhmasphuṭa-siddhānta,* XII, 27 [Sharma (ed.) 1966, III, 834; Gupta 1974b, 173]. (The *KKK* itself proves it separately (pp. 365–366). It is also used and proved in another Indian work called

Yuktibhāṣā (=*YB*) which is attributed to Jyeṣthadeva (c. 1500–1610) [Sarma 1972, 59–60; Thampuran and Aiyar 1948, 231, 243–246].

Lemma II has been very popular in India since it was first stated by Brahmagupta in his *Brāhmasphuṭa-siddhānta* (=BSS), XII, 21 [Sharma (ed.) 1966, III, 816; Gupta 1974a, 34–35]. According to Dr. K.S Shukla (a great authority on Hindu astronomy and mathematics), Brahmagupta and other early Indian Mathematicians have committed an error in declaring the formula (3) "as applicable to all quadrilaterals (with unequal altitudes), when in fact it is applicable to cyclic quadrilaterals only" [Shukla (ed.) 1959, p. 90 (translation)]. However, a recent scholar has been "unable to accept that Brahmagupta could have imagined that his rules would apply to all quadrilaterals whatsoever" [Pottage 1974, 354]. The whole difficulty arises out of the fact that Brahmagupta himself has neither explicitly specified the correct range of application of his rule (3) nor given any derivation for it. But this state of affairs was not an unusual feature of ancient Indian mathematical texts.

Gaṇeśa in his commentary (AD 1545) on the *Līlāvatī* [Apte (ed.) 1937, 156–157] attempted to prove the rule (3) but the demonstration is incorrect [Inamdar 1946, 36–42].

A detailed proof of Lemma II is found in the *YB* (pp. 247–257). When the product of the two diagonals is needed in the course of this proof, it is derived by making use of the following (the so-called Brahmagupta's expressions for diagonals of a cyclic quadrilateral):

$$x = \sqrt{\frac{(ac + bd)(ad + bc)}{(ab + cd)}} \tag{4}$$

$$y = \sqrt{\frac{(ac + bd)(ab + cd)}{(ad + bc)}} \tag{5}$$

These results are given by Brahmagupta in his *BSS*, XII, 28 [Sharma (ed.) 1966, III 836] and are considered to be the "most remarkable in Hindu Geometry and solitary in its excellence" by a recent historian of mathematics [Eves 1969, 187]. The formula (5) is stated to be rediscovered in Europe by W. Snell who gave it in his edition (1619) of Van Ceulen's work [Smith 1958, 287]. In fact the expressions (4) and (5) are separately derived in the *YB* (p. 233) from Lemma III which we now consider.

The Indian discussion of Lemma III is quite interesting because of the concept of the third diagonal of a cyclic quadrilateral. Bhāskara II had shown that the interchange of two adjacent sides of a (cyclic) quadrilateral alters the length of one of the diagonals (thereby getting a third diagonal), and this area and perimeter preserving construction appears in his *Līlāvatī* [Apte (ed.) 1937, II, 187; Colebrooke (tr.) 1967, 110; Pottage 1974, 306].

The geometry of the three diagonals of a cyclic quadrilateral is discussed in greater detail by Nārāyaṇa Paṇḍita (not to be confused with Mahiṣamaṅgalam Nārāyaṇa

mentioned above) in his *Gaṇita-kaumudī* (c. 1356). For instance, Rule 52 from the *Kśetravyavahāra* portion of the work runs as follows [Dvivedi (ed.) 1942, 59]:

द्विगुणव्यासविभक्ते त्रिकर्णघातेऽथवा गणितम् ।
त्रिभुजे चतुर्भुजे वा व्यासस्य दलं प्रजायते हृदयम् ॥ ५२ ॥

The product of the three diagonals divided by twice the diameter (of the circumscribed circle) is the area of a triangle or quadrilateral; half of the diameter becomes the *hṛdayam* (circum-radius).

That is, Area $S = \frac{xyz}{4R}$ for a cyclic quadrilateral as well as a triangle (in which case the three sides themselves will be its three diagonals).

Rule $137\frac{1}{2}$ from the same portion of the work gives the above relation in the form $R = \frac{xyz}{4}$ (area). It is interesting to note that, after stating this rule, Nārāyaṇa criticized Brahmagupta's rule for the circum-radius [*BSS*, XII. 26; Pottage 1974, 334–335] as being *avyāpaka* ('not universal') and further said that Lalla (c. 748 AD) and Śrīpati (c. 1039) blindly followed Brahmagupta in this respect [Dvivedi (ed.) 1942, 175].

The discussion of the three diagonals as found in the *KKK* is more subtle. Firstly, it shows that in a cyclic quadrilateral more than three diagonals are not possible. The arguments given are substantially as follows (p. 351):

Let $\alpha, \beta, \gamma, \delta$ be the angular measures of the arcs corresponding to the sides a, b, c, d (respectively) of a cyclic quadrilateral. Now a sum of any two arcs can be made to define a diagonal. Hence there can be six cases. But because $\alpha + \beta + \gamma + \delta = 360°$ there will be only three final possibilities (e.g. if $\alpha + \beta$ defines one diagonal, $\gamma + \delta$ will define the same diagonal). Hence only three diagonals are possible (our x, y, z will be found to correspond to $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ respectively).

The complete proof of Lemma III as given in the *KKK* (pp. 349–351) may be briefly mentioned in terms of modern symbols as follows.

Simple geometrical proofs of the following two preliminary results are given

$$ch^2\theta - ch^2\phi = ch(\theta + \phi) \cdot ch(\theta - \phi) \tag{6}$$

$$ch\lambda \cdot ch\mu = ch^2\left\{\frac{(\lambda + \mu)}{2}\right\} + ch^2\left\{\frac{(\lambda - \mu)}{2}\right\}, \tag{7}$$

where *ch* stands for the chord of the arc (for the sine function also, similar results hold good). Now we have, with reference to the accompanying figure,

$$\begin{aligned} ab + cd &= ch\,\alpha \cdot ch\,\beta + ch\,\gamma \cdot ch\,\delta \\ &= ch^2\left\{\frac{(\alpha + \beta)}{2}\right\} - ch^2\left\{\frac{(\beta - \alpha)}{2}\right\} + ch^2\left\{\frac{(\gamma + \delta)}{2}\right\} - ch^2\left\{\frac{(\gamma - \delta)}{2}\right\}. \end{aligned} \tag{8}$$

by one of the above results.

If E and W be the mid-points of the arcs ABC and ADC, respectively, then

$$ch\left\{\frac{(\alpha+\beta)}{2}\right\} = AE, \text{ and } ch\left\{\frac{(\gamma+\delta)}{2}\right\} = AW.$$

Also AEW is a right-angled triangle with hypotenuse EW equal to the diameter of the circle. Hence (8) gives

$$\begin{aligned} ab + cd &= (2R)^2 - ch^2\left\{\frac{(\beta-\alpha)}{2}\right\} - ch^2\left\{\frac{(\gamma-\delta)}{2}\right\} \\ &= ch^2\left(180^\circ - \frac{\beta-\alpha}{2}\right) - ch^2\left\{\frac{(\gamma-\delta)}{2}\right\} \\ &= ch\left(180^\circ - \frac{\beta-\alpha}{2} + \frac{\gamma-\delta}{2}\right)\cdot ch\left(180^\circ - \frac{\beta-\alpha}{2} - \frac{\gamma-\delta}{2}\right) \end{aligned}$$

by the formula (6). Because $\alpha+\beta+\gamma+\delta = 360^\circ$, we finally get

$$\begin{aligned} ab + cd &= ch\ (\alpha+\gamma)\cdot ch\ (\alpha+\delta) \\ &= (\text{chord of the arc } BAD')\cdot(\text{chord of the arc } BAD) \\ &= BD'\cdot BD = z\cdot y \end{aligned}$$

which is the first equation of Lemma III. The other equations can be derived similarly, and the proof of the Lemma III is thus completed. The proof given in the *YB* (pp. 228–233) is somewhat similar to this.

These Indian proofs of the so-called Ptolemy's theorem are radically different from that given about 1500 years earlier by Ptolemy in his *Almagest* [Taliaferro 1952, 16–17].

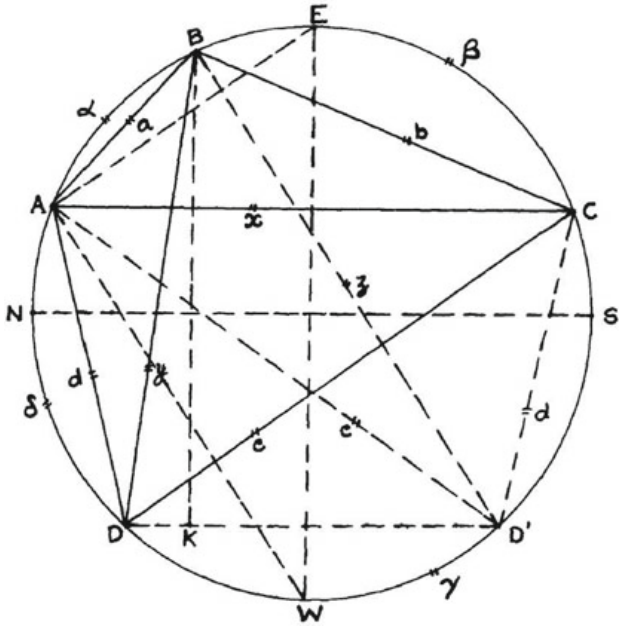
After proving Lemma III, the *KKK* (p. 351) derives the expressions for the squares of the two diagonals (x and y) from it, that is, from the equation in Lemma III. These expressions are equivalent to the famous Indian formulas (4) and (5). Finally, a similar expression for the third diagonal is also derived but "it is not given here (that is, in the original text) because of its non-utility (*anupayoga*)", the KKK says. Almost the same discussion is found in the *YB* (p. 233). These Indian derivations may be contrasted with the conjectural Brahmaguptan proofs as suggested by Pottage [1974, 344–349].

The derivation of the main result (1) may now be presented briefly.

The *KKK* starts (p. 364) by asking us to draw a diagram similar to the accompanying figure. In it EW and NS are east-west and north-south lines (east was represented upwards by Indians). BK is drawn perpendicular to DD' (which is parallel to AC). Other details are self-evident in the figure.

By Lemma I, applied to the triangle BDD', we get

$$\text{perp. } BK = \frac{yz}{2R}\ldots \tag{9}$$



Three diagonals of a cyclic quadrilateral

This perpendicular BK will be the sum of the altitudes of the two triangles BAC and DAC into which the quadrilateral $ABCD$ is divided by the diagonal AC (which becomes their common base). Thus, the area of the quadrilateral

$$s = \left(\frac{1}{2}\right) BK \cdot x \ldots \tag{10}$$

Therefore, by (9) and (10), we get

$$R = \frac{xyz}{4S} \ldots . \tag{11}$$

As stated above, this result was already known to Nārāyaṇa Paṇḍita (c. 1356).

Parameśvara's rule (1) now immediately follows from (11) by using Lemma II and Lemma III, that is, by multiplying the equations in Lemma III to get xyz as needed in (11).

Just after completing the proof, the *KKK* (p. 365) adds an intelligent remark which renders unnecessary the alternate reading (involving the work *vadha* or *ghāta,* that is, 'product') of the original Sanskrit stanza, as mentioned by the editor and found quoted elsewhere [Saraswathi 1969, 69; Sarma 1972, 19].

Acknowledgements The author is indebted to Professor M. B. Gopalakrishnan for helping him in consulting the Malayalam *YB*.

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C. T. Rajagopal and M. S. Rangachari (1978)

On an Untapped Source of Medieval Keralese Mathematics

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C. T. Rajagopal

For a biographical sketch, see p. 247.

M. S. Rangachari

M. S. Rangachari was C. T. Rajagopal's student, then his colleague and collaborator. M. S.Rangachari was appointed as the Director of the Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai in July 1996 and continued in that position till his retirement in June 1998.

Source: National and Kapodistrian University of Athens University in Zografou, Greece

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On an Untapped Source of Medieval Keralese Mathematics

C. T. RAJAGOPAL & M. S. RANGACHARI

Dedicated to the late Rāma Varma Maru Tampurān, our scholarly helpmate and ever our inspiration

Communicated by D. T. WHITESIDE

1. Introduction

It is nowadays generally accepted[1] that medieval Keralese mathematics may be credited with a number of achievements, above all the infinite-series expansions whose modern equivalents are

$$[\mathrm{A}] \quad \theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots \qquad (\tan\theta \leqq 1)$$

and

$$[\mathrm{B}_1] \quad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$[\mathrm{B}_2] \quad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \qquad (\theta \leqq \pi/2).$$

The Keralese origin of these series is independent of the discovery in a later century of the inverse-tangent expansion [A] by JAMES GREGORY (1671) and GOTTFRIED WILHELM LEIBNIZ (1673), of the sine and cosine series by ISAAC NEWTON (1670) and LEIBNIZ (1676). The Malayāḷam work *Yukti-bhāṣa*, where these series are found, belongs unquestionably to the sixteenth century, and there is not the slightest possibility of their being a subsequent interpolation in the text, drawn from a later European source and rephrased with only a veneer of local colouring.

The *Yukti-bhāṣa* also states, in the particular case $\theta = \pi/4$ of the series [A], three successively closer approximations to $\pi/4$ of form

$$[\mathrm{C}] \quad 1 - \tfrac{1}{3} + \tfrac{1}{5} - \cdots \pm \frac{1}{n} \mp f_i(n+1), \qquad i = 1, 2, 3,$$

[1] See, for instance, pp. 168–73 of A.P. JUSHKEVICH's standard history of medieval mathematics [1] and the fuller accounts by MUKUNDA MARAR & RAJAGOPAL [2] and by RAJAGOPAL & VENKATARAMAN [3].

where

$$f_1(n)=\tfrac{1}{2}n,\quad f_2(n)=\frac{\frac{1}{2}n}{n^2+1}\quad\text{and}\quad f_3(n)=\frac{(\frac{1}{2}n)^2+1}{\frac{1}{2}n(n^2+4+1)}.$$ [2]

In generalisation of our expressions of $f_2(n)$ and $f_3(n)$, as continued fractions, D.T. WHITESIDE has shown (in private correspondence with us) that the correcting function $f(n)$ which renders the equation

$$\frac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\cdots\mp\frac{1}{n}\mp f(n+1)$$

exact can be represented as the infinite continued-fraction

$$\frac{1}{2}\cdot\frac{1}{n+}\,\frac{1^2}{n+}\,\frac{2^2}{n+}\,\frac{3^2}{n+}\cdots$$

whose first three convergents are precisely the $f_i(n)$, $i=1, 2, 3$, of the approximations $[C_i]$. This, for $n=2$, would give at once

$$\frac{2}{4-\pi}=2+\frac{1^2}{2+}\,\frac{2^2}{2+}\,\frac{3^2}{2+}\cdots,$$

an infinite continued-fraction having no European parallel till WILLIAM BROUNCKER's celebrated reworking in 1654 of JOHN WALLIS' related continued-product as

$$\frac{4}{\pi}=1+\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2^+}\cdots.$$

As a preliminary to a detailed and comprehensive survey which we are making of all that is known of the Keralese series-expansions, this article has the limited aim of introducing to modern scholars a hitherto unexplored medieval Sanskrit mansuscript, the *Tantrasaṅgraha-vyākhyā*, which contains all these results save the first rounding-off approximation given by $f_1(n)$ in [C], much as in the *Yukti-bhāṣa*, unlike any other available work of its own period. The importance of this *Vyākhyā* (or 'Commentary'), of which there is a manuscript on paper in the Government Oriental Manuscripts Library at Madras,[3] can be

[2] See [2], p. 72. The expressions for $f_2(n)$ and $f_3(n)$ are given explicitly in the *Yukti-bhāṣa*, but that for $f_1(n)$ appears there only as a step in the argument leading to $f_2(n)$. For convenience, [C] in its three instances is denoted in the sequel by $[C_i]$, $i=1, 2, 3$.

[3] The authors are indebted to the Curator of the Library for allowing them access to the manuscript, catalogued R 2505, and especially to T.A.K. VENKATACHARIAR, for useful observations upon it. An edition of the *Tantrasaṅgrahā-vyākhyā*, based on four known manuscripts, is to be brought out shortly by K.V. SARMA of the Vishveshvaranand Institute of Sanskrit and Indological Studies, Panjab University. In this article we have largely relied on Professor SARMA's invaluable bibliographic research in the field of medieval Keralese mathematics, collected in [4] and scattered otherwise in Sanskrit works which he had edited, to supply the dates of the several medieval Keralese works and authors whom we cite.

fully appreciated only in its context. The *Yukti-bhāṣa*, now accepted as a work of JYEṢṬHADEVA (c. 1500–c. 1610) of Paraṅṅoṭṭu (the Malayāḷam adaptation of Sanskrit house-name Parakroḍa),[4] professes to concern itself with the mathematics of a classic work, the *Tantrasaṅgraha*, composed c. 1500, with fair certainty by the versatile NĪLAKAṆṬHA (c. 1445–c. 1545). It is therefore surprising that the series [A], [B] and [C] are not to be found in well-known versions of the *Tantrasaṅgraha* itself.[5] Indeed it may seem to be doubtful that there ever had existed any more comprehensive original version of the *Tantrasaṅgraha* including [A], [B] and [C], which had been orally transmitted in the traditional fashion by a succession of teachers to their pupils but never actually committed to writing. But such doubts are robbed of importance by the *Tantrasaṅgrahavyākhyā*, composed by an unknown student of JYEṢṬHADEVA (who is referred to as a Brahmin of Parakroḍa house), admittedly embodying the teaching of that master and very likely written down during his lifetime. This *Vyākhyā* presents not only the texts of the series [A], [B] and [C] but also that of an alternative form of [B] denoted by [B′] in the sequel, which is not found in the *Yukti-bhāṣa* but proves to be of crucial significance in determining the authorship of [B], and which scholars have been hitherto able to locate in its entirety only in an anonymous work, the *Karaṇapaddhati*, of the mid-eighteenth century.[6] The *Tantrasaṅgraha-vyākhyā*, too, is anonymous, but, by its very title and its repeated claim (in chapter endings) that its content is derived from JYEṢṬHADEVA, reveals that the younger mathematicians of the NĪLAKAṆṬHA era looked upon all these series as being germane to the *Tantrasaṅgraha*.

2. The series [A], [B] and [C] as enunciated in the *Tantrasaṅgraha-vyākhyā*

There are several noteworthy features of the series [A], [B] and [C] as they are presented in the text of the *Tantrasaṅgraha-vyākhyā*.[7] For one thing, the series are there more conveniently arranged, with [C] coming after [A] along with the variant transformations of the series for $\pi/4$ from [A] into more rapidly convergent forms which we call [C'_1], [C'_2] in the sequel, bearing in mind their genesis from [C_1], [C_2][8]; the two mutually related series [B] occur apart, preceded by their essentially alternative forms [B′] which would seem to have been more widely known among medieval Keralese mathematicians (as witness the pointed reference to them in the verses enunciating [B]). Again, the enunciations given of series [A] and [B] are fuller than in other works: the *Vyākhyā* adds the sufficient conditions for series [A] to be valid, $\tan\theta < 1$ explicitly and $\tan\theta = 1$ implicitly, which is not present in the *Karaṇapaddhati*

[4] See [4], pp. 59–60].

[5] A variorum text, based on ten manuscripts, has been published in the Trivandrum Sanskrit Series (No. 188, Trivandrum, 1958).

[6] See, for instance, [3], p. 3.

[7] Our following references to this work are keyed to the pages of the Madras manuscript referred to in footnote 3.

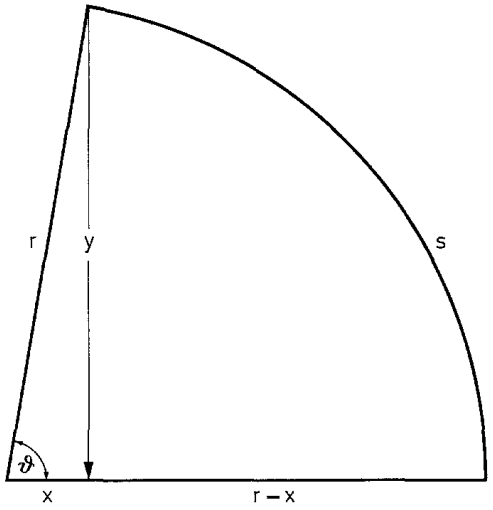
[8] As explained in [2], p. 72.

version[9]; and it explicitly states the series [B] for the sine and cosine functions which, in our previous quotation,[10] are defined only implicitly in the sequence of constructions of the individual terms of the series.

Terminology. In the following Sanskrit verses (and our translations of them into English), a *cāpam* or circular arc less than a quadrant has three elements which define it: the *bhujajyā*, the *koṭijyā* and the *śaram* or *utkramajyā* which are respectively the sine, the cosine and *sagitta* or versine of the arc, where the last named element may be replaced by the *trijyā* or the radius of the arc in an alternative definition. *Bhujajyā* and *koṭijyā* are shortened respectively to *jyā* and *koṭi* as a rule, and, as Professor T.S. KUPPANNA SASTRY has pointed out to us, to *dorjyā* (where *doḥ=bhuja* in its sense of 'arm') and *kojyā* on occasion. Sometimes *jyā* and *kojyā* are considered together as the two *jyās* of an arc, with implied recognition of their interchanged roles for the complement of the arc.

To obtain familiar modern forms of the formulae verbally expressed in the Sanskrit verses, we may suppose, as in the sketch below, that θ (in radians) is the acute angle subtended at the centre of a circle of radius r by the circular arc s and hence substitute for the arc, its sine y and its cosine x:

$$s=r\theta, \quad y=r\sin\theta, \quad x=r\cos\theta.$$



[A] (=p. 103, ll. 13–17): Series for an arc in terms of its *jyā* and *koṭi*

इष्टज्या त्रिज्ययोर्घाताल्कोट्याप्तं प्रथमं फलम्
ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम्

[9] See, for instance, [2], p. 77.

[10] See [3], p. 2. This quotation, as from a version of the *Tantrasaṅgraha*, was furnished by the late RĀMA VARMA MARU TAMPURĀN (mentioned again in footnote 14), who was one of the Princes of Cochin at the time [2] was written and referred to as such in [2]. By his generous supply of manuscript material, then practically unknown, he was, in fact, the prime mover behind [2] and [3]. A search for extant manuscripts of the *Tantrasaṅgraha* which contain some at least of the series [A], [B] and [C] would, though difficult to make, be of great value in its potential findings.

प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहुः
एकज्यादोर्ज्यसंख्याभिर्भक्तेष्वेतेष्वनुक्रमात्
ओजानां संयुते त्यक्ते युग्मयोगे धनुर्भवेत्
द्वोःकोट्योरल्पमेवेषुं कल्पनीयमिह स्मृतम्
मुहुःकृतेऽपि लब्धानां नान्यथा वस्तिर्भवेत्।

Translation. The first result(ing term) is the product of the *trijyā* and the *jyā* of the desired arc (=*iṣṭajyā*) divided by the *koṭi* of the arc. The succeeding terms are gained by a process of iteration where the first term is repeatedly multiplied by the square of the *jyā* and divided by the square of the *koṭi*. All the terms are then divided in order by the odd numbers. And the arc is obtained by adding and subtracting [respectively] the terms of odd rank and those of even rank. Of the *jyā* and the *koṭi* (which are thought of together as the two *jyās*, *doḥ* and *koṭi*, in the Sanskrit text at this point) the former is understood to be the smaller here.[11] Otherwise the terms will not be bounded [ultimately] obtained though they be by our iterative process.

[This gives, in modern notation:

$$r\arctan\frac{y}{x}=\frac{1}{1}\cdot\frac{ry}{x}-\frac{1}{3}\frac{ry^3}{x^3}+\frac{1}{5}\frac{ry^5}{x^5}-\cdots, \quad \text{where } \frac{y}{x}\leqq 1\Big]$$

[C_2] (=p. 109, l. 1↑–p. 110, l. 6): a finite-series approximation to circumference

व्यासे वारिधिनिहते रूपहृतव्यासपादगुणभिहते
त्रिशरादिविषमसंख्याभक्तमृणं पृथक्क्रमं कुर्यात्
यत्संख्ययात्र हरणे कृते निवृत्ताहृतिस्तु जामित्या
कस्या ऊर्ध्वगतायाः समसंख्यातद्दलं गुणेऽन्ते स्यात्
तद्वर्गे रूपयुतो हारो व्यासाब्धिघाततः
प्राग्वत्ताभ्यामाप्तं स्वमृणे कृते धनक्षेप एव करणीयः
लब्धः परिधिः सूक्ष्मो बहुकृत्वो हरणेऽतिसूक्ष्मः स्यात्।

[11] y (the *jyā*) and x(the *koṭi*) are conditioned by $y<x$ expressly and $y\leqq x$ by implication. For, the 'otherwise' of the next sentence is the case $y>x$ where the terms of the series for the arc form an unbounded sequence. Furthermore, the proof of the series for arc tan y/x in the *Yukti-bhāṣa*, referred to in footnote 22, is recognized to be applicable to both the cases $y<x$ and $y=x$.

Translation. Let four times the diameter be succeeded by terms, alternately subtracted and added, got by dividing this quadrupled diameter in order by odd numbers 3, 5, ... Let this process of division terminate at any particular [odd] number. Then let four times the diameter be multiplied by the next even number halved and divided by unity added to that number squared. This quotient is to be added to, or subtracted from, the product of the earlier procedure according as the last term of it was subtracted from, or was added to, the last but one term of it. The final result so obtained is the circumference reckoned more exactly than by going on with the first procedure [so as to include a few more terms].

[This is the approximation:

$$\pi d \approx 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots \mp \frac{4d}{n} \pm 4d \frac{(n+1)/2}{(n+1)^2+1}, \quad \text{where } d = 2r.]$$

[C_3] (=p. 112, ll. 5–8): An approximation better than [C_2]

एभ्यः सूक्ष्मतरोऽन्यो विलिख्यते कश्चनापि संस्कारः
अन्ते समसंख्यादलवर्गस्त्रिको गुणःस एव पुनः
युगगुणितो रूपयुतः समसंख्यादलहृतो भवेद्धारः
त्रिशरादिविषमसंख्याहरणात्परमेतदेव वा कार्यम्।

Translation. Another method more refined than the preceding is now given. [We retain] the first procedure inolving division of the diameter by odd numbers. [But] we subsequently add or subtract the quotient where there is [attached to four times the diameter] a multiplier which is unity added to the next even number halved and squared and a divisor which is unity added to four times the preceding multiplier [with this] multiplied by the even number halved.

[That is, more accurately than [C_2]:

$$\pi d \approx 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots \mp \frac{4d}{n} \pm 4d \frac{\{(n+1)/2\}^2+1}{\{(n+1)^2+4+1\}(n+1)/2},$$

where $d = 2r$.]

[C_1'] (=p. 111, ll. 5–4): The circumference as a series from [C_1]

व्यासाद्वारिधिनिहतात् पृथगाप्ते व्यासाद्युणिमूलघनैः
त्रिघ्न्यांसे स्वमृणं क्रमशः कृत्वापि परिधिरनेयः।

(*N.B.* The manuscript reading त्रिघ्न at the beginning of the second line is obviously a misprint for त्रिघ्न्य .)

Translation. The circumference is likewise obtained when four times the diameter is [successively] divided by the cubes of the odd numbers, beginning with 3, diminished by these numbers themselves, and the [respective] quotients are alternately added to, and subtracted from, the diameter multiplied by 3.

$$\left[\pi d = 3d + \frac{4d}{3^3-3} - \frac{4d}{5^3-5} + \frac{4d}{7^3-7} - \cdots,\ d = 2r.\right]$$

[C_2'] (= p. 111, ll. 11–13): Circumference as a series from [C_2]

समपञ्चाहतायोगा रूपाढ्ययुजाश्चतुर्घ्नमूलयुताः
ताभिः षोडशगुणिता द्व्यासात्पृथगाहृतेषु विषमयुते
समफलयोगे त्यक्ते स्याद्दिष्टव्यास संभवः परिधिः ।

Translation. The fifth powers of the odd numbers [1, 3, 5, ...] are increased by 4 times themselves; 16 times the diameter is successively divided by all such numbers [so gotten]; the results [of division] of odd rank are added and those of even rank are subtracted. The circumference corresponding to the diameter is [thereby] obtained.

$$\left[\pi d = \frac{16d}{1^5+4.1} - \frac{16d}{3^5+4.3} + \frac{16d}{5^5+4.5} - \cdots,\ d = 2r.\right]$$

[B_1] (= p. 127, ll. 8–11): Series for *jyā* of an arc

[B_2] (= p. 127, ll. 12–16): Series for *śaram* of an arc

निहत्य चापवर्गेण -चापं तत्तत्फलानि च
हरेत्समूलयुग्वर्गैस्त्रिज्यावर्गहृतैः क्रमात्
-चापं फलानि चाधोऽधो न्यस्योपर्युपरि त्यजेत्
जीवार्धे संग्रहोऽस्यैवं विद्वान्यादिना कृतः ।
निहत्य चापवर्गेण रूपं तत्तत्फलानि च
हरेद्विमूलयुग्वर्गैस्त्रिज्यावर्गहृतैः क्रमात्
किन्तु व्यासदलेनैव द्विघ्नेनाद्यं विभज्यताम्
फलान्यधोऽधः क्रमशः न्यस्योपर्युपरि त्यजेत्
शरार्धे संग्रहोऽस्यैवं स्तेनः स्त्रीत्यादिना कृतः ।

(*N.B.* In the third line, रूपं of the MS has been corrected to चापं.)

[B_1]: *Translation.* The arc is to be repeatedly multiplied by the square of itself and is to be divided [in order] by the square of each even number increased by itself and multiplied by the square of the radius. The arc and the terms obtained by these repeated operations[12] are to be placed in sequence in a column, and any last term is to be subtracted from the next above, the remainder from the term then next above, and so on, to obtain the *jyā* of the arc. It was this procedure which was briefly mentioned in the verse starting with '*Vidvān*'.

[There results, on *inverting* the procedure,

$$y = s - s \cdot \frac{s^2}{(2^2+2)\,r^2} + s \cdot \frac{s^2}{(2^2+2)\,r^2} \cdot \frac{s^2}{(4^2+4)\,r^2} - \cdots$$
$$= s - \frac{1}{3!}\frac{s^3}{r^2} + \frac{1}{5!}\frac{s^5}{r^4} - \cdots.]$$

[B_2]: *Translation.* The radius is to be repeatedly multiplied by the square of the arc and is to be divided [in order] by the square of each even number diminished by itself and multiplied by the square of the radius. These repeated operations yield terms of which the first involves only 2.[13] All such terms are to be placed sequentially in a column, any last term is to be subtracted from the one next above, the remainder from the one then next above, and so on, to obtain the *śaram* of the arc. Only a brief instance of this procedure was set out in the verse starting with '*Stenastri*'.

[The procedure yields, on inversion,

$$r - x = r \cdot \frac{s^2}{(2^2-2)\,r^2} - r\,\frac{s^2}{(2^2-2)\,r^2} \cdot \frac{s^2}{(4^2-4)\,r^2} + \cdots$$
$$= \frac{1}{2!}\frac{s^2}{r} - \frac{1}{4!}\frac{s^4}{r^3} + \cdots.]$$

Alternative versions of the series [B], in effect as series of powers of s/c where (as before) s is a given arc and c is the quadrant, are also stated, *viz.*:

[B′]
$$y = s - \frac{1}{3!}\left(\frac{\pi}{2}\right)^2 \frac{s^3}{c^2} + \frac{1}{5!}\left(\frac{\pi}{2}\right)^4 \frac{s^5}{c^4} - \cdots$$
$$r - x = \frac{1}{2!}\left(\frac{\pi}{2}\right)\frac{s^2}{c} - \frac{1}{4!}\left(\frac{\pi}{2}\right)^3 \frac{s^4}{c^3} + \frac{1}{6!}\left(\frac{\pi}{2}\right)^5 \frac{s^6}{c^5} - \cdots.$$

---

[12] The terms thus obtained are successively

$$s \cdot \frac{s^2}{(2^2+2)\,r^2}, \qquad s \cdot \frac{s^2}{(2^2+2)\,r^2} \cdot \frac{s^2}{(4^2+4)\,r^2},$$

and so on.

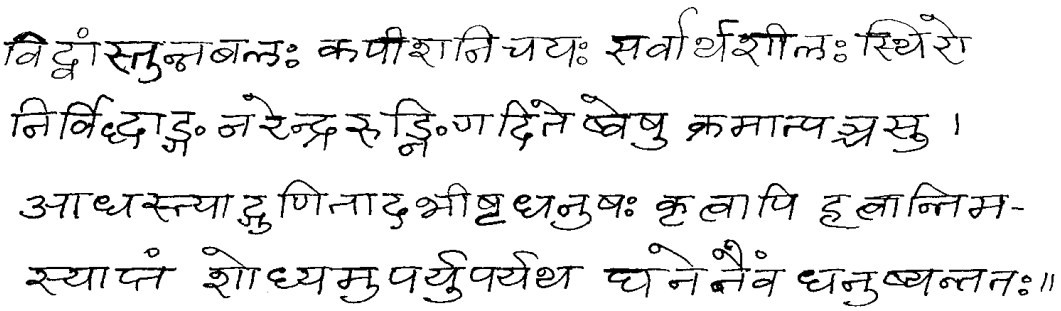
[13] After this first term $r \cdot \dfrac{s^2}{(2^2-2)\,r^2}$, we have the second one

$$r \cdot \frac{s^2}{(2^2-2)\,r^2} \cdot \frac{s^2}{(4^2-4)\,r^2}$$

followed successively by other similarly constructed terms.

In interpreting the Sanskrit verses for the series [B′], we may note that they use a *sexagesimal* angular measurement, taking a quarter circumference to be $90° = 5400'$, the corresponding radius being taken equal to $10{,}800'/\pi$ where the constant π is to be approximated by the same mode of computation used also to calculate approximately the coefficients of the powers of s/c in [B′]. RĀMA VARMA MARU TAMPURĀN has pointed out that the coefficients in these approximate numerical evaluations are assigned individual Sanskrit names (based on the *kaṭapayādi* notation)[14], *viz. vidvān* (=coefficient of s^{11}/c^{11}), *tunnabala* (=coefficient of s^9/c^9), and so on; and also *stena* (=coefficient of s^{12}/c^{12}), *stripisuna* (=coefficient of s^{10}/c^{10}), *etc.* This valuable insight allows us to view the original Sanskrit verses for [B′] divested of the apparent concreteness with which they specify the (approximate) coefficients of powers of s/c, and so take these verses in their proper generality of meaning.

[B_1'] (= p. 126, ll. 9–16): Alternative forms of [B_1]



Translation. In the sequence of five numbers *vidvān, tunnabala, ...*, which are respectively 44 [=44″′], 3306 [=33″06″′], 160541 [=16′05″41″′], 2735747 [=273′57″47″′] and 22203940 [=2220′39″40″′] in this order, the first number is multiplied by the square of the given arc [in degrees] and divided by the square of 5400, and the quotient is subtracted from the second number. The result of the subtraction is next multiplied by the square of the given arc and divided by the square of 5400 and the resulting quotient is subtracted from the third number. The process of multiplication and division followed by subtraction is repeated till all the five numbers are gone through. The final result is then multiplied by the cube of the given arc and divided by the cube of 5400. If the

[14] In his editorial notes (pp. 183–198) to the Malayālam version of the *Yukti-bhāṣa*, edited by him and A.R. AKHILESWARA IYER (Trichur: Mangalodayam Press, 1948). What is important in the present context is his identification of the individual numerical values of the coefficients of powers of s/c, given by the '*Vidvān*' and '*Stena*' verses, with the theoretically exact values in p. 96, [B′]. A discussion of the approximation of π, used in the evaluation of *vidvān*, etc., would take us beyond the purview of this article. We may notice, however, that there is a significant deficiency of 1″′ in the value of the coefficient *mīnāngonarasimha* from its accurately rounded-off are; all the other coefficients are true to the nearest third minute.

quotient then resulting is subtracted from the given arc, what remains will be the required *jyā*.[15]

[There thus ensues (in minutes, corresponding to the value 5400′ for the quadrant *c*)

$$y \approx s - \left(\frac{s}{c}\right)^3 \left[u_3 - \left(\frac{s}{c}\right)^2 \left\{u_5 - \left(\frac{s}{c}\right)^2 \left(u_7 - \left(\frac{s}{c}\right)^2 \left[u_9 - \left(\frac{s}{c}\right)^2 u_{11}\right]\right)\right\}\right],$$

where $c = 5400'$ and also

$$u_{11} = vidvān \approx \frac{1}{10 \times 11} \left(\frac{\pi}{2}\right)^2 u_9^* = u_{11}^* \quad \text{(say)},$$

$$u_9 = tunnabala \approx \frac{1}{8 \times 9} \left(\frac{\pi}{2}\right)^2 u_7^* = u_9^*,$$

$$u_7 = kapīśanicaya \approx \frac{1}{6 \times 7} \left(\frac{\pi}{2}\right)^2 u_5^* = u_7^*,$$

$$u_5 = sarvārthaśīlasthira \approx \frac{1}{4 \times 5} \left(\frac{\pi}{2}\right)^2 u_3^* = u_5^*,$$

and

$$u_3 = nirviddhāṅganarendrarugṇigadita \approx \frac{1}{2 \times 3} \left(\frac{\pi}{2}\right)^2 \times 5400' = u_3^*.]$$

[B_2'] (= p. 126, l. 17 – p. 127, 1.4): Alternative form of [B_2]

स्तेनः स्तीपिशुनः सुगन्धिनगनुद्गदाङ्गभव्यासनो
मीनाङ्गो नरसिंह अनधनकुन्दूरेव षट् स्वेषु तु ।
आद्यस्थाद्गणिनादशीघ्रधनुषः कृत्या विहृत्यानिम-
स्यार्धं शोध्यमुपर्युपर्यथ फलं स्याद्व्युत्क्रमस्थान्यजम् ॥

Translation. In the sequence of six numbers *stena*, *stripiśuna*, ..., which are respectively 6 [=6‴], 512 [=5″, 12‴], 30937 [=3′09″37‴], 714324 [=71′43″24‴], 8720305 [=872′03″05‴] and 42410900 [=4241′09″00‴] in

[15] Here notice that, since the *jyā* is given only to the nearest third minute, the further terms in its full series cannot improve upon this degree of accuracy. Incidentally, the process for determining the *jyā* in [B_1'] and the *śaram* in [B_2'] by series (each of six terms) written backwards is readily seen to be an abbreviation of the process of determining the *jyā* and the *śaram* by alternative forms of the series in [B_1] and [B_2], written backwards from any last term. This remark, in the verses for [B_1] and [B_2], carries with it the implication that the series [B_1'] and [B_2'] in the Sanskrit text may be extended to any number of terms.

this order, the first number is multiplied by the square of the given arc (in degrees) and divided by the square of 5400, and the quotient is subtracted from the second number. The remainder after the subtraction is next multiplied by the square of the given arc and divided by the square of 5400 and the quotient thus resulting is subtracted from the third number. This process of multiplication and division followed by subtraction is repeated till all six numbers are used up. The final result will be the *utkramajyā* or *śaram*[16].

[Hence there ensues

$$r-x\approx\left(\frac{s}{c}\right)^2\left(u_2-\left(\frac{s}{c}\right)^2\left[u_4-\left(\frac{s}{c}\right)^2\left\{u_6-\left(\frac{s}{c}\right)^2\left[u_8-\left(\frac{s}{c}\right)^2\left\{u_{10}-\left(\frac{s}{c}\right)^2 u_{12}\right\}\right]\right\}\right),$$

where again $c=5400'$ and also

$$u_{12}=stena\approx\frac{1}{11\times12}\left(\frac{\pi}{2}\right)^2 u_{10}^*=u_{12}^* \quad \text{(say)},$$

$$u_{10}=strīpiśuna\approx\frac{1}{9\times10}\left(\frac{\pi}{2}\right)^2 u_8^*=u_{10}^*,$$

$$u_8=sugandhinaganud\approx\frac{1}{7\times8}\left(\frac{\pi}{2}\right)^2 u_6^*=u_8^*,$$

$$u_6=bhadrāṅgabhavyāsana\approx\frac{1}{5\times6}\left(\frac{\pi}{2}\right)^2 u_4^*=u_6^*,$$

$$u_4=mīnāṅgonarasimha\approx\frac{1}{3\times4}\left(\frac{\pi}{2}\right)^2 u_2^*=u_4^*,$$

and

$$u_2=ūnadhanakṛdbhūreva\approx\frac{1}{1\times2}\left(\frac{\pi}{2}\right)^2 5400'=u_2^*=4241'\,9''\,0.296'''+.]$$

3. The author of the series [A], [B] and [C] tentatively identified

The *Tantrasaṅgraha-vyākhyā* merely enunciates all three series without proof or mention of their discoverers. Demonstrations, acceptable to modern eyes, are given in the *Yukti-bhāṣa* alone.[17]

In seeking to determine the authorship of the three series, we must be guided for the present by scattered but specific statements in unexpected sources, in default of more cogent attribution. NĪLAKAṆṬHA's *Āryabhaṭīya-bhāṣya*, a late work of his, quotes the verses for $[B_1']$ as those of a mathematician named MĀDHAVA.[18] It is natural to suppose that $[B_2']$ as also $[B_1]$ and $[B_2]$ may be

[16] Here again, since the *śaram* is given to the nearest third minute, the further terms in its expansion are negligible to this order of accuracy.

[17] These are reproduced in [2] and [3] with certain (inessential) anachronistic touches which serve to clarify their arguments for us.

[18] See p. 113 of the printed edition of the work (Trivandrum Sanskrit Series, No. 101, Trivandrum, 1930), also p. 180 where NILAKAṆṬHA refers to MĀDHAVA as a '*Ganitagṛhā-cārya*'. This Sanskrit word conveys the idea of 'Practitioner and Preceptor' (of mathe-

attributed to him, especially since, as we have seen, the verses for $[B'_1]$ and $[B'_2]$ are said to enunciate their mathematical procedures in the same manner as, but more 'briefly' than, the essentially identical procedures in the verses for $[B_1]$ and $[B_2]$.[19] In a discussion of the remaining series, K.V. SARMA[20] has attributed [A], $[C_2]$ and $[C_3]$ to MĀDHAVA, citing in support the *Kriyākramakarī*, which is said to be a commentary on BHĀSKARA II's *Līlāvatī* and of date not later than 1534. However, we would remark that this work, though it contains [A] and $[C_2]$, expressly attributing the latter to MĀDHAVA,[21] does not have $[C_3]$. It is a very plausible step from this to suppose that $[C_3]$ and *a fortiori* the infinite series for $\pi/4$ implied by $[C_2]$ are MĀDHAVA's discoveries. Equally plausible is the further conjecture that the series [A] for $\theta<\pi/4$, like that for $\theta=\pi/4$, is again due to MĀDHAVA, particularly since both series are proved in the *Yukti-bhāṣa*, without distinction being made between them, by a method essentially equivalent to the quadrature term by term of the series-expansion

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$

with respect to the base variable $x=\tan\theta\leqq 1$, the individual integrals of the powers of x being evaluated as limit-sums, essentially as in our modern Riemannian manner.[22]

The mathematician MĀDHAVA to whom we attribute all the series [A], [B] and [C] has been identified by bibliographical researchers as the one who lived

---

matics) much the same as 'Professor' in academic parlance today, but without any suggestion of professional status. However, as pointed out in [4], p. 52, it is usual in medieval Keralese literature to honour MĀDHAVA somewhat differently as a '*golavid*' (=master of global science). A justification for this may be found in his only two works (of astronomical computations) which have come down to us intact, *viz.* the *Veṇvāroha* (on the moon's motion) and the *Agaṇita* (on planetary positions). Further, the use of sexagesimals in the verses for [B'] implies an astronomical context. That the rules of approximation in the verses may indeed be a prelude to an astronomical treatise of some sort (as customarily) appears likely from the fragmentary manuscript mentioned in footnote 19, which is one of the sources of [B'].

[19] At the suggestion of Professor Y. SITARAMAN of the Mathematics Department at Kerala University, the Director of the Oriental Research Institute and Manuscripts Library there has kindly furnished a transcript of an extremely fragmentary and damaged manuscript (8358-G: *Jīvānayanam*) earlier brought to our attention by Professor K.V. SARMA and Doctor N. PARAMESWARAN UNNI of the Sanskrit Department, Kerala University. This manuscript contains a significant remark in the first person at the end of the Sanskrit verses for the series [B] and [B']: "This procedure [embodied in the verses referred to] is briefly set down by me after a fashion *(ityatra yuktissaṅkshepāt-katañcillikhitā mayā)* so that scholars may follow it in all its ups-and-downs *(unātirekata śodyā vicārya viduṣām ganaiḥ)*". The tone of this would suggest that it is an authentic aside by the originator of [B] and [B'], which reaches out to us across the centuries.

[20] See [4], footnotes on pp. 20, 24.

[21] See p. 379 of the Sanskrit text edited by K.V. SARMA (Vishveshvaranand Indological Series, No. 66, Hoshiarpur, 1975).

[22] See [2], p. 68.

about 1340–1425 at Saṅgamagrāma, the modern Irinjālakkuḍa, near Cochin. This MĀDHAVA, even if he be credited with only the discoveries of the series [A] and [B] at so unexpectedly early a date, assuredly merits a permanent place among the great mathematicians of the world. There were before him many who had made notable contributions to algebra and geometry. But it was he who, in India at least, took the decisive step onwards from the finite procedures of 'ancient' mathematics to treat their limit-passage to infinity which is the kernel of modern classical analysis.

The authors are indebted to Dr. D.T. WHITESIDE for his improvement of their original manuscript in several significant points of detail.

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2. K. MUKUNDA MARAR & C.T. RAJAGOPAL, "On the Hindu quadrature of the circle", *Journal of the Bombay Branch of the Royal Asiatic Society, N.S.*, **20** (1944). pp. 65–82.
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4. K.V. SARMA, *A History of the Kerala School of Hindu Astronomy* (Vishveshvaranand Indological Series, No. 55, Hoshiarpur, 1972).

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Added: 20 September 1977. The information in our article about the date and authorship of the *Yukti-bhāṣa*, and about the date of the anonymous *Karaṇapaddhati is* at variance with the information in papers [2], [3] cited under References, but essentially in accord with later bibliographic research as presented, for instance, in a paper by

K.V. SARMA: Jyeṣṭhadeva and his identification as the author of the *Yukti-bhāṣa*, *Adyar Library Bulletin*-vol. 22 (1958), pp. 35–40.

and in the Introduction in English to the

Karaṇapaddhati edited by S.K. NAYAR, Madras Government Oriental Series No. 98 (1956).

Further relevant information may be gathered from a paper by

K. KUNJUNNI RAJA: Astronomy and mathematics in Kerala, *Adyar Library Bulletin* – vol. 27 (1963), pp. 118–167.

A pioneer effort in the same direction, of special interest even today, is the paper by

CHARLES M. WHISH: On the Hindú Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four Śástras, the

Tantra Sangraham, Yucti Bháshá, Carana Padhati, and *Sadratnamálą, Transactions of the Royal Asiatic Society of Great Britain and Ireland* – vol. 3 (1835) pp. 509–523.

K.V. SARMA's edition of the *Tantrasaṅgraha-vyākhyā* is Part (ii) of the complete work comprising the (i) *Tantrasaṅgraha,* (ii) *Yuktidīpikā* (iii) *Laghuvivṛti* (Panjab University Indological Series – 10, 1977, Vishveshvaranand Vishva Bhandu Institute of Sanskrit and Indological Studies, Panjab University, Hoshiarpur). He makes the following positive affirmations in his Introduction. (a) The *Tantrasaṅgraha-vyākhyā* is professedly related to the *Yukti-bhāṣa* because it presents in lucid and succinct texts the mathematics expounded in the later work, this exposition being (as usually regarded) of the basic mathematics needed for the computation of planetary motions as in the *Tantrasaṅgraha* (p. xlv); and the author of the *Tantrasaṅgraha-vyākhyā* is actually referring to the *Yukti-bhāṣa,* not to the *Tantrasaṅgraha,* when he speaks of his work being concerned with the "exposition [80] well stated by the Parakroḍa brahmin" (p. xxxix). (b) There is nevertheless a less obvious but significant relation between the *Tantrasaṅgraha-vyākhyā* and the *Tantrasaṅgraha* in that the former rearranges the material presented in the *Yukti-bhāṣa* in a sequence of groups of verses, in the same order, and headed by the same key words, as verses in the latter (p. xlvi). (c) The author of the *Tantrasaṅgraha-vyākhyā,* to whom is due also the early major part of the *Kriyākramakarī* and the entire *Laghuvivṛti,* is ŚANKARA VĀRIYAR (p. xlix); and the date of the *Tantrasaṅgraha-vyākhyā* is inferentially prior to c. 1534 (p. lxi). SARMA's negative conclusion is that "it is impossible even to visualize the possibility of a larger version of the *Tantrasaṅgraha*" (p. xliv) containing the texts of significance for us. He, however, concedes that the *Tantrasaṅgraha-vyākhyā* is "a repository of the mathematical and astronomical tradition prevalent in Kerala about 1500" (p. lxxvi). We may therefore conclude finally: even if SARMA's position be fully granted, viz. that both WHISH and RĀMA VARMA mistook for the complete *Tantrasaṅgraha* that portion of the work generally named as such today, plus some suggested or actual interpolations from the *Tantrasaṅgraha-vyākhyā,* the statements made by us, about the origin, and especially the authorship, of some major Keralese achievements in mathematical analysis, neither gain nor lose in credibility.

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Paper 35

Miklos Horvath (1983)

On the Leibnizian quadrature of the circle

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Miklos Horvath

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ON THE LEIBNIZIAN QUADRATURE OF THE CIRCLE

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The purpose of this paper is to present the original – and probably less known – Leibnizian deduction of the so-called Leibniz-series $\left(\frac{\pi}{4} =\right) 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ (mostly on the basis [12], [13] and [15], [16], [17], [18]).

The efforts to calculate π[1] as exactly as possible and the history of mathematics both are almost of the same age[2]. We have chosen the Leibniz' arithmetical quadrature of the circle[3] from the manifold intellectual production accumulated in this field for centuries not only because of its characteristically individual and interesting order of ideas but also thus we can get an insight into the important process of an outstanding period (i.e. the late

[1] Leibniz uses the term "circumferencia (circuli), si diameter circuli sit 1" (= "circumference of a circle, of which the diameter is equal to unity") instead of the symbol π ([10], [13], [15], [17]). The π (in the sence just quoted) first appeared in print at the beginning of the eighteenth century ([33], p. 243.), but the wide-spread application of it is due to the influence of Euler's works ([42], p. 41.; [48], p. 484.; [49]. p. 347.).

[2] For the details see [35], [42], [44], [50]. Leibniz deduced the infinite series mentioned above in 1673, but it were the Hindu mathematicians, who first obtained it before 1550 ([45], [46], [47]). Similar results were achieved (before Leibniz, though by a methodologically different route from his) by J. Gregory (1638–1675) (in [22], which was first published in print in [34] p. 25.), Newton (1642–1727) ([30]) and Brouncker (1620–1684) ([25]), too. The latter transformed Wallis' infinite product for $\frac{4}{\pi}$ ([25], pp. 181–182) into continued fraction. Euler showed (in [39]) that Brouncker came very close to Leibniz' result because the reciprocal partial fractions of the continued fraction of Brouncker turn out to be the partial sums of the Leibniz-series (see further [36], [37], [38], [40], [41]).

[3] Leibniz remarked: "J'appelle cette Quadrature, Arithmetique, par ce qu'elle exprime exactement la granduer du Cercle, et de ses portions par un rang infini de nombres rationaux ou commensurables a une grandeur donnée." ([15], p. 338.) (See also [4], [14], [16], [17].)

seventeenth century) in the history of mathematics in the course of which the handling of the mathematical problems developed from the older geometrical methods to the (Leibnizian) analitical solution of them.

Leibniz found the quadrature of the circle "in arithmetical way" in the summer of 1673 ([1], [2])[4]. He sent the first detailed description of his result to Huygens in October 1674 ([12], [13]).

Leibniz' procedure in question can be summarized as follows.

1. Leibniz defined the auxiliary curve AEZUF (fig. 1).

Let O be the centrum of circle with radius $AO = a$. Let the tangentes of this circle at A and B meet at P. In the fig. 1. $BD \perp AD$, $LQ \| NR \| BD$, $KL \| AO$, $HG \perp AH \perp OU$, $TV \perp AV$, $VZ \perp AV$, $CZ \perp AC$ and $BS \perp AP$. Let us introduce the following notations: $x = AD$, $y = AP = BP = DE$, $z = BD$, $\Psi = AT$, $c = AV$, $b = VZ$, $v = TV$, $M =$ the moment of arc AT about the AP-axis, $dx = KL$, $dy =$ the infinitely small element of the interval AC[5], $dz = NK$, $d\Psi = NL$.

Moreover, $T_1 =$ the (shaded) area $AVZA$[6],

$T_2 =$ the area of the (shaded) circular segment ATA,
$T_3 =$ the area $AZCA$,
$T_4 =$ the area of the circular sector $AOTA$.

If we perform the construction shown in fig. 1. (i.e. the construction of the point E) in every point of the semi-circle ABH, than the obtained points define a locus, a new curve $AEZUF$ – the so-called "figura segmentorum" or "figura anonyma"[7] – with equation[8]

[4] Leibniz wrote: "Est autem magni momenti haec Circuli reductio ad Rationalitatem, qua nemo quicquam maius ad circuli dimensionem praestitit." ("It is of great importance to reduce the quadrature of the circle to defining it by rational way. This result is the most considerable progress in this field.") ([3], p. 245^v) Moreover, Leibniz ascertained: "Quadratura Circuli Arithmetica a nemine ante me data est ..." ("It was I who first found the quadrature of the circle by arithmetical way ...") ([4], p. 7^v).
In December of 1673 Leibniz met Huygens and – without giving the details – told him about this invention ([43], pp. 688–689.; [51], p. 63.; [52], p. 62.).
In his correspondence Leibniz first alluded explicitly his quadrature of circle only in summer of 1674 ([10], p. 117.; [11], p. 120.).
The circle-quadrature of Leibniz appeared in print for the first time in 1682 (see [23]).

[5] The rectangular triangle KLN with infinitely small (that is negligible with respect to finite quantities, but unequal to zero) sides was called by Leibniz "triangulum characteristicum" ("characteristic tringle") ([20], p. 217.; [24], pp. 399–400.; i.e. "triangle caracteristique", [21], p. 259.). Leibniz only introduced the symbols dx, dy and dz for the sides of the triangle in question later (see [9]). During 1673–75, under the influence of Pascal's and Wallis' works, Leibniz applied the so-called „infinitesimal unity", therefore he omitted it, when it acted as a multiplier ([4], [9]).

[6] Leibniz did not use indices.

[7] J. Gregory had already introduced this curve ([29], pp. 23–24.), but Leibniz did not noticed it ([51], p. 69.). (Leibniz bought [29], when he was at London for the first time in the spring of 1673, see [51], p. 75.) You can find the "figura segmentorum" (before Leibniz) at P. Fermat (1601–1665), too, but Leibniz could not know this fact in 1674, because the Fermat's paper (i.e. [26]) was in deposit at Carcavy (1600–1684) and it was published only in 1679 for the first time (in [32], pp. 44–57.).

[8] On the basis the equalities $\frac{y}{a} = \frac{x}{z}$ and $z^2 = x(2a - x)$.

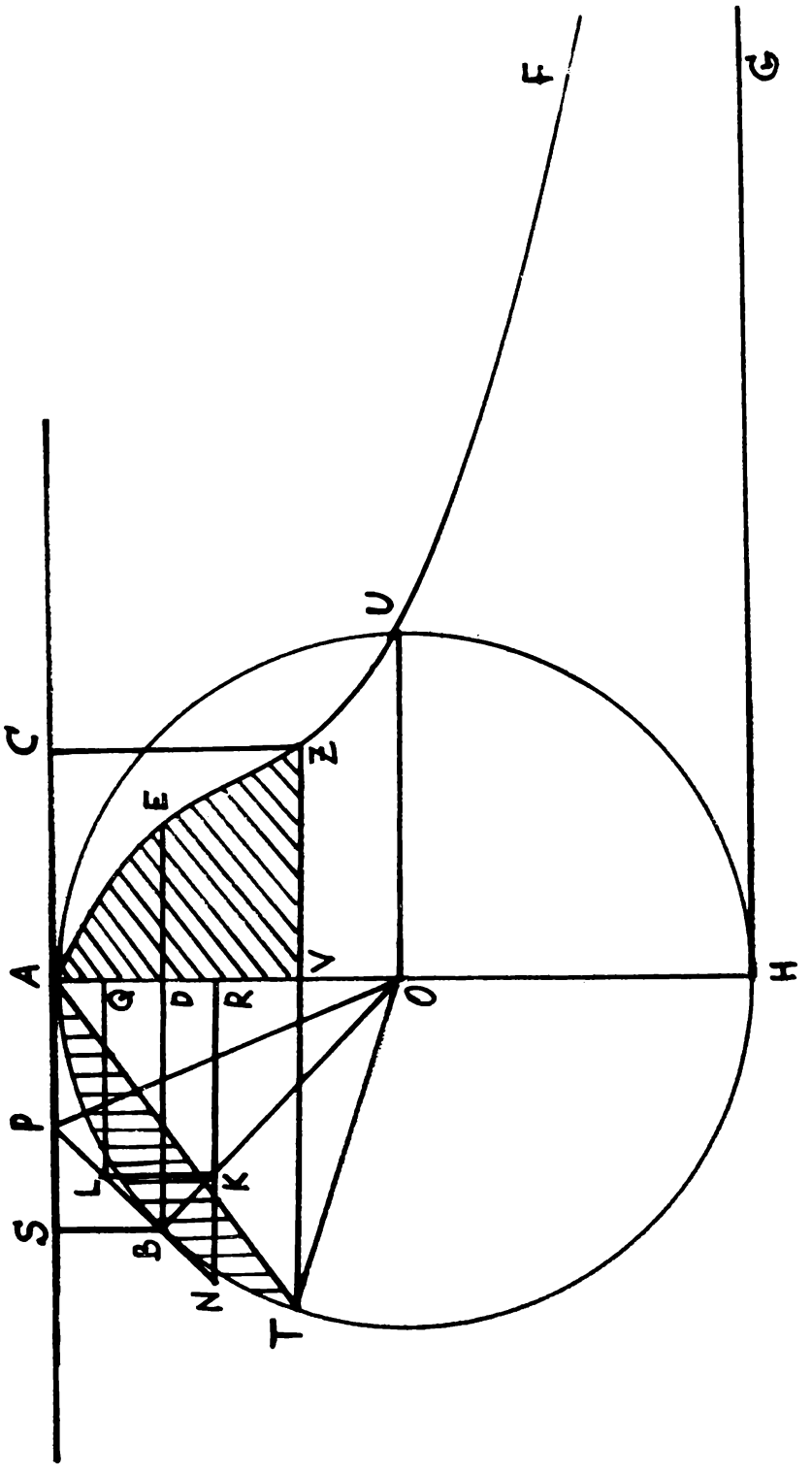

Fig. 1.

$$x = \frac{2ay^2}{a^2+y^2}. \qquad (a)$$

2. Leibniz proved that $M = T_1$.

Proof: The moment M is equal to the sum of the elementary moments $(m =) NL \cdot x$[9]. But, since

$$\frac{NL}{KL} = \frac{a}{BD},$$

therefore

$$NL \cdot x = \frac{ax}{BD} \cdot KL,$$

further considering

$$y = \frac{ax}{BD},$$

we have $M =$ summa omnium $NL \cdot x =$ summa omnium $\frac{ax}{BD} \cdot KL =$ summa omnium $y \cdot KL = T_1$.[10]

3. Proposition: $M = 2T_2$.

Proof: From the equality $\frac{NL}{NK} = \frac{a}{a-x}$ we have $(m =) NL \cdot x = a(NL - NK)$, so that

$$M = \text{summa omnium } m = \text{summa omnium } NL \cdot x =$$

$$= \text{summa omnium } a \cdot NL - \text{summa omnium } a \cdot NK = a\Psi - av = a(\Psi - v).$$[11]

This completes the proof of the proposition in question because the area T_2 is equal to $\frac{1}{2} a(\Psi - v)$.

4. From the latter two propositions it follows that $T_1 = 2T_2$.[12]

5. Leibniz calculates the arae T_3 that T_1 and (finally) T_2 are to be known.

[9] In Leibnizian term: $(M =)$ summa omnium $NL \cdot x$.

[10] In modern form: $M = \int_0^{\Psi} x \, d\Psi = \int_0^{c} y \, dx = T_1$.

[11] In modern style: $M = \int_0^{\Psi} x \, d\Psi = \int_0^{\Psi} a \, d\Psi - \int_0^{v} a \, dz = a(\Psi - v)$.

[12] This proposition is one of the most useful special cases of Leibniz' notable "transmutation theorem", see later.

T_3 = summa omnium $x\,dy$. But $x = \dfrac{2ay^2}{a^2+y^2}$ (see (a)) and

$$\frac{y^2}{a^2+y^2} = \frac{y^2}{a^2+y^2}\cdot\frac{a^2-y^2}{a^2-y^2} = \frac{(ay)^2}{a^4-y^4} - \frac{y^4}{a^4-y^4} =$$

$$= \left(\frac{y}{a}\right)^2\cdot\frac{1}{1-\left(\frac{y}{a}\right)^4} - \left(\frac{y}{a}\right)^4\cdot\frac{1}{1-\left(\frac{y}{a}\right)^4} =$$

$$= \left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)^4 + \left(\frac{y}{a}\right)^6 - \left(\frac{y}{a}\right)^8 + \ldots,$$

so that

$$T_3 = \text{summa omnium } \frac{2ay^2}{a^2+y^2}dy = 2\left(\frac{b^3}{3a} - \frac{b^5}{5a^3} - \frac{b^7}{7a^5} - \ldots\right).^{13}$$

Thus, we have $T_2 = \dfrac{1}{2}T_1 = \dfrac{1}{2}(bc - T_3)$.

6. Lastly, $T_4 = T_2 + \dfrac{av}{2} = \dfrac{bc+av}{2} - \dfrac{T_3}{2}$.

But, in our case: $bc + av = 2ab$,[14] so

$$T_4 = ab - \frac{b^3}{3a} + \frac{b^5}{5a^3} - \frac{b^7}{7a^5} + \ldots. \qquad (b)$$

7. In the special case $a = b = 1$, we get from (b) the area $\dfrac{\pi}{4}$ of the quarter circle with unity radius:

$$\left(\frac{\pi}{4} =\right) 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots.$$

* * *

It is remarkable that after 1674 Leibniz proved the equality $T_1 = 2T_2$ using no more the moments (as we have shown), but he considered it ([15], [16], [17], [18], [19] and [24]) as an immediate consequence of his so-called "transmutation theorem".

Leibniz obtained this theorem as follows:

Let an arbitrary (smooth convex) curve AT be given (fig. 2.). Now we are going to find the area of the curve-segment ATA.

[13] Leibniz performed the long-division and the integration term by term on the basis of Mercator's and Wallis' works ([27], [25]).

[14] It follows from the formulae of the remark (8), if $x = c$, $y = b$, $z = v$.

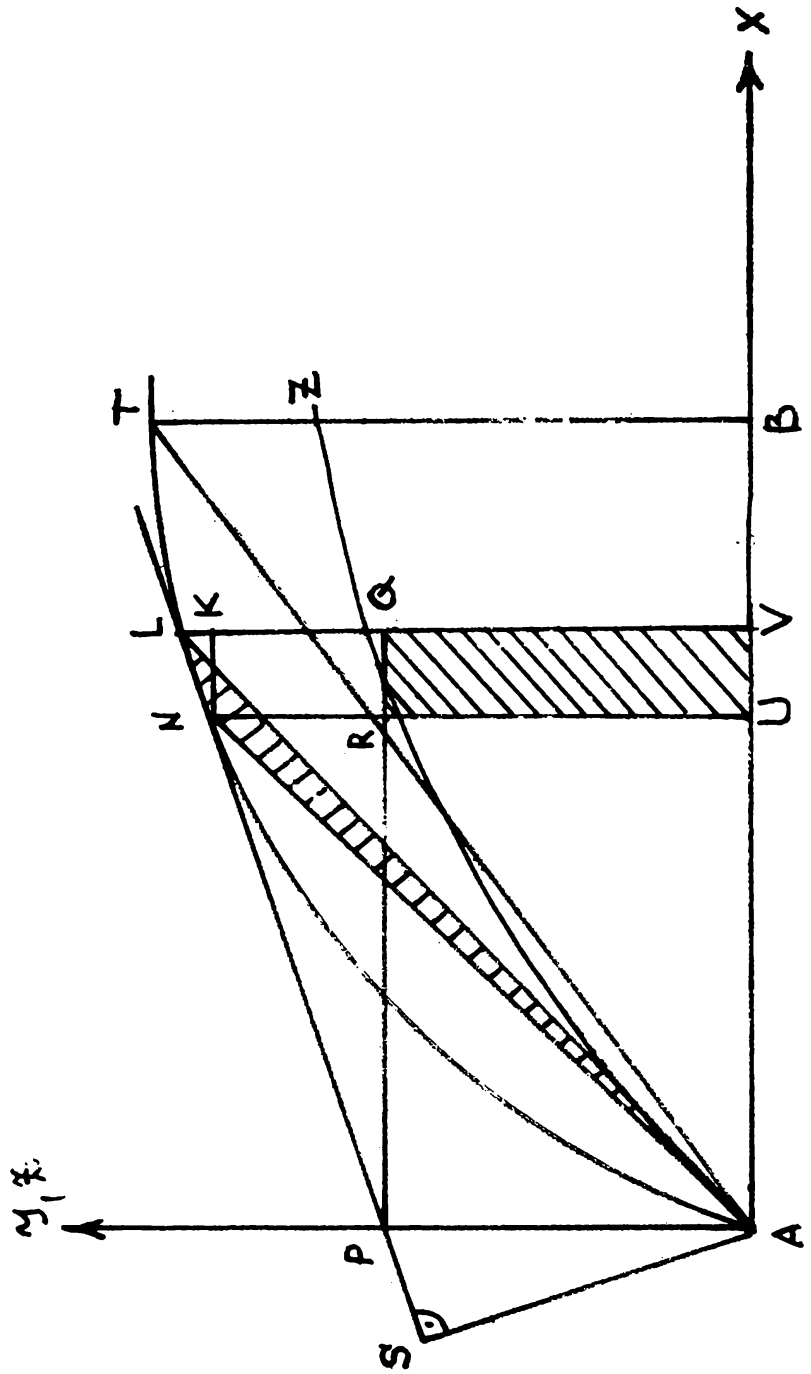

Fig. 2.

For this purpose let us divide the given arc AT into infinitely small parts (whose number is infinite). Let us join the points of this partition to the origin A, at the same time through them draw a set of parallels to the z-axis to their intersection with the x-axis.

Let now N, L be two neighbouring points on the arc AT. Let the straight line NL (that is the one of the tangents[15] of the arc AT) meet the z-axis at P. Let the parallel to the x-axis through P meet NU in R, LV in Q.

From our construction, it follows that the area of the (shaded) rectangle $RQUV$ is equal to the double-area of the (shaded) triangle ANL.[16]

If we perform the construction just mentioned in every tangent of the arc AT, then the corresponding points R (or – because of the partition of the arc AT into infinitely small parts – points Q) define a new curve AZ. One half of the area under the arc AZ of this curve is equal to the area of the segment ATA. This statement is the famous transmutation theorem (of Leibniz).[17]

The above equality $2T_2 = T_1$ is one of the special cases of this theorem.

The transmutation theorem was the first truly independent achievement of Leibniz.[18] Leibniz was stimulated by the efficiency of this theorem to solve further quadrature-problems. These investigations of his considerably contributed to invent convenient symbols for infinitesimals ([7], [9]).

At the same time, the experience gained by him in calculations with moments led him to discover the fundamental, general rules of the (Leibnizian) infinitesimal calculus ([5], [6], [7], [8]).

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[15] This tangent-concept of Leibniz see e.g. [4], pp. 1^r, 2^v.

[16] Because of the similarity of the triangles NLK, APS.

[17] Leibniz discovered this theorem is 1673 ([1], [2]) inspired by the works of Pascal and Desargues (1593–1662) (see [15], p. 342.; [17], p. 358.).

[18] But, in a slightly different way, it had occured in the works of Gregory and Barrow (1630–1677) ([28], [31]), too, though Leibniz studied the details (in question) of their works only later. (See [51], pp. 69, 76.)

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Paper 36

Bhibhutibhusan Datta and Avadhesh Narayan Singh (1984)

Use of calculus in Hindu mathematics

Revised by K. S. Shukla. Indian Journal of History of Science, Vol. 19, No. 2, 1984, pp. 95–104.

Bhibhutibhusan Datta

For a biographical sketch of Bhibhutibhusan Datta see p. 179.

Avadhesh Narayan Singh

For a biographical sketch of Avadhesh Narayan Singh see p. 215.

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Use of calculus in Hindu mathematics *

1 Differential calculus

1.1 A controversy

Attention was first drawn to the occurrence of the differential formula

$$\partial(\sin\theta) = \cos\theta\,\partial\theta$$

in Bhāskara II's (1150) *Siddhāntaśiromaṇi* by Pandit Bapu Deva Sastri[1] in 1858. The Pandit published a summarised translation of the passages which involve the use of the above formula. His summary was defective in so far as it did not bring into prominence the idea of the infinitesimal increment which underlies Bhāskara's analysis. Without making clear to his readers, the full significance of Bhāskara's result, the Pandit made the mistake of asserting—what was plain to him—that Bhāskara was fully acquainted with the principles of the differential calculus.

The Pandit was adversely criticised by Spotiswoode,[2] who without consulting the original on which the Pandit based his conclusions, remarked (1) that Bapu Deva Sastri had overstated his case in saying that Bhāskarācārya was fully acquainted with the principles of the differential calculus, (2) that there was no allusion to the most essential feature of the differential calculus, viz. the infinitesimal magnitudes of the intervals of time and space therein employed, and (3) that the approximative character of the result was not realised.

Since the above controversy took place no serious investigation of the subject seems to have been made by any scholar.[3] In order that the reader may be better able to judge the merit of the Hindu claim to the invention of the differential calculus, it is desirable that the problems which required the use of the above differential formula be stated first.

* Bhibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 19, No. 2 (1984), pp. 95–104.

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1.2 Problems in astronomy

The calculation of eclipses is one of the most important problems of astronomy. In ancient days this problem was probably more important than it is now, because the exact time and duration of the eclipses could not be foretold on account of lack of the necessary mathematical equipment on the part of the astronomer. In India, the Hindus observed fast and performed various other religious rites on the occasion of eclipses. Thus their calculation was a matter of national importance. It afforded the Hindu astronomer a means of demonstrating the accuracy of his science and his own ability to the public who patronised him. The problem of the calculation of conjunction of planets and occultation of stars was equally important both from scientific as well as religious view points.

In problems of the above nature it is essential to determine the true instantaneous motion of a planet or star at any particular instant. This instantaneous motion was called by the Hindu astronomers *tātkālika-gati*. The formula giving the *tātkālika-gati* (instantaneous motion) is given by Āryabhaṭa and Brahmagupta in the following form:

$$u' - u = v' - v \pm e(\sin w' - \sin w) \tag{1}$$

where u, v, w are the true longitude, mean longitude, mean anomaly respectively at any particular time and u', v', w' the values of the respective quantities at a subsequent instant; and e is the eccentricity or the sine of the greatest equation of the orbit. The *tātkālika-gati* is the difference $u' - u$ between the true longitudes at the two positions under consideration. Āryabhaṭa and Brahmagupta used the sine table to find the value of $(\sin w' - \sin w)$. The sine table used by them was tabulated at intervals of $3°45'$ and thus was entirely unsuited for the purpose. To get the values of sines of angles not occurring in the table, recourse was taken to interpolation formulae, which were incorrect because the law of variation of the difference was not known.

1.3 A differential formula

Mañjula (932) was the first Hindu astronomer to state that the difference of the sines,

$$\sin w' - \sin w = (w' - w)\cos w,$$

where $(w' - w)$ is small.

He says:

> True motion in minutes[4] is equal to the cosine (of the mean anomaly) multiplied by the difference (of the mean anomalies) and

[4]This clearly shows that the formula is intended for use when difference is small, the result being expressible in minutes.

> divided by the *cheda*,[5] added or subtracted contrarily (to the mean motion).[6]

Thus according to Mañjula formula (1) becomes

$$u' - u = v' - v \pm e(w' - w)\cos w, \tag{2}$$

which, in the language of the differential calculus, may be written as

$$\partial u = \partial v \pm e\cos\theta\,\partial\theta.$$

We cannot say exactly what was the method employed by Mañjula to obtain formula (2). The formula occurs also in the works of Āryabhaṭa II (950),[7] Bhāskara II (1150),[8] and later writers. Bhāskara II indicates the method of obtaining the differential of sine θ, His method is probably the same as that employed by his predecessors.

1.4 Proof of the differential formula

Let a point P (See Figure 1) move on a circle. Let its position at two successive intervals be denoted by P and Q. Now, if P and Q are taken very near each other, the direction of motion in the interval PQ is the same as that of the tangent at P. Let PT be measured along the tangent at P equal to the arc PQ. Then PT would be the motion of the point P if its velocity at P had not changed direction.

Discussing the motion of planets, Bhāskarācārya says:

> The difference between the longitudes of a planet found at any time on a certain day and at the same time on the following day is called its (*sphuṭa*)*gati* (true rate of motion) for that interval of time.
>
> This is indeed rough motion (*sthūlagati*). I now describe the fine (*sūkṣma*) instantaneous (*tātkālika*) motion.[9] The *tātkālika-gati* (instantaneous motion) of a planet is the motion which it would have, had its velocity during any given interval of time remained uniform.

During the course of the above statement, Bhāskara II observes that the *tātkālika-gati* is *sūkṣma* ("fine" as opposed to rough), and for that the interval

[5] Here *cheda* (divisor) $= \frac{1}{e}$. According to Hindu astronomers $\frac{1}{e} = \frac{360}{P}$, where P is the periphery of the epicycle.
[6] *Laghumānasa*, ii. 7.
[7] *MSi* (=*Mahā-siddhānta*), iii. 15f.
[8] *SiŚi* (=*Siddhāntaśiromaṇi*), *Gaṇitādhyāya*, *Spaṣṭādhikāra*, 36–7.
[9] *SiŚi* (=*Siddhāntaśiromaṇi*), *Gaṇitādhyāya*, *Spaṣṭādhikāra*, 36 (c–d).

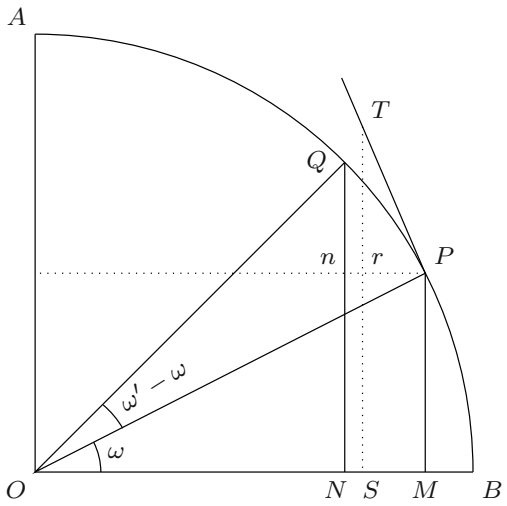

Figure 1

must be taken to be very small, so that the motion would be very small. This small interval of time has been said to be equivalent to a *kṣaṇa*[10] which according to the Hindus is an infinitesimal interval of time (immeasurably small).[11] It will be apparent from the above that Bhāskara did really employ the notion of the infinitesimal in his definition of *tātkālika-gati*.

But in actual practice, the intervals that are considered are not infinitesimal. How are we, then, to apply the notion of *tātkālika-gati* to actual problems? The answer to the above question is given by Bhāskara II as follows:

In equation (1) we have to consider the sine-difference $(\sin w' - \sin w)$. Let an arc of 90° be divided into n parts each equal to A, and let us consider the sine differences $R(\sin A - \sin O)$, $R(\sin 2A - \sin A)$, $R(\sin 3A - \sin 2A)$, etc. These differences are termed *bhogya-khaṇḍa*. Bhāskara II says:

> These are not equal to each other but gradually decrease, and consequently while the increase of the arc is uniform, the increment of the sine varies—on account of deflection of the arc.

In the figure given above, let the arc $PQ = A$. Then

$$R(\sin \angle BOQ - \sin \angle BOP) = QN - PM = Qn$$

which is the *bhogya-khaṇḍa*. Bhāskara introduces the notion of *tātkālika bhogya-khaṇḍa* (instantaneous sine difference) in order to find the variation

[10] The smallest unit of time, according to Bhāskara II is a *truṭi* (*SiŚi*, *Gaṇita*, *Madhyamādhikāra*, *Kālamānādhyāya*, 6), which is equivalent to $\frac{1}{33750}$ of a second. The *kṣaṇa* is smaller, in fact the smallest interval of time that can be imagined.

[11] These remarks are made with reference to the motion of the moon. As the motion of the moon is comparatively quicker, so the *tātkālika-gati* will not give correct result unless the time interval is taken small enough.

of the sine at P. According to him if the arc BP instead of being deflected towards Q, be increased in the direction of the tangent, so that $PT = PQ = A$, then $TS - PM = Tr$ is the *tātkālika bhogya-khaṇḍa* of the sine PT, i.e. the "instantaneous sine difference". By having recourse to this artifice Bhāskara II avoids the use of the infinitesimal in his analysis. It should be borne in mind that the "instantaneous sine difference" for a finite arc PQ, is a purely artificial quantity created with a special end in view, and is different from the actual "sine difference" $R(\sin BOQ - \sin BOP)$.

Now from the similar triangles PTr and PMO, we at once derive the proportion[12]

$$R : PT :: R\cos w : Tr. \tag{3}$$

Therefore $Tr = PT\cos w$. But

$$Tr = R(\sin w' - \sin w) \quad \text{and} \quad PT = R(w' - w).$$

Therefore

$$(\sin w' - \sin w) = (w' - w)\cos\ w.$$

Thus the *tātkālika bhogya-khaṇḍa* (the instantaneous sine difference) in modern notation is

$$\partial(\sin\ \theta) = \cos\theta\,\partial\theta.$$

This formula has been used by Bhāskara to calculate the *ayana-valana* ("angle of position").[13]

If the above were the only result occurring in Bhāskara II's work, one would be justified in not accepting the conclusions of Pandit Bapu Deva Sastri. There is however other evidence in Bhāskara II's work to show that he did actually know the principles of the differential calculus. This evidence consists partly in the occurrence of the two most important results of the differential calculus:

(i) He has shown that when a variable attains the maximum value its differential vanishes.

(ii) He shows that when a planet is either in apogee or in perigee the equation of the centre vanishes. Hence he concludes that for some intermediate position the increment of the equation of centre (i.e. the differential) also vanishes.[14]

[12] It should be noted that for the purpose of the following proof, it is immaterial, whether we take PQ small or not, because it is PT that we are considering and not PQ. Bhāskara actually takes the $\angle POQ = \left(3\frac{3}{4}\right)^\circ = 225'$ for exhibiting equation (3). The notion of the infinitesimal is here involved in the definition of *tātkālika bhogya-khaṇḍa*.

[13] *SiŚi*, *Golādhyāya*, *Grahaṇa, Grahaṇa-vāsanā*; see also Sen Gupta, *l.c.*, p. 11 ff.

[14] These results occur in the *Golādhyāya*, *Spaṣṭādhikāra vāsanā* of the *Siddhāntaśiromaṇi*, and were first noted by Sudhakara Dvivedi.

The second of the above results is the celebrated Rolle's Theorem, the mean value theorem of the differential calculus.

1.5 Remarks

The use of a formula involving differentials in the works of ancient Hindu mathematicians has been established beyond the possibility of any doubt. That the notion of instantaneous variation of motion entered into the Hindu idea of differentials as found in works of Mañjula, Āryabhaṭa II, and Bhāskara II is apparent from the epithet *tātkālika* (instantaneous) *gati* (motion) to denote these differentials. The main contribution of Bhāskara II to the theory of these differentials, which were already worked out by his predecessors, seems to be his proof of the formula by the rule of proportion without actually using the infinitesimal or varying quantities. He has, however, made it quite clear that the differentials give true results only when very small variations are concerned.

1.6 Nīlakaṇṭha's result

Nīlakaṇṭha (c. 1500) in his commentary on the *Āryabhaṭīya* has given proofs, on the theory of proportion (similar triangles) of the following results:

(i) The sine-difference $\sin(\theta + \partial\theta) - \sin\theta$ varies as the cosine and decreases as θ increases.

(ii) The cosine-difference $\cos(\theta + \partial\theta) - \cos\theta$ varies as the sine negatively and numerically increases as θ increases.

He has obtained the following formulae:

(i) $\sin(\theta + \partial\theta) - \sin\theta = 2\sin\frac{\partial\theta}{2}\ \cos\left(\theta + \frac{\partial\theta}{2}\right)$

(ii) $\cos(\theta + \partial\theta) - \cos\theta = -2\sin\frac{\partial\theta}{2}\sin\left(\theta + \frac{\partial\theta}{2}\right)$.

The above results are true for all values of $\partial\theta$ whether big or small. There is nothing new in the above results. They are simply expressions as products of sine and cosine differences.

But what is important in Nīlakaṇṭha's work is his study of the second differences. These are studied geometrically by the help of the property of the circle and of similar triangles. Denoting by $\Delta_2(\sin\theta)$, and $\Delta_2(cos\theta)$, the second differences of these functions, Nīlakaṇṭha's results may be stated as follows:

(i) The difference of the sine-difference varies as the sine negatively and increases (numerically) with the angle.

(ii) The difference of the cosine-difference varies as the cosine negatively and decreases (numerically) with the angle.

For $\Delta_2(\sin\theta)$, Nīlakaṇṭha[15] has obtained the following formula:

$$\Delta_2(\sin\theta) = -\sin\theta\left(2\sin\frac{\Delta\theta}{2}\right)^2.$$

Besides the above, Nīlakaṇṭha, has made use of a result involving the differential of an inverse sine function.[16] This result, expressed in modern notation, is

$$\partial(\sin^{-1} e\sin w) = \frac{e\cos w}{\sqrt{1-e^2\sin^2 w}}\partial w.$$

In the writings of Acyuta (1550–1621 AD) we find use of the differential of a quotient[17] also

$$\partial\left[\frac{e\sin w}{1\pm\cos w}\right] = \frac{e\cos w \pm [(e\sin w)^2/(1\pm e\cos w)]}{1\pm e\cos w}\partial w.$$

2 Method of infinitesimal-integration

2.1 Surface of the sphere

For calculating the area of the surface of a sphere Bhāskara II (1150) describes two methods which are almost the same as we usually employ now for the same purpose.

First method

> Make a spherical ball of clay or of wood. On it take a (vertical) circumference circle and divide this into 21600 parts. Mark a point on the top of it. With that point as the centre and with the radius equal to the 96th part of the circumference, i.e. to 225′, describe a circle. Again with same point as the centre with twice that arc as radius describe another circle; with thrice that another circle; and so on up to 24 times. Thus there will be 24 circles in all. The radii of these circles will be the *jyā* 225′ (= $R\sin 225'$), etc. From them the lengths of the circles can be determined by proportion. Now the length of the extreme circle is 21600′ and its radius is 3438′. Multiplying the *R*sines (of 225′, 450′, etc.) by 21600 and dividing by 3438, we shall obtain the lengths of the circles. Between two

[15] This together with the results given above are proved by Nīlakaṇṭha in the commentary on the *Āryabhaṭīya*, ii. 12.
[16] *Cf. Tantrasaṅgraha*, ii. 53–4.
[17] *Cf. Sphuṭa-nirṇaya-tantra*, iii, 19–20; *Karaṇottama*, ii. 7.

> of these circles there lie annular strips and there are altogether 24 such. They will be many more in case of many *R*sines being taken into consideration (*bāhujyā-pakṣe-bahūni syuḥ*). In each annulus considering the larger circle at the lower end as the base and the smaller circle at the top as the face and 225′ as the altitude (of the trapezium), find its area by means of the rule 'half the sum of the base and the face multiplied by the altitude etc.'[18] Similarly the areas of all the annular figures severally can be found. The sum of all these areas is equal to the area of the surface of half of the sphere. So twice that is the area of the surface of the whole sphere. And that is equal to the product of the diameter and the circumference.[19]

In other words, if T_n denotes the nth *jyā* (or *R*sine), C_n the circumference of the corresponding circle, A_n the area of the nth annulus, and S the area of the surface of the sphere, then we shall have

$$\begin{aligned} C_n &= \frac{21600}{3438} \times T_n \\ A_n &= \frac{C_n + C_{n-1}}{2} \times 225 \\ &= \frac{225 \times 21600}{2 \times 3438}(T_n + T_{n+1}). \end{aligned}$$

Therefore,

$$\frac{1}{2}S = \sum A_n = \frac{225 \times 21600}{2 \times 3438} \sum (T_n + T_{n+1})$$

the summation being taken so as to include all the *R*sines in a quadrant of the circle. Since there are ordinarily 24 *R*sines in a Hindu trigonometrical table, we have

$$\begin{aligned} \frac{1}{2}S &= \frac{225 \times 21600}{3438} \sum (T_1 + T_2 + \cdots + T_{23} + \frac{1}{2}T_{24}) \\ &= \frac{21600 \times 225 \times 52513}{3438} \\ &= 21600 \times 3436.7\ldots \end{aligned}$$

Hence approximately

$$S = 21600 \times 2 \times 3437.$$

Bhāskara II states:

> Area of the surface = circumference × diameter.

[18] The rule quoted here for finding the area of a trapezium is that given by Śrīdhara (*Triś*, R. 42). Bhāskara II's rule is defined slightly differently (*vide L*, p. 44).

[19] *SiŚi*, *Gola*, *Bhuvanakośa*, verses 55–7 (gloss).

Second method

> Suppose the (horizontal) circumference-circle on the surface of the sphere to be divided into parts as many as four times the number of *R*sines (in a quadrant). As the surface of an emblic myrobalan is seen divided into *vapras* (i.e. lunes) by lines passing through its face (or top) and bottom, so the surface of the sphere should be divided into lunes by vertical circles as many as the parts of the above mentioned (horizontal) circumference-circle. Then the area of each lune should be determined by (breaking it up into) parts. And this area of a lune is equal to the sum of all the *R*sines diminished by half the radius and divided by the semi-radius. Since that is again equal to the diameter of the sphere, so it has been said that the area of the surface of a sphere is equal to the product of its circumference and diameter.[20]

The method has been further elucidated by him in his gloss thus:

> As many as are *R*sines in the table of any particular work selected, take four times that number, and suppose the (horizontal) circumference-circle on the sphere is to be divided into, as many parts. Like the natural lines seen on the surface of a round emblic myrobalan passing through its face and base and thus dividing it into lunes, draw circles on the surface of the given sphere, passing through its top and bottom and thereby dividing it into lunes as many as the number of parts into which the (horizontal) circumference-circle is divided. Next the area of each lune has to be determined. It can be done thus: For instance in the *Dhīvrddhida*,[21] there are 24 *R*sines. So suppose the (horizontal) circumference-circle measures 96 cubits. On drawing the vertical circles through every cubit, there will be as many lunes. Then the upper half of any one lune on drawing the transverse arcs at distances of every cubit, will be divided into portions equal to the number of *R*sines, that is, 24. The lengths of these transverse lines will be obtained by dividing the *R*sines severally by the radius. Of these the lowest line measures one cubit; but the upper and upper ones are a little smaller and smaller according to the *R*sines. But the altitude is all along one cubit in length. Now by finding the area of each portion in accordance with the rule, "half the sum of the top and the base multiplied by the altitude etc." they should be added together. This sum gives the area of half a lune; twice

[20] *Ibid*, verses 58–61.
[21] That is *Śiṣyadhīvṛddhida* of Lalla.

> that is the area of a lune. For the determination of that the rule is, "the sum of all the *R*sines minus half the radius etc." Now the sum of all the *R*sines, 225 etc., is 54233.[22] This diminished by the semi-radius becomes 52514. Dividing the result by the semi-radius we get the area of each lune as 30;33. Now 30;33 is equal to the diameter of a circle whose circumference measures 96. And as the number of lunes is equal to the number of portions of the circumference it is consequently proved that the area of the surface of a sphere is equal to the product of its circumference and diameter.

If l_n denotes the length of the nth transverse arc, we have

$$l_n = \frac{T_n \times 1}{R}.$$

Therefore,

$$\begin{aligned}\text{area of a lune} &= 2 \times \sum \frac{1}{2}(l_n + l_{n+1}) \times 1 \\ &= 2 \sum \frac{1}{2R}(T_n + T_{n+1})\end{aligned}$$

the summation being taken so as to include all the *R*sines. Hence

$$\begin{aligned}\text{area of a lune} &= 2 \times \frac{1}{R}\left(T_1 + T_2 + \cdots + T_{23} + \frac{1}{2}T_{24}\right) \\ &= \frac{1}{R/2}\left(T_1 + T_2 + \cdots + T_{24} - \frac{R}{2}\right) \\ &= 30; 32, 94 \ldots\end{aligned}$$

Now

$$96 \times \frac{1250}{3927} = 30; 33, 46 \ldots$$

Hence the area of a lune is numerically equal to the diameter of the sphere. As the number of lunes is equal to the number of parts of the circumference of the sphere, we get

$$\text{Area of the surface} = \text{circumference} \times \text{diameter}.$$

2.2 Volume of the sphere

To find the volume of a sphere Bhāskara II states the following method:

> Consider on the surface of the sphere pyramidal excavations, each of a base of a unit area having unit sides and of a depth equal to

[22] According to Lalla the sum of the *R*sines is 54233.

> the radius, as many as the number of units of area in the surface. The apices of these pyramids meet at the centre of the sphere. The sum of the volumes of the pyramids is equal to the volume of the sphere. So it is proved (that the volume of a sphere is equal to the sixth part of the product of the surface area and diameter).[23]

The above results are the nearest approach to the method of the integral calculus in Hindu Mathematics. It will be observed that the modern idea of the "limit of a sum" is not present. This idea, however, is of comparatively recent origin so that credit must be given to Bhāskara II for having used the same method as that of the integral calculus, although in a crude form.

[23] *SiŚi*, *Gola*, *Bhuvanakośa*, verses 58–61, (gloss).

Paper 37

C. T. Rajagopal and M. S. Rangachari (1986)

On Medieval Kerala Mathematics

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C. T. Rajagopal

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M. S. Rangachari

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On Medieval Kerala Mathematics

C. T. RAJAGOPAL† & M. S. RANGACHARI

Dedicated by M. S. Rangachari to the memory of the late T. S. Kuppanna Sastry

Communicated by A. YOUSCHKEVITCH

§ 1. On the correspondence in content between three late medieval works

In an earlier article [1] we identified a number of infinite trigonometrical series in the text of the Tantrasaṅgraha-vyākyā, a Sanskrit work of the 16th century. In modern terms these are equivalent to the following familiar expansions:

[A] $$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots$$

and

[B_1] $$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots,$$

[B_2] $$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

The two latter series are also found stated in the Tantrasaṅgraha in the related forms

[B_1'] $$r\sin\theta = s - \frac{1}{3!}\left(\frac{\pi}{2}\right)^2\frac{s^3}{c^2} + \frac{1}{5!}\left(\frac{\pi}{2}\right)^4\frac{s^5}{c^4} - \dots,$$

and

[B_2'] $$r\cos\theta = r - \frac{1}{2!}\left(\frac{\pi}{2}\right)\frac{s^2}{c} + \frac{1}{4!}\left(\frac{\pi}{2}\right)^3\frac{s^4}{c^3} - \dots,$$

where, in a circle of radius r, the angle θ is understood to be subtended at the centre by an arc $s \leqq c = \frac{1}{2}\pi r$. Furthermore, the Tantrasaṅgraha announces the three approximations (to the length of the quadrant of unit diameter)

[C_i] $$\frac{\pi}{4} \approx 1 - \tfrac{1}{3} + \tfrac{1}{5} - \dots \mp \frac{1}{n} \pm f_i(n+1), \quad i = 1, 2, 3,$$

† The second author is sorry to have to give notice of the demise of his collaborator on 25th April 1978 and for the inordinate delay in the posthumous publication.

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where[1]

$$f_1(n) = \tfrac{1}{2}/n, \quad f_2(n) = \tfrac{1}{2}\, n/(n^2 + 1)$$

and

$$f_3(n) = [(\tfrac{1}{2}\, n)^2 + 1]/[\tfrac{1}{2}\, n(n^2 + 4 + 1)];$$

and also related forms[2]

$$[C_i'] \quad \frac{\pi}{4} = [1 - f_i(2)] - [\tfrac{1}{3} f_i(2) - f_i(4)] + [\tfrac{1}{5} - f_i(4) - f_i(6)] - \ldots \quad (i = 1, 2),$$

the two infinite series being implicit in $[C_i']$ for $i = 1, 2$, with their partial sums precisely as the sequences on the right side of $[C_i]$ for $i = 1, 2$, in turn and for $n = 1, 2, 3, \ldots$

The Tantrasaṅgraha-vyākhyā also contains two further series which may in modern equivalents be stated thus:

$$[D] \quad \frac{\pi - 2}{4} \approx \frac{1}{2^2 - 1} - \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} - \ldots \mp \frac{1}{n^2 - 1} \pm \frac{1}{2[(n+1)^2 + 2]}$$

and[3]

$$[B^*] \quad \sin^2\theta = \theta^2 - \frac{\theta^4}{(2^2 - 2/2)} \pm \frac{\theta^6}{(2^2 - 2/2)(3^2 - 3/2)} - \frac{\theta^8}{(2^2 - 2/2)(3^2 - 3/2)(4^2 - 4/2)} + \ldots$$

---

[1] On p. 90 of [1] the value $\frac{1}{2}\, n$ stated for $f_1(n)$ should read $\frac{1}{2}/n$. The Yuktibhāsa ([2], Chapter VI, last section but one) adduces the approximation $[C_1]$ before $[C_2]$ and gives the 'correction' term $f_1(n + 1)$ as $\frac{1}{2}(n + 1)$, where of course the denominator $2(n + 1)$ is the number next greater than the last odd number $(2n + 1)$.

[2] To be precise (compare [1], p. 95), the Tantrasaṅgraha-vyākhyā states the series $[C_1']$ and $[C_2']$ in the respective equivalent forms

$$\pi = 3 + \frac{4}{3^3 - 3} - \frac{4}{5^3 - 5} + \frac{4}{7^3 - 7} - \ldots,$$

and

$$\pi = \frac{16}{1^5 + 4.1} - \frac{16}{3^5 + 4.3} + \frac{16}{5^5 + 4.5} - \ldots$$

[3] The Sanskrit enunciation of [B*] (given for instance in [3], p. 119, verses 455–6) is similar to that of $[B_1]$ and $[B_2]$ (for which see [1], p. 95). It may be translated as follows.

"Obtain first the sequence whose terms are the squares of a given arc multiplied by itself once, twice, thrice and so on. Next obtain the continued product whose any factor is the radius squared and multiplied by the square of an integer diminished by half of itself, the integer being chosen successively as 2, 3, 4, ... Write down in a column the arc squared followed by the quotients resulting from division of the terms of the sequence first got by the corresponding partial products of the continued product next got. If any last term in the column is subtracted from the next one above, the remainder

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As will be seen in Section 2 below the series [D] is akin to [C_2]; and we may guess the analogues to [C_1] and perhaps analogue to [C_3] also once existed. But [B*] is of little novelty or interest doubtless being obtained straight from [B_2] $\left(\text{making implicit restriction } \theta \leqq \frac{\pi}{4}\right)$ merely applying the standard formula $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$.

About the same time that we wrote our first paper [1], K. V. SARMA brought out his edition of Tantrasaṅgraha-vyākhyā – he calls it the Yuktidīpikā by an alternative name given to it by its author – seeking to fill its manifest lacunas by adjoining to his text a portion of a second commentary on the Tantrasaṅgraha, the Laghuvivṛti and offering also a text of this basic work obtained by collating an already printed text and two manuscripts (see [3], xix). As we stated at the end of our earlier article ([1], page 101), SARMA's composite text is of considerable use for exhibiting the many links which connect the Yuktidīpikā with the Tantrasaṅgraha on the one hand, and with the Yuktibhāṣa on the other; it further supplied weighty internal support both for identifying the author of the Yuktidīpikā (and, so SARMA argues, Laghuvivṛti) as SANKARA VARIYAR (c. 1550–c. 1560) and also for dating the composition of this work with reasonable certainty to the early 1530's. We have ourselves previously expressed the broad view that the absence of the series [D] from the text of the Tantrasaṅgraha used by SARMA does not weaken the probability that it was known in Kerala some time before the Tantrasaṅgraha was composed. The likelihood that this was so would now seem strengthened practically into certainty through a recent collation by us of two further copies of this work[4], neither considered by SARMA, which list in common the series [A], [C_i] and [C'_i]. The following table listing these series in the

---

from the one next above this, and so on, there will finally result the squared jyā of the given arc".

On denoting the arc, its radius and its jyā (or sine) by s, r, and y respectively (as in [1], p. 92) and inserting the above procedure, we obtain [B*] in the form

$$y^2 = s^2 - \frac{s^4}{r^2(2^2 - 2/2)} + \frac{s^6}{r^4(2^2 - 2/2)(3^3 - 3/2)} - \cdots$$

This series is followed in the Sanskrit text by what is essentially its equivalent expansion in powers of s/c where c is the quadrant $\frac{1}{2}\pi r$. This variant, evidently analogous to [B'_1] and [B'_2], is found stated in the 'Śaurirjayati' rule in [3], pp. 119–20, verses 457–8.

[4] One of these (a palm-leaf) copy in Grantha script has (so we are told) been preserved since 1925 in the Adyār Library at Madras, where it is numbered 75706; the other is a copy which CHARLES M. WHISH claimed nearly a century and a half ago to have been "obtained with some difficulty" (see [4], p. 522). It is very likely that this copy is now in the Whish collection at the Royal Asiatic Society in London. However, WHISH'S statement that "he obtained with some difficulty" the copy used by him need not mean that it came permanently into his possession. (The WHISH collection of manuscripts, as it now exists, is one, it would appear, assembled after his death by his brother J. C. WHISH and by him donated to the Society; see the Journal of the Royal Asiatic Society, 3 (1836), p. 1 xiii, where report is given of the pertinent proceedings of the Society on 2 July 1836.)

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order in which they appear, shows at a glance the precise correspondences which can be made between these two versions of the Tantrasaṅgraha and also the Yuktidīpikā and the Yuktibhāsa [2].

The order of appearance of the series

[A], [B] $\equiv$ [B_i], [B′] $\equiv$ [B'_i] ($i = 1, 2$), [C] $\equiv$ [C_i], [C′] $\equiv$ [C'_i] ($i = 1, 2, 3$) in:			
1. WHISH's copy of the Tantrasaṅgraha (see [4]) [C_2] [C_3] [A] for $\theta = \pi/4$ [C'_2] [C'_1] [D] all on pp. 512–5 [A]: pp. 520–1	2. Adyār copy of the Tantrasaṅgraha, Chapter II. The series in column 1 here occur in the same order (on ff. 10–1 except the 1st which is found on ff. 13–4) [B'_1] given by the 'Vidvān' verse (but not [B'_2] given by the 'Stena' verse) appears on f. 12 and is then followed by [B].	3. The Yuktidīpikā [3] Chapter II [A]: pp. 95–6 verses 206b–9 [A]: (case $\theta = 2\pi/6$): p. 96 verses 212a–4b [C_2]: p. 101 verses 271–4 [C'_2]: p. 102 verses 287–8 [C'_1]: p. 103 verse 290 [D]: p. 103, Verse 292 [C'_3]: p. 103 verses 295–6 [B′]: given by vidvān-Stena verses on pp. 117–8 verses 437–8 [B*]: p. 118, verses 440–4a [B*]: p. 119, verses 455–6 [B*]: (in form analogous to [B′]): pp. 119–20 verses 457–8.	4. The Yuktibhāṣa [2] Chapters VI, VII. These series in col. 3 occur in the same order in Chapter VI last four sections. [B] (but not [B′]) occurs in Chapter VII, seventh and ninth sections where it is followed by [B*] (but not accompanied by its analogue [B′])

Note. There is a subtle indication that the statements of [A] in Yuktidīpikā and in the Tantrasaṅgraha have a differing textual history despite their essential verbal equivalence. In the former the enuniciation in verse of [A] is prefaced by the line "Athaitanyāyato jyānām cāpīkaranamucyate" while in the existing copies of the latter it is introduced by the phrase "Vṛttānayana mārgina breṣṭa jyānā dhanurnayet". That WHISH does not in his paper [4] refer to [B_1] and [B], does not prove that they were lacking in the copy of Tantrasaṅgraha used by him, for, the series were, it will be clear, not pertinent to the limited objective of his

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paper. That they were, in fact, present in it would seem strongly suggested by its including the other results [A], [C], [C′] in the same order as they are found in the Adyār copy of the Tantrasaṅgraha. Furthermore, in both copies of Tantrasaṅgraha the following minor corollary to the infinite series $[C_i]$ for $\frac{\pi}{4}$,

$$[D_0] \qquad \frac{\pi}{4} = \frac{2}{2^2 - 1} + \frac{2}{6^2 - 1} + \frac{2}{10^2 - 1} + \ldots$$

$$= 1 - \left(\frac{2}{4^2 - 1} + \frac{2}{8^2 - 1} + \frac{2}{12^2 - 1} + \ldots\right),$$

is placed after [D], maybe because the infinite series implied by [D] could be seen to result from the term-by-term addition of the two infinite series of $[D_0]$[5].

We may now summarize: (i) It is almost certain that WHISH had access to a text of the Tantrasaṅgraha the same in its essentials as the Adyār copy. (ii) There is no historical possibility that the Adyār copy was an admixture of a 'pure' version of the Tantrasaṅgraha, as presented, for instance, by SARMA in [3], filled out subsequently with interpolations from the Yuktidīpikā. For, like all other copies, this has eight chapters ending with the identifying sentence "Chapter (first, second, ..., eighth) of the Tantrasaṅgraha composed by Kerala Gārgya Nīlakaṇṭha is now concluded". Of particular relevance here is SARMA's remark (see [3], p. xxv) that the prefix 'Kerala' probably refers to the name of Nīlakaṇṭha's house and not to the familiar name of his Indian State as it has come down to us. (iii) The Yuktidīpikā, while not directly concerning itself with the source of any of its series expansions, does, by tagging on the word 'iti' to their enunciation in verse, clearly indicate their derivation for the most part from some common original (very probably the Tantrasaṅgraha, our findings would suggest).

That the series in which we are presently interested are found in identical sequence in the Yuktidīpika and the Yuktibhāṣa, one differing from their order in the Adyār and WHISH copies of the Tantrasaṅgraha, strongly indicates that the first two works borrowed these series from a common earlier source which was not the Tantrasaṅgraha, and we may further conjecture that this no longer surviving parent text was richer in content than either of the works to which it gave rise. Though the series are enunciated in much the same form in both the Yuktidīpikā and the Yuktibhāṣā, only in the latter work are they supplied with full proofs or outlines of them—demonstrations found, indeed, nowhere else in these early Hindu texts, while on the other hand some rather obscure allusions in the Yuktibhāṣa notedly the 'Vidvān-Stena' variant [B′] of the series [B], and the analogous one of [B*] presented in the 'Śaurirjayati' verses (see footnote 3) are clarified or fully stated only in the Yuktidīpikā. Both the Yuktidīpikā and the Yuktibhāṣa agree, however, in linking their mathematical subject matter to the text of the Tantrasaṅgraha, the outstanding contemporary digest of general mathematical formulae of use in medieval Hindu astronomy. While,

[5] The series $[D_0]$ is given after [D], but before $[C_3]$, in both the Yuktidīpikā (see [5], p. 103, verses 293–4) and the Yuktibhāṣa.

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probably, both the Yuktidīpikā and Yuktibhāṣā have unmistakable characteristic Keralese touches, we are led to infer that they borrow the essence of their content from a fuller, mathematically more complete, traditional source which was in existence in 1500, when Tantrasaṅgraha was composed.

§ 2. Further approximations to π in the style of the series (D)

The Sanskrit verse for [D] in the Yuktidīpikā ([3], p. 103, verse 292) reads:

Dvyādiyujām va kṛtayo vyekā hārād dvinighna viṣkambhe

Ghanamrnamantasyordhvagataujakrtirdvisahitā hārasyārdham.

In English translation: "Four times the diameter is first divided successively by the even numbers beginning with 2 squared and diminished by unity. Next, four times the diameter is divided by double the result of squaring the odd number next greater than the last even number and adding 2 to it. Lastly, when all the primary quotients are alternately added to and subtracted from twice the diameter, then results an approximation to the circumference, with a correction which is the last term in the series thus defined."

Or, if d is the diameter of a circle, then

$$\pi d \approx 2d + \frac{4d}{2^2 - 1} - \frac{4d}{4^2 - 1} + \frac{4d}{6^2 - 1} - \ldots \mp \frac{4d}{n^2 - 1} \pm \frac{4d}{2(n+1)^2 + 2}.$$

Let us make some observations which (in their main part) go beyond the evidence of the surviving Keralese texts but are perhaps not irrelevant. The final term applied in correction to the main series (as shown in [5], p. 73) is in fact but one of a whole sequence which may similarly be adduced yielding closer approximation to the circumference πd of the circle and hence to π itself. To elaborate this point a little, if

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \ldots \mp \frac{1}{n} \pm f(n)$$

$$= 1 - \tfrac{1}{3} + \tfrac{1}{5} - \ldots \pm \frac{1}{n+1} \mp f(n+1),$$

then

$$f(n+1) + f(n-1) = \frac{1}{n},$$

and, if we write

$$f(n) = \frac{1}{2r_n}, \quad r_n = n + \frac{1}{s_n}, \quad s_n = n + \frac{2^2}{t_n}, \quad t_n = n + \frac{3^2}{v_n}, \ldots,$$

we have successively

$$n^2 = (2r_{n+1} - n)(2r_{n-1} - n) = (2s_{n+1} - \overline{n-2})(2s_{n+1} - \overline{n+2}) =$$
$$= (2t_{n+1} - \overline{n-4})(2t_{n-1} - \overline{n+4}) = \ldots$$

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Analogously, if

$$\frac{\pi - 2}{4} = \frac{1}{2^2 - 1} - \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} - \ldots \mp \frac{1}{n^2 - 1} \pm g(n + 1),$$

and, if we write

$$g(n) = \frac{1}{2n \cdot R_n}, \quad R_n = n + \frac{2}{S_n}, \quad S_n = n + \frac{2\times 3}{T_n}, \quad T_n = n + \frac{3\times 4}{V_n}, \ldots,$$

from the fact that

$$g(n - 1) + g(n + 1) = \frac{1}{n^2 - 1}$$

we have

$$n^2 - 1 = (2R_{n+1} - \overline{n - 1})(2R_{n-1} - \overline{n + 1}) = (2S_{n+1} - \overline{n - 3})(2S_{n-1} + \overline{n + 3})$$

$$= (2T_{n-1} - \overline{n - 5})(2T_{n-1} - \overline{n + 5}) = \ldots$$

We thus have the representations of $f(n)$ and $g(n)$ as continued fractions:

(1) $$f(n) = \tfrac{1}{2} \cdot \frac{1}{n+} \frac{1^2}{n+} \frac{2^2}{n+} \frac{3^2}{n+} \ldots,$$

(2) $$g(n) = \frac{1}{2n} \cdot \frac{1}{n+} \frac{1.2}{n+} \frac{2.3}{n+} \frac{3.4}{n+} \ldots;$$

the $f_i(n + 1)$ occurring in $[C_i]$, $i = 1, 2, 3$, are precisely the first three convergents of the continued fraction in (1). The first two convergents of (2) are

$$g_1(n) = \frac{1}{2n^2} \quad \text{and} \quad g_2(n) = \frac{1}{2(n^2 + 2)}.$$

In terms of the latter, the approximate equality

(3) $$\frac{\pi - 2}{4} \approx \frac{1}{2^2 - 1} - \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} - \ldots \mp \frac{1}{n^2 - 1} \pm g_2(n + 1)$$

gound in Tantrasaṅgraha is at once seen to hold true. Though no other convercents to $g(n)$ seem to occur in the surviving medieval Sanskrit texts, the prime torrection in terms of $g_1(n + 1)$ must also have been known, even if not in ferms of $g_3(n + 1)$. Let us add, wholly anachronistically, that (3), with one more term supplied at the beginning of its series, may be written as

(4) $$\frac{\pi - 2}{4} + 1 = -\frac{1}{0^2 - 1} + \frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} \ldots \mp \frac{1}{n^2 - 1} \pm g(n + 1).$$

Further, that the particular cases of (3) with $n = 2$ and of (4) with $n = 0$ are respectively

$$\frac{\pi - 2}{4} = \frac{1}{2^2 - 1} - g(3) \quad \text{and} \quad \frac{\pi - 2}{4} + 1 = -\frac{1}{0^2 - 1} + g(1).$$

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These, fully written out, are the continued fractions

$$\frac{\pi-2}{4}=\tfrac{1}{3}\left\{1-\tfrac{1}{2}\cdot\frac{1}{3+}\,\frac{1.2}{3+}\,\frac{2.3}{3+}\,\frac{3.4}{3+}\cdots\right\}$$

and

$$\frac{\pi-2}{4}=\tfrac{1}{2}\left\{\frac{1}{1+}\,\frac{1.2}{1+}\,\frac{2.3}{1+}\,\frac{3.4}{1+}\cdots\right\};$$

the second of these was afterwards independently derived by LEONHARD EULER in 1739 (see [6], Vol. 3, p. 699) and published in 1750 (Comm. Ac. Petrop., Vol. XI, p. 48).

The following remarks of Dr. WHITESIDE are worth recording here. (i) The rounding-off $f_i(n)$ enormously improves the slow convergence of the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$, because $\frac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots-\frac{1}{19}+f_3$ (20) errs from $\frac{1}{4}\pi$ by only 1 in the 8th decimal place, and $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{1}{49}-f_3(50)$ is $\frac{1}{4}\pi$ accurate to the first 11 decimal places.

(ii) Among the many theorems which HARDY received from RAMANUJAN in January 1913 was the following evaluation (see [7], p. xxvii):

$$(5)\qquad \frac{4}{x+}\,\frac{1^2}{2x+}\,\frac{3^2}{2x+}\,\frac{5^2}{2x+}\,\frac{7^2}{2x+}=\left(\Gamma\left(\frac{x+1}{4}\right)\Big/\Gamma\left(\frac{x+3}{4}\right)\right)^2.$$

The left-hand side of (5) is $4/\phi(x)$ where $\phi(x)$ is BROUNCKER's continued fraction of 1654:

$x+\dfrac{1^2}{2x+}\,\dfrac{3^2}{2x+}\,\dfrac{5^2}{2x+}\ldots$, desired by him to satisfy $\phi(x+1)\times\phi(x-1)\equiv x^2$. RAMANUJAN seems to have followed (with or without the knowledge of BROUNCKER's continued fraction) BROUCNKER's procedure to obtain the equivalent expression for $4/\phi(x)$ on the right-hand side of (5). One of RAMANJUAN's Hindu ancestors (as was set out just above) arrived at the same procedure of successive reductions

$$\phi(x)=x+\frac{1^2}{\alpha_x},\qquad \alpha_x=2x+\frac{3^2}{\beta_x},\qquad \beta_x=2x+\frac{5^2}{\gamma_x},\ldots$$

which maintain

$$x^2\equiv(\alpha_{x+1}-(x-1))\,(\alpha_{x-1}-(x+1))$$

$$=(\beta_{x+1}-(x-3))\,(\beta_{x-1}-(x+3))=(\gamma_{x+1}-(x-5))\,(\gamma_{x-1}-(x+5))=\ldots$$

From the known background of RAMANUJAN, it is very unlikely that he knew the work of this ancestor of his. But the satisfying 'sameness' in all these is remarkable.

(iii) In view of the fact (see footnote 15 to [1]) that the quantities arising in the 'Vidvān' verse ([B_1'] of [1]) are given only to the nearest third minute, *i.e.* only accurate to the 8th decimal place, it would be interesting to know what value of π was used in calculating them.

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It seems to us that we have lost some wider theory of accelerating the convergence of singly infinite series by applying such corrections as have now come down to us in the medieval Hindu texts, the Tantrasaṅgraha and Yuktibhāṣa, as we know them. Whoever be the author of these penetrating researches, Mādhava or a predecessor of his, one cannot hesitate to give him due credit.

The authors were helped by the late Professor T. S. KUPPANNA SASTRY with exact translations of several verses in Chapter II of Tantrasaṅgraha and in the Yuktidīpikā. The paper owes its present form to several significant emendations in details by Dr. D. T. WHITESIDE, to whom the authors are beholden. It also incorporates some suggestions of Professor A. P. YOUSCHKEVITCH.

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Paper 38

R. C. Gupta (1987)

South Indian Achievements in Medieval Mathematics

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R. C. Gupta

For a biographical sketch of R. C. Gupta, see p. 341.

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South Indian Achievements in Medieval Mathematics

1 Introduction

The development of Hindu mathematics did not come to a standstill after the famous Bhāskarācārya or Bhāskara II (circa 1150 AD) although many scholars believed and still believe that.

This is mainly because the historians failed to notice or take cognizance of the findings published by C.M. Whish (as early as 1835),[1] C.T. Rajagopal and others.[2]

Thus for quite some time the view that there was no progress in Indian mathematics and astronomy after Bhāskara II continued to prevail and still prevails in some quarters due to ignorance. Of course one reason for this is that the important works (some of which are in Malayalam and other regional languages) of the post-Bhāskara II period have not been translated into English.

However, nice editions of several important south Indian works (original texts as well as commentaries) are now available. They are *Vyavahāra-gaṇitam* (in Kannada of Rājāditya (c. 1190), a native of Puvina Bage (in North Karnataka); the commentary by Sūryadeva Yajvan (born 1191) of Gangaikonda Colapuram on the *Āryabhaṭīya (= AB);* some works of Mādhava of Saṅgamagrāma near Cochin (c. 1340–1425), and of Parameśvara (between 1360 and 1460) of Alattur village in Kerala; the *Tantrasaṅgraha (= TS)* and *Āryabhaṭīya-bhāṣya (= NAB)* by Nīlakaṇṭha Somayāji (c. 1500) of Kundapura (near Tirur, South Malabar); the Malayalam *Yukti-bhāṣā (= YB)* of Jyeṣṭhadeva (c. 1500–1610); the *Kriyākramakarī (= KKK)* which is an elaborate commentary on Bhāskara's *Līlāvatī* and composed by Śankara Vāriyar (1500–1560) and, after his demise, by Mahiṣamaṅgalam Nārāyaṇa (c. 1530–1610), both of Kerala; and a host of others.

The purpose of the present paper is to give a topic-wise survey of the south Indian achievements in medieval mathematics (twelfth to seventeenth century) based mostly on primary sources.

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https://doi.org/10.1007/978-981-13-1229-8_40

2 Decimal Place Value Names

Śrīdhara in his *Pāṭīgaṇita (= PG)* (c. 750 AD), 7–8, gives names of 18 decimal notational places,[3] the last being *Parārdha* which stood for 10^{17}. These names (sometimes with slight variations) are later on given by Śrīpati (c. 1040), Bhāskara II, and Nārāyaṇa Paṇḍita (1356). This shows that the practice of 18 notational place-names became more or less standard (especially in north India).

In south India, however, we find bigger lists of names of notational places. Thus the *Gaṇitasārāsaṅgraha (= GSS)*, 1, 63–68 of Mahāvīra (850) written under the Rāṣṭrakūṭa king Amoghavarṣa (who ruled in the Kanarese area of south), extends the list to 24 places ending with[4]

> महाक्षोभं चतुर्नयम् ।
>
> The twenty-fourth (place) is *Mahākṣobha* (= 10^{23}).

Pāvaḷūri Mallikārjuna in his *Gaṇita-śāstra* gives[5] the list up to 36th place which is called *mahābhūri*. The author seems to be same as Pāvaḷūri Mallana (c. 1100), son of Sivanna, who wrote a Telugu version of the *GSS*[6]. But there was another south Indian author called Mallikārjuna Sūri (c. 1178) who composed a Telugu commentary on the *Sūrya-siddhāntā (= SS)* and some other works.[7]

Yallaya (c. 1480) of Skanda-somesvara (S.E. of Srisaila in Kurnool District of Andhra Pradesh) restricts his list to 29 places. He gave the list in his commentary on the *AB*, II, 2 where list of only 10 names occurs.[8]

The biggest south Indian list of notational places is found in the Kannada work, *Vyavahāra-gaṇitam* of the Jaina author Rājāditya (twelfth century). It extends to 40 places and the names are[9]

1 *ekam*	21 *kṣiti*
2 *dāham*	22 *mahākṣiti*
3 *śatam*	23 *kṣobha*
4 *sābira*	24 *mahākṣoba*
5 *dāsābira*	25 *nadī*
6 *lakṣa*	26 *mahānadī*
7 *dālakṣa*	27 *naga*
8 *koṭi*	28 *mahānaga*
9 *dākoṭi*	29 *ratha*
10 *śatakoṭi*	30 *mahāratha*
11 *arbuda*	31 *hari*
12 *nyarbuda*	32 *mahāhari*
13 *kharva*	33 *phaṇi*
14 *mahākharva*	34 *mahāphaṇi*
15 *padma*	35 *kratu*
16 *mahāpadma*	36 *mahākratu*
17 *kṣoṇī*	37 *sāgara*
18 *mahākṣoṇī*	38 *mahāsāgara*

19 *śaṅkha* 39 *parimita*
20 *mahāśankha* 40 *mahāparimita*$= 10^{39}$

3 Geometrico Algebra

Inheriting the tradition of the *Śulba-sūtras,* the full bloom of the elementary geometric algebra is found developed in the works of the late Āryabhaṭa School in South India.

The *Yuktibhāṣā (= YB)*[10] demonstrates geometrically the multiplication identity

$$ab = (a+c)\left[b - \frac{b.c}{(a+c)}\right]$$

for

$$a = 12,\ b = 20,\ \text{and}\ c = 4.$$

If one side of a rectangle is increased arbitrarily, in what proportion the other side should be decreased in order to preserve the area unchanged. The above identity provides the answer to the question. Such questions were relevant to construction of altars.

An algebraic rule from the *Līlāvatī* says that $a^2 \pm b^2 - 1$ will be a perfect square if $a = 8c^4 + 1$, and $b = 8c^3, c$ being arbitrary.

The *Kriyākramakarī (=KKK)* (c. 1534) commentary on this rule presents an interesting graphical demonstration which may be outlined as follows.[11]

The quantity $(8c^4)^2$ can be represented as a square $PQRS$, and $b^2 = (8c^3)^2 = (8c^4)\cdot(8c^2)$ as a rectangle $ABCD$ (see the accompanying Fig. 1) of sides $8c^4$ and $8c^2$.

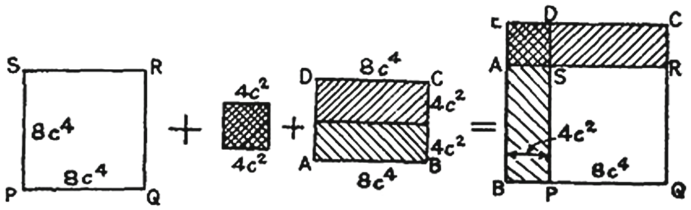

Fig. 1 Geometrical demonstration (case I)

$ABCD$ is divided into two equal rectangular strips each of sides $8c^4$ and $4c^2$. Applying these strips to the square $PQRS$ so as to form a square of side $(8c^4 + 4c^2)$ but with a square of side $4c^2$ remaining unfilled at one corner. However,

$$a^2 = (8c^4 + 1)^2 = (8c^4)^2 + 16c^4 + 1$$

Hence $(16c^4+1)$ is still to be used. The part $16c^4$ can be used to fill the empty corner square of side $4c^2$ needed above. Hence when unity is subtracted from the sum of their squares, the result is a perfect square. Or

अतस्तयोर्वर्गयोगो, व्येको मूलीकर्तव्य एव

as the *KKK* (p. 139) puts it. That is,

$$(8c^4+1)^2+(8c^3)^2-1=(8c^4+4c^2)^2.$$

A similar demonstration will hold for the other case (see Fig. 2).

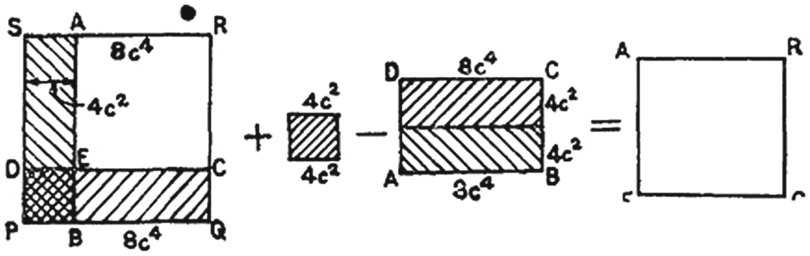

Fig. 2 Geometrical demonstration (case II)

The geometrical demonstration of the formula

$$S=\frac{a(r^n-1)}{(r-1)}$$

for the sum of a geometrical progression is more interesting. The *KKK* (pp. 263–264) gives it for the case when r is equal to 4 somewhat as follows.

Let a long rectangular strip $ABCD$ (see the Fig. 3) be taken to represent the $(n+1)$th term, T_{n+1}, of the series. Let it be divided into 3 (i.e. $r-1$) equal parts.

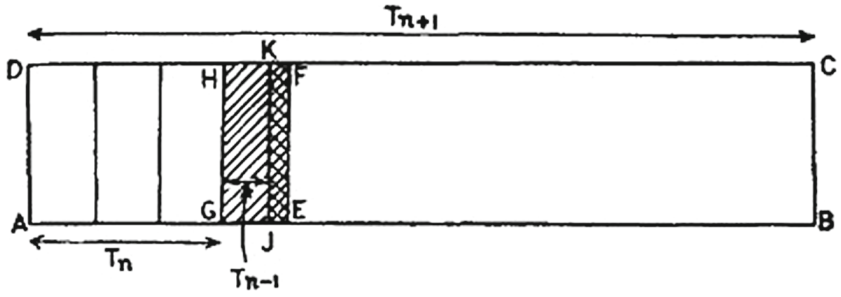

Fig. 3 Sum of a G.P.

One of these parts is $AEFD$ which is further divided into 4 (i.e. r) equal parts. Three (i.e. $r-1$) of these smaller divisions will be equal to

$$\left(\frac{3}{4}\right)\cdot\left(\frac{1}{3}\right)\cdot T_{n+1}=\left(\frac{1}{r}\right)\cdot T_{n+1}=T_n$$

Thus, the rectangle $AGHD$ represents the nth term of the series and the rectangle $GEFH$ is one-third of $AGHD$, just as the rectangle $AEFD$ was one-third of $ABCD$ which represented T_{n+1}.

So we repeat the above process of dividing $GEFH$ into 4 equal parts and choose 3 of them to get the rectangle $GJKH$ to represent T_{n-1}.

The remaining rectangle $JEFK$ can be similarly treated. This process is repeated till first term is reached. The part ultimately left will be thus one-third of T_1, Hence

$$\begin{aligned}\left(\frac{1}{3}\right)T_{n+1}&=\text{rectangle } AEFD\\&=T_n+T_{n-1}+\ldots+T_1+\left(\frac{1}{3}\right)\cdot T_1\\&=(\text{sum of } n \text{ terms})+\left(\frac{1}{3}\right).T_1\end{aligned}$$

which gives the required S. With the general common ration r, the above implies

$$\frac{(ar^n)}{(r-1)}=S+\frac{a}{(r-1)}$$

giving the general formula for S.

The enunciation of the general formula for the sum of an infinite geometrically progressing convergent series and its proof in particular cases seems to be first (?) given by Nīlakaṇṭha Somayāji (c. 1500) of Kerala.[12]

4 Solution of Algebraic Equations

Citrabhānu (1475–1550) of the village Covvaram (or Sivapuram) near Trichur in Central Kerala has not yet found place in books on Indian mathematics. His work on solution of equations is quoted in that portion of the KKK which was composed by Śaṅkara Vāriar who not only provides detailed explanations of the rules but speaks as if he had received instructions from the former.

Citrabhānu gave method of solving each of a set of 21 pairs of simultaneous equations in two unknowns, say x and y. The 21 pairs arise by taking, at a time, any two of the following seven quantities as known:[13]

(i) $x+y=a$, say;

(ii) $x-y=b$

(iii) $xy = p$
(iv) $x^2 + y^2 = m$
(v) $x^2 - y^2 = n$
(vi) $x^3 + y^3 = r$
(vii) $x^3 - y^3 = s$

Thus there will be ${}^7C_2 = 21$ cases all of which are dealt by Citrabhānu in his work called *Ekaviṃśati-praśnottara* ("Twenty-one Question-Answers"). They are

1. $x+y=a,\quad x-y=b$
2. $x+y=a,\quad xy=p$
3. $x+y=a,\quad x^2+y^2=m$
4. $x+y=a,\quad x^2-y^2=n$
5. $x+y=a,\quad x^3+y^3=r$
6. $x+y=a,\quad x^3-y^3=s$
7. $x-y=b,\quad xy=p$
8. $x-y=b,\quad x^2+y^2=m$
9. $x-y=b,\quad x^2-y^2=n$
10. $x-y=b,\quad x^3+y^3=r$
11. $x-y=b,\quad x^3-y^3=s$
12. $xy=p,\quad x^2+y^2=m$
13. $xy=p,\quad x^2-y^2=n$
14. $xy=p,\quad x^3+y^3=r$
15. $xy=p,\quad x^3-y^3=s$
16. $x^2+y^2=m,\ x^2-y^2=n$
17. $x^2+y^2=m,\ x^3+y^3=r$
18. $x^2+y^2=m,\ x^3-y^3=s$
19. $x^2-y^2=n,\ x^3+y^3=r$
20. $x^2-y^2=n,\ x^3-y^3=s$
21. $x^3+y^3=r,\ x^3-y^3=s$

The exact solution in 15 of the above 21 cases is more elementary. As a sample, Citrabhānu's rule for solution in the 5th case starts with (*KKK*, p. 112) the stanza

योगघनाद् घनयोगे त्यक्ते त्रिगुणेन राशियोगेन । शिष्टे हृतेऽथ राश्योर्द्धयोर्भवेदिष्टयोर्घातः ॥

From the cube of the sum, subtract the sum of the cubes and divide the remainder by three times the sum of the quantities. The result is the product of the two quantities.

That is,

$$\frac{\left[(x+y)^3 - (x^3+y^3)\right]}{3(x+y)} = xy.$$

In this way the problem reduces to case 2 and the solution is easily obtained.

To take a sample of non-elementary type, the second equation in case 6 can be written as

$$(x-y)\left[3(x+y)^2 + (x-y)^2\right] = 4s$$

or, with $x - y = u$, as

$$u(3a^2 + u^2) = 4s$$

The exact solution, therefore, depends on the method of solving a general cubic which does not seem to be discussed by ancient or medieval Indians. Citrabhānu's procedure (*KKK*, p. 112) may be translated as:

> The (given) difference of the cubes (of the unknowns), multiplied by four, be divided by three times the square of the (given) sum (of the unknowns). The quotient (thus determined) is the (estimated) difference of the (desired) quantities, if the remainder (left in the above division) is capable of being cancelled by the cube of the (estimated) difference.

This rule is easily seen to be based on the identity

$$\begin{aligned} \frac{4(x^3 - y^3)}{3(x+y)^2} &= (x-y) + \frac{(x-y)^3}{3(x+y)^2} \\ &= Q + \frac{R}{3(x+y)^2}, \qquad \text{say.} \end{aligned}$$

We shall get the exact result, if, as the commentary states, the remainder R is equal to the cube of the quotient Q. This is illustrated by taking an example in which $a = 25$ and $s = 2375$. We get $(x - y) = 5$ exactly, and so $x = 15$ and $y = 10$.

It can be easily seen that the method will also be useful for the integral values of the involved quantities if

$$(x-y)^3 \text{ is less than } 3(x+y)^2.$$

In other situations, this empirical method will be rather impractical, e.g. when one tries to solve

$$x + y = 11,\ x^3 - y^3 = 999.$$

We have to follow trial-and-error method to get correct solution. Anyway, the effort to discuss all the 21 possible cases indicates south Indian interest in pure mathematics in medieval times.

5 Values of π

The best ancient Indian approximation, as given in the *AB*, II, 10, is[14]

$$\pi = \frac{C}{D} = \frac{62832}{20000}$$

This is later on put by the Andhra Pradesh commentator Yallaya (fifteenth century) in the popular south Indian alphabetic system, called Kaṭapayādi Nyāya, as[15]

रङ्गेहरिस्तुपरिधेर्विष्कम्भोज्ञानिनोनरः ।

The Chinese value $\frac{355}{113}$ for π, which was given by Tsu Chhung-chih (fifth century AD), occurs in India explicitly in the *Tantra-samuccaya* of Nārāyaṇa (fifteenth century),[16] a priest of Travancore for temple construction. The same approximation is given by Nīlakaṇṭha (c. 1500) in his *TS*, II, 7; as well as in his *Golasāra*, III, 12 where the rule is[17]

विश्वैकसमोव्यासः परिधेः प्रायोऽर्थबाणगुणभागः ।

(When) the diameter equals 113, the circumference has 355 parts nearly.

It is interesting to note that the *KKK* (p. 377) quotes a variant reading of a rule from Bhāskara II in order to credit him with the above value.

A very good value is given by Mādhava (c. 1340–1425) in the verse[18]

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥

In a circle of diameter nine *nikharva* (that is, 9×10^{11}), the measure of the circumference is taken to be 2827, 4333, 88233 by the learned.

That is,

$$\pi = \frac{(2827433388233)}{9\times10^{11}}$$

which yields a value correct to 11 decimal places (after rounding off).

When expressed as a continued fraction, Mādhava's value yields the Chinese approximation $\frac{355}{113}$ as its fourth convergent. The six convergent comes

$$\frac{104348}{33215}$$

for which a Sanskrit stanza is quoted in the *KKK* (p. 377) but with a wrong remark that this value is a closer approximation than Mādhava's (*ato'api sukṣmatama*).

The *Karaṇapaddhati (=KP),* VI, 7 composed by Putumana Somayāji (c. 1600–1740) of Sivapuram (Trichur) gives a value of π which is correct to 10 decimal places.[19] It might have been obtained from that of Mādhava.

A value correct to 17 decimal places is found in a late south Indian work, namely, the *Sadratnamālā* of Śaṅkara Varmā (1800–1838) who belonged to a royal house of north Malabar.[20]

Before going on to next topic it may be worthwhile to quote the *NAB* on the incommensurability of π as follows:[21]

...इत्येकेनैवमानेन मीयमानयोरुभयोः क्वापि न निरवयवत्वं स्यात् ।
महान्तमध्वानं गत्वापि अल्पावयवत्वमेव लभ्यम् । निरवयवत्वं तु क्वापि न लभ्यम् ।...

…Hence the two (that is, the diameter and the circumference of a circle) measured by the same unit will never be without remainder. By carrying out the process further, smallness in the remainder may be obtained. But remainderlessness can never be achieved.…

Thus irrationality of π is clearly stated. Series for π are dealt below.

6 Cyclic Quadrilaterals

The cyclic quadrilateral has significant place in the history of Indian mathematics. The concept of the third diagonal is quite interesting and reflects a sort of abstract mathematical thinking. When two adjacent sides of a (cyclic) quadrilateral are interchanged, the length of one diagonal is altered (thereby we get the third diagonal). This area and perimeter preserving property is mentioned by Bhāskara II, and the geometry of three diagonals of a cyclic quadrilateral is discussed in greater details by Nārāyaṇa Paṇḍita (fourteenth century)[22].

The discussion of the three diagonals as found in the KKK is more subtle. It shows that in a cyclic quadrilateral, more than three diagonals are not possible. The arguments given are substantially as follows (KKK, p. 351).

Let $\alpha, \beta, \gamma, \delta$ be the angular measures of the arcs corresponding to the sides a, b, c, d (respectively). Now a sum of any two arcs can be made to define a diagonal. Thus there will be six cases. But because of

$$\alpha + \beta + \gamma + \delta = 360^\circ$$

there will be only three final possibilities (for example, if $\alpha + \beta$ defines one diagonal, $\gamma + \delta$ will define the very same diagonal). Hence only three diagonals are possible.

The KKK also proves the following beautiful generalization of the so-called Ptolemy's theorem:

THEOREM: Let $ABCD'$ be the quadrilateral formed from the cyclic quadrilateral $ABCD$ by interchanging the sides AD and CD. That is, by taking $AD' = CD = c$, and $CD' = AD = d$, and $AB = a$, $BC = b$. If x, y, z denote the three diagonals AC, BD and BD', respectively, then

$$yz = ab + cd$$
$$zx = bc + da$$
and $$xy = ca + bd$$

We have already published[23] the KKK (pp. 349–351) proof in modern language and notation and pointed out that it is different from Ptolemy's proof.

The *KKK* (p. 351) then obtains the expressions for the three diagonals from the above theorem. The resulting expressions for the diagonals AC and BD were already known to Brahmagupta (628 AD) who[24] gave them in his *Brāhmasphuṭa-siddhānta (= BSS),* XII, 28. These two expressions are considered to be the "most remarkable to Hindu geometry and solitary in its excellence" by a modern historian and one of them was rediscovered in Europe by W. Snell (seventeenth century).[25]

The usual expressions for the area of a cyclic quadrilateral in terms of its sides, namely,

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

were first stated in *BSS*, XII, 21 by Brahmagupta who did not, as usual, prove it.[26] And the proof attempted by Gaṇeśa (c. 1545) of Nandigram (Nanded, Gujarat) is found to be incomplete.[27]

A detailed and complete proof is given in the Malayalam *YB* (pp. 247–257) which may be briefly outlined as follows:

By adding the areas of the triangles BAC and DAC (see Fig. 4) and squaring, we easily get

$$\begin{aligned} S^2 &= \frac{(AC)^2(BE+DF)^2}{4} \\ &= \frac{(AC)^2(BD^2-EF^2)}{4} \end{aligned}$$

Now from the individual triangles BAC and ADC we can show that

$$\begin{aligned} EM &= \frac{(b^2-a^2)}{2\cdot AC} \\ FM &= \frac{(c^2-d^2)}{2\cdot AC} \end{aligned}$$

where M is the mid-point of AC. Thus, the *lambanipātāntara* is given by

$$EF = \frac{\left[(a^2+c^2)-(b^2+d^2)\right]}{2.AC}$$

Hence we get

$$S^2 = \left(\frac{1}{4}\right)\cdot(AC)^2\cdot(BD)^2 - \frac{\left[(a^2+c^2)-(b^2+d^2)\right]^2}{16}$$

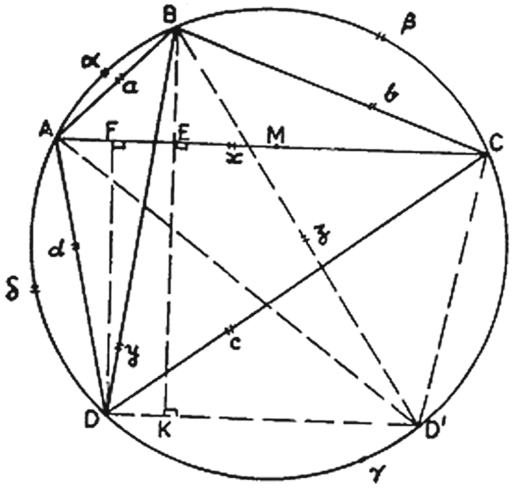

Fig. 4 Area of a cyclic quadrilateral

Using Brahmagupta's expressions for the diagonals AC and BD (or using the expression for xy from the above Theorem) and simplifying, we get the required result in terms of sides only.

Another glorious achievement of medieval south Indian mathematics is the expression for the circum-radius of the cyclic quadrilateral. This is given by Parameśvara in his commentary (before 1432) on the *Lilāvatī* and the rule is also quoted in the KKK (p. 363). The rule is[28]

> दोष्णां द्वयोर्द्वयोर्घातयुतीनां तिसृणां वधे । एकैकोनेतरत्र्यैक्यचतुष्केण विभाजिते ॥
> लब्धमूलेन यद् वृत्तं विष्कम्भार्धेन निर्मितम् । सर्वं चतुर्भुजं क्षेत्रं तस्मिन्नेवावतिष्ठते ॥
>
> The three sums of the products of the sides taken two at a time are to be multiplied together and divided by the tetrad formed by diminishing one (of the sides) at a time form the sum of the other three. If a circle is drawn with the square-root of the quotient (just obtained) as semi-diameter, the whole quadrilateral figure will be located therein.

That is

$$R = \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}}$$

This formula for the circum-radius was rediscovered in Europe about 350 years later on by S. A. J. L'Huilier (c. 1780).[29] The KKK (pp. 364–365) has an elegant derivation. Applying a well-known result to triangle BDD' (see Fig. 4), it gets

$$\text{perpendicular } BK = \frac{yz}{2R}$$

Also

$$S = \left(\frac{1}{2}\right) BK.x$$

Hence

$$R = \frac{xyz}{4S}$$

which was already known to Nārāyaṇa Paṇḍita (c. 1356) and from which the required result follows.[30]

7 Sine Tables

The AB, I, 10 contains differences corresponding to a set of 24 tabular Sines ($h = 225$ minutes) for the radius[31]

$$R = \frac{21600}{2\pi}$$
$$= 3438 \text{ to nearest minute,}$$

which can be derived by using the value of π given in the AB itself (see above). The fractional parts beyond minutes are either left out or rounded off. The fractional parts consisting of (sexagesimal) seconds and thirds are given by a south Indian, Govindasvāmin (c. ninth century), who lived in a place which had a latitude of about 10° (in his gloss on the *Māhā-Bhāskarīya*).[32] Parameśvara in his commentary (1408) on the *Laghubhāskarīya* quotes a sine table in which the *sinus totus* is 3437; 44 minutes.[33]

Employing a better value of π, Mādhava (c. 1400) obtained, from the above relation, the value

$$R = 3437; 44, 48 \text{ (to nearest second).}$$

Mādhava's tabular Sines given in the *Kaṭapayādi* system are contained just in six couplets which are quoted in NAB (Part I, p. 55) and in a commentary on TS.[34] We reproduce the Sines in the accompanying Table.

Table: Mādhava's Mahājyā $\left(R = \frac{21600}{2\pi} \text{ and } h = 225'\right)$

(*Note:* Here ಱ, ಱಾ, that is, *ḷa, ḷā,* denote 9.)

n	in Devanāgarī	in Transliteration	Implied $R\sin nh$
1	श्रेष्ठं नाम वरिष्ठानाम्	*śreṣṭhaṃ nāma variṣṭhānām*	0224; 50, 22
2	हिमाद्रिर्वेदभावनः	*himādrirveda bhāvanaḥ*	0448; 42, 58
3	तपनो भानुसूक्तज्ञो	*tapano bhānusūktajño*	0670; 40, 16
4	मध्यमं विद्धि दोहनम्	*madhyamaṃ viddhi dohanam*	0889; 45, 15
5	धिगाज्यो नाशनं कष्टम्	*dhigājyo nāśanaṃ kaṣṭam*	1105; 01, 39
6	छन्नभोगाशयाम्बिका	*channabhogāśayāmbikā*	1315; 34, 07
7	मृगाहारो नरेशोऽयम्	*mṛgāhāro nareśo'yam*	1520; 28, 35
8	वीरो रणजयोत्सुकः	*vīro raṇajayotsukaḥ*	1718; 52, 24
9	मूलं विशुद्धं नाळस्य	*mūlaṃ viśuddhaṃ nāḷasya*	1909; 54, 35
10	गानेषु विरळा नराः	*gāneṣu viraḷā narāḥ*	2092; 46, 03
11	अशुद्धिगुप्ता चोरश्रीः	*aśuddhiguptā coraśrīḥ*	2266; 39, 50
12	शङ्कुकर्णो नगेश्वरः	*śaṅkukarṇo nageśvaraḥ*	2430; 51, 15
13	तनुजो गर्भजो मित्रम्	*tanujo garbhajo mitram*	2584; 38, 06
14	श्रीमानत्र सुखी सखे	*śrīmānatra sukhī sakhe*	2727; 20, 52
15	शशी रात्रौ हिमाहारो	*śaśī rātrau himāhāro*	2858; 22, 55
16	वेगज्ञः पथि सिन्धुरः	*vegajñaḥ pathi sindhuraḥ*	2977; 10, 34
17	छायालयो गजो नीलो	*chāyālayo gajo nīlo*	3083; 13, 17
18	निर्मलो नास्ति सत्कुले	*nirmalo nāsti satkule*	3176; 03, 50
19	रात्रौ दर्पणमभ्राङ्गम्	*rātrau darpaṇamabhrāṅgam*	3255; 18, 22
20	नगस्तुङ्गनखो बली	*nagastuṅganakho balī*	3320; 36, 30
21	धीरो युवा कथालोलः	*dhīro yuvā kathālolaḥ*	3371; 41, 29
22	पूज्यो नारीजनैर्भगः	*pūjyo nārījanairbhagaḥ*	3408; 20, 11
23	कन्यागारे नागवल्ली	*kanyāgāre nāgavallī*	3430; 23, 11
24	देवो विश्वस्थली भृगुः	*devo viśvasthalī bhṛguḥ*	3437; 44, 48

8 Second-Order Differences of Sines

It can be easily seen that

$$D_n - D_{n+1} = F.S_n \tag{1}$$

where (for positive integral values of n)

$$S_n = R\sin nh$$
$$D_1 = S_1$$
$$D_{n+1} = S_{n+1} - S_n$$

and F is the proportionality factor, a constant independent of R and n. The relation (1) expresses the simple property that second-order differences of sines are proportional to sines themselves, a fact which was known to Indians from almost the beginning of their trigonometry, e.g. see the AB, II, 12 and the SS, II, 15–16 (according to the interpretations of Mallikārjuna Sūri and Rāmakṛṣṇa), *Golasāra*, III, 13–14, and NAB (Part I, p. 53).[35] The TS, II, 4 gives the value of the reciprocal of F as 233.5 and the commentator thereof even gives this as

$$233 + \frac{32}{60}$$

which is almost equal to the actual value.[36]

More details of the rule (1) are given in NAB (part I, 48–53) which contains a beautiful simple geometrical proof of it. We have already published the proof in modern language and notation[37] but a condensed mathematical version may be given as follows:

From the similar triangles OAQ and NKM (see Fig. 5), we have

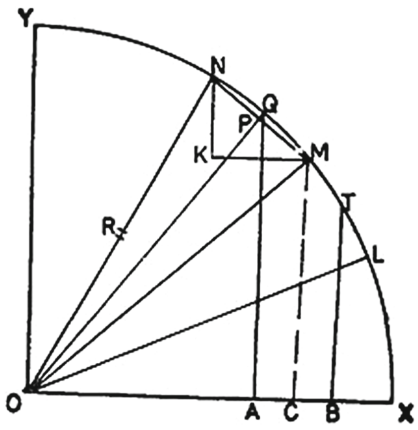

Fig. 5 Second order Sine-difference

$$NK = \frac{OA \cdot MN}{OQ} \tag{2}$$

and

$$MK = \frac{QA \cdot MN}{OQ} \tag{3}$$

That is,

$$D_{n+1} = R\cos\left(nh + \frac{h}{2}\right) \cdot \frac{(\text{crd } h)}{R} \tag{4}$$

and

$$E_{n+1} = R\sin\left(nh + \frac{h}{2}\right) \cdot \frac{(\text{crd } h)}{R} \tag{5}$$

where

$$C_n = R\cos nh$$
$$E_{n+1} = C_n - C_{n+1}$$

and

$$E_1 = C_1;\ \text{arc } MX = nh;\ \text{arc } MN = h;$$
$$\text{crd } h = \text{ chord of arc } h$$

Now, using (2),

$$D_n - D_{n+1} = \frac{OB \cdot (\text{crd } h)}{R} - \frac{OA \cdot (\text{crd } h)}{R}$$
$$= \frac{(\text{crd } h)(OB - OA)}{R}$$

But by (5) applied to the elemental arc TQ, we get

$$OB - OA = \frac{(\text{Sine at the middle of } TQ) \cdot (\text{crd } h)}{R}$$
$$= \frac{MC \cdot (\text{crd } h)}{R}$$

Hence

$$D_n - D_{n+1} = \frac{(\text{crd } h)^2 MC}{R^2}$$
$$= \frac{(\text{crd } h)^2 \cdot (S_n)}{R^2}$$

which proves the required property (1).

9 Addition and Subtraction Theorems for Sine

Although these were already known to Bhāskara II, in the Āryabhaṭa School of South India these theorems, called *Jīveparaspara-nyāya*, were attributed to the famous Mādhava (c. 1400) who gave two forms, namely[38]

$$R\sin(A \pm B) = \frac{(R\sin A)\cdot(R\cos B)}{R} \pm \frac{(R\cos A)\cdot(R\sin B)}{R} \tag{6}$$

and

$$R\sin(A \pm B) = \sqrt{(R\sin A)^2 - (lamba)^2} \pm \sqrt{(R\sin B)^2 - (lamba)^2}$$

where

$$lamba = \frac{(R\sin A)(R\sin B)}{R}$$

Geometrical proofs of the theorems are found in the *NAB* (Part I, pp. 58–61) and the Malayalam *YB* (pp. 206–208, 212–213, 237–238). We have already brought out these in modern notation in a paper[39] from which one proof (*YB*, 237–238) may be briefly extracted as follows:

Referred to Fig. 6, arc $PE = A$; arc $PQ = B$

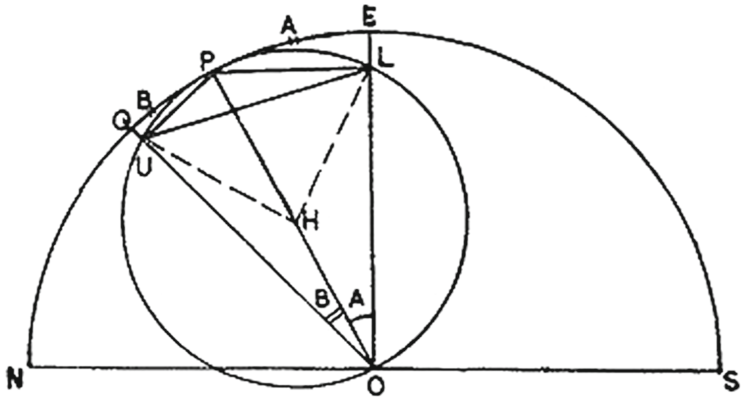

Fig. 6 Proof of addition theorem for Sine

Thus PL and OL are Sine and Cosine of A, and PU and OU those of B. By applying the *bhujā-pratibhujā* rule (i.e. Ptolemy's theorem) to the cyclic quadrilateral $LPUO$, we get

$$PL\cdot OU + OL\cdot PU = LU\cdot OP$$

or

$$(R\sin A)\cdot(R\cos B) + (R\cos A)\cdot(R\sin B) = LU\cdot R.$$

Now LU is the full chord of the arc $(LP+PU)$ in the circle which circumscribes the quadrilateral $LPUO$ whose radius is $\frac{R}{2}$. Thus

$$\begin{aligned} LU &= 2\cdot\left(\frac{R}{2}\right)\cdot\sin\left\{\frac{(2A+2B)}{2}\right\} \\ &= R\sin(A+B) \end{aligned}$$

Hence the theorem follows from the above relation.

10 Solution of Spherical Triangles

Consider the spherical triangle SPZ on the celestial sphere, where S, P, Z are the positions of the sun, north pole and the zenith. It has following five useful elements:

(i) side PZ, the co-latitude, $90^\circ - \phi$
(ii) side ZS, the co-altitude, $90^\circ - \alpha$
(iii) side SP, the co-declination, $90^\circ - \delta$
(iv) angle SPZ, the hour angle, H
(v) angle PZS, the azimuth (measured from the north), $A = 90^\circ \pm \gamma$.

When any three out of the above five elements are given, the remaining two can be found out. This leads to ten different cases all of which are discussed systematically in the TS (Chapter III).[40] The rules are fully explained and the examples worked out in the commentary of TS.[41]

The solutions given are equivalent to what one will get by using modern spherical trigonometry. We have already shown this in our detailed paper.[42] Here we take, as a sample, the case II, namely, given H, A and ϕ, to find α and δ. For finding the altitude TS, III, 68–71, says:

> The product of the Sine of the hour angle and Cosine of the latitude divided by the radius is the Sine of the local hour angle. Divide the product of the Sines of the hour angle and the latitude by the Cosine of the local hour angle and multiply by the Sine of the directional amplitude (azimuthal angle measured from the east). The result should be added to, in case the directional amplitude is towards north, or subtracted from, in case the directional amplitude is towards south, the product of their *'koṭis'* (i.e. their uprights when they are taken as bases and the radius is taken as the hypotenuse in each case). The result (now obtained) be divided by the radius and the quotient (thus obtained) multiplied by the Cosine of the local hour angle be put separately (at two places).
>
> At one place divide (the quantity) by the radius and add the square (of the result) to the square of the sine of the local hour angle. By the square root of that (the sum of the squares just obtained) divide the quantity (placed) separately. (The final result) becomes the Gnomon (the Sine of the altitude)....

That is

$$\text{Sine of the local hour angle} = \frac{(R\sin H)\cdot(R\cos\phi)}{R} = J, \text{ say}$$

$$\text{Cosine of the local hour angle} = \sqrt{R^2 - J^2} = C, \text{ say}$$

Then we form the quantity

$$\left(\frac{C}{R}\right)\cdot\left[R\cos\gamma\cdot\sqrt{R^2 - \left\{\frac{(R\sin H)\cdot(R\sin\phi)}{C}\right\}^2} \pm \frac{R\sin\gamma\cdot(R\sin H)\cdot(R\sin\phi)}{C}\right] = Q, \text{say}$$

The rule then gives

$$R\sin\alpha = \frac{Q}{\sqrt{J^2 + \left(\frac{Q}{R}\right)^2}}.$$

By combining the various steps, the solution given can be seen to be equivalent to

$$\sin\alpha = \frac{\sin A\cdot\cos H + \cos A\cdot\sin H\cdot\sin\phi}{\sqrt{(\sin H\cdot\cos\phi)^2 + (\sin A\cdot\cos H + \cos A\cdot\sin H\cdot\sin\phi)^2}}$$

which is a transformed form of the following result obtained by using the modern cotangent formula of spherical trigonometry:

$$\tan\alpha\cdot\cos\phi = \sin A\cdot\cot H + \cos A\cdot\sin\phi.$$

The complicated Indian solution does not make use of tangent and cotangent functions. Another complication results by deriving the rules from working 'inside' the celestial sphere instead of 'on its surface'.

We have brought out these complicated rules into modern forms but the more challenging task of publishing, in modern forms, their ancient rationales from the south Indian elaborate commentary of TS still remains to be done.[43]

11 Series for π

Several such series were known to medieval south Indians. We shall give a few of them here. One verse, which is found in the *Gaṇita-Yuktibhāṣā (= GYB),* the *Yuktidīpikā* commentary or *Vyākhyā* on TS (= TSV), and in the KP, says:[44]

व्यासाद्वारिधिनिहतात् पृथगाप्तं त्र्याद्ययुग्विमूलघनैः ।
त्रिघ्नव्यासे स्वमृणं क्रमशः कृत्वापि परिधिरानेयः ॥

The circumference is likewise obtained when four times the diameter is (successively) divided by the cubes of the odd numbers, beginning with 3, diminished by these numbers themselves, and the (respective) quotients are alternately added to, and subtracted from, the diameter multiplied by three.

That is,

$$C = \pi D = 3D + \frac{4D}{3^3-3} - \frac{4D}{5^3-5} + \frac{4D}{7^3-7} - \ldots$$

Another Sanskrit text says[45]

The fifth powers of odd numbers are increased by 4 times themselves; 16 times the diameter is successively divided by all such numbers (so obtained); the results (of division) of odd rank are added and those of even rank are subtracted. The circumference corresponding to the diameter is (thereby) obtained.

That is

$$\pi D = \frac{16D}{1^5+4\cdot 1} - \frac{16D}{3^5+4\cdot 3} + \frac{16D}{5^5+4\cdot 5} - \ldots$$

Now to give a sample of finite series, we have Sanskrit stanzas which contain series[46]

$$\pi D \approx 4D - \frac{4D}{3} + \frac{4D}{5} - \ldots \mp \frac{4D}{n} \pm 4D \cdot f(n)$$

where

$$f(n) = \frac{\frac{(n+1)}{2}}{(n+1)^2 + 1}$$

or, more accurately,

$$f(n) = \frac{[\frac{(n+1)}{2}]^2 + 1}{[(n+1)^2 + 5]\frac{(n+1)}{2}}.$$

These two closer approximations are also stated in the sixteenth century Malayalam work *YB* which is said to have given a third form of the closing term but as an intermediate step (see *AHES*, Vol 18, pp. 89–90; cf. ref. 44). All these can be put in an alternative modern form as

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{n} \mp f_i(n+1)$$

where

$$f_1(n) = \frac{1}{(2n)}$$

$$f_2(n) = \frac{\left(\frac{n}{2}\right)}{(n^2+1)}$$

and

$$f_3(n) = \frac{\left(\frac{n^2}{4}\right) + 1}{\left(\frac{n}{2}\right) \cdot (n^2+5)}$$

It is said that Prince Rāma Varma of Cochin had verified, by taking $n = 55$, that π obtained by f_1, f_2 and f_3 is correct to 1, 6 and 10 decimal places, respectively.

Rounding off $f_i(n)$ enormously improves slowness of convergence of the series used. It is therefore rightly remarked that apparently "we have lost some wider theory of accelerating the convergence of singly infinite series by applying such corrections as have now come down to us in medieval Hindu texts the *TS* and *YB*" (*AHES*, Vol. 35, p. 99).

12 Series for Sine and Cosine

For the Sine series we quote the following stanzas (nos. 440–441) from the *TSV* of Saṅkara Vāriar (1st half of sixteenth century)[47].

निहत्य चापवर्गेण चापं तत्तत्फलानि च ।
हरेत् समूलयुग्वर्गैस्त्रिज्यावर्गहतैः क्रमात् ॥
चापं फलानि चाधोऽधोऽन्यस्योपर्युपरि त्यजेत् ।
जीवाप्यै संग्रहोऽस्यैवं 'विद्वान्' इत्यादिनाकृतः ॥

The arc is to be repeatedly multiplied by the square of itself and is to be divided (in order) by the square of each even number increased by itself and multiplied by the square of the radius. The arc and the terms obtained by these repeated operations are to be placed in sequence in a column and any last term is to be subtracted from the next above, the remainder from the term then next above, and so on, to obtain the Sine of the arc. It was this procedure which was briefly mentioned in the verse starting with *'Vidvān'*.

That is,

$$\sin s = s - s \cdot \frac{s^2}{(2^2+2)r^2} + s \cdot \frac{s^2}{(2^2+2)\cdot r^2} \cdot \frac{s^2}{(4^2+4)\cdot r^2} - \ldots$$

The interesting reference to another set of rules starting with *'Vidvān'* points out to a pair of famous stanzas which contain the rule[48]

$$\sin s = s - t^3[Q_5 - t^2[Q_4 - t^2\{Q_3 - t^2(Q_2 - t^2 Q_1)\}]]$$

where

$$\begin{aligned} t &= \frac{s}{h} \\ h &= 5400;0,0 \\ R &= 3437;44,48 \\ Q_1 &= 0;00,44 \\ Q_2 &= 0;33,06 \\ Q_3 &= 16;05,41 \\ Q_4 &= 273;57,47 \\ Q_5 &= 2220;39,40. \end{aligned}$$

The important point to note is that the *'Vidvān'* stanzas are attributed to the famous Mādhava (c. 1400) of Saṅgamagrāma (near Cochin) in the *NAB* (Part I, p. 113). This shows that the power series expansion of sine was known already in India more than 200 years before it was known in Europe.

There are corresponding stanzas[49] for the cosine series which are given side by side with the above rules.

South Indian medieval derivations of these series have been published and to reproduce them here will take us too far.[50]

13 The Mādhava-Gregory Series

Three and a half Sanskrit couplets containing series for inverse tangent function have been translated as:[51]

> The product of the given Sine and the radius divided by the Cosine is the first result. From the first, (and then, second, third), etc., results, obtain (successively) a sequence of results by taking repeatedly the square of the Sine as the multiplier and the square of the Cosine as the divisor. Divide (the above results) in order by the odd numbers one, three, etc., (to get the full sequence of terms). From the sum of the odd terms, subtract the sum of the even terms. (The result) becomes the arc. In this connection, it is laid down that the (Sine of the) arc or (that of) its complement, whichever is smaller, should be taken here (as the 'given Sine'); otherwise, the terms, obtained by the (above) repeated process will not tend to the vanishing-magnitude.

That is,

$$R\theta = \frac{R\cdot(R\sin\theta)^1}{1\cdot(R\cos\theta)^1} - \frac{R\cdot(R\sin\theta)^3}{3\cdot(R\cos\theta)^3} + \frac{R\cdot(R\sin\theta)^5}{5\cdot(R\cos\theta)^5} - \dots$$

or

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots.$$

The Sanskrit text translated is found in the KKK as well as TSV both of which belong to the first half of sixteenth century while James Gregory knew the series about 1670 in Europe.

But there are reasons to believe that the series was known still one more century earlier to Mādhava (c. 1340–1425). First, the KKK (p. 379) expressly attributes to him certain Sanskrit stanzas which contain the series[52]

$$\pi D = 4D - \frac{4D}{3} + \frac{4D}{5} - \dots \mp \frac{4D}{n} \pm 4D\cdot f(n)$$

where (see Sect. 11)

$$f(n) = \frac{\frac{(n+1)}{2}}{(n+1)^2+1}.$$

This shows that Mādhava knew the Gregory series for $\theta = \frac{\pi}{4}$. Also the YB (pp. 113–116) proves both these (the general and the particular cases) without making any distinction between them by essentially same approach.[53]

This south Indian proof starts with a geometrical derivation of a rule which is basically equivalent to

$$d\theta = \frac{d(\tan\theta)}{1+\tan^2\theta}.$$

Then it consists of steps which are equivalent to expansion and term-by-term integration in modern analysis. Since the details of the proof (from the Malayalam YB) which belongs to pre-calculus period have been already brought out in modern

notation, it is not necessary to reproduce it here.[54] The Indian proofs of $\sin x, \cos x$, and $\tan^{-1} x$ show that the seeds of modern analysis were first sown in India.

14 Taylor Series for the Sine and Miscellaneous Matters

Rules equivalent to second-order Taylor series expansions of the sine and the cosine were known to Mādhava whose Sanskrit stanzas are given in the *TS*, II, 10–13 (p. 112) and quoted in the *NAB* (Part I, pp. 54–55).[55]

A sort of third-order Taylor series expansion of the sine is implied in the following verses which are quoted anonymously by Parameśvara of Kerala (c. 1360–1460) in his supercommentary on the *Mahābhāskarīya*.[56]

> दोश्चापखण्ड-संभक्तं व्यासार्धं भाजको भवेत् ।
> दोर्ज्यामतीतचापान्ते कोटिज्यां च न्यसेत् पुनः ॥
> कोटिज्यातो भाजकेन लब्धस्यार्धेन संयुतात् ।
> दोर्गुणाद्, भाजकाप्तार्धं हित्वा कोटिगुणात् पुनः ॥
> तस्मादाप्तं भाजकेन दोर्ज्याखण्डः स्फुटो भवेत् ।
> चापान्तदोर्ज्या तद्युक्ता स्यादिष्टज्या भुजोद्भवा ॥
>
> The semi-diameter divided by the residual arc becomes the divisor. Put down the Sine and again the Cosine at the end of the arc traversed. From the Cosine, subtract half the quotient obtained from the divisor-divided Sine (which is) increased by half the quotient obtained from the Cosine by the divisor. Again (the quotient) obtained from that (above difference) by dividing by the divisor becomes the true Sine-difference. The Sine at the end of the arc traversed increased by that (true Sine-difference) becomes the desired Sine for a (given) arc.

That is, let

$$\text{divisor} = \frac{R}{\theta} = D, \text{ say;}$$
$$\text{true Sine-difference} = R\sin(\alpha+\theta) - R\sin\alpha.$$

Then the above rule gives[57]

$$R\sin(\alpha+\theta) = R\sin\alpha + \left[R\cos\alpha - \left\{R\sin\alpha + \frac{R\cos\alpha}{2D}\right\}\frac{1}{2D}\right]\frac{1}{D}$$

which on simplification becomes

$$R\sin(\alpha+\theta) = R\sin\alpha + \left(\frac{\theta}{R}\right)\cdot R\cos\alpha - \left(\frac{\theta}{R}\right)^2\cdot\frac{(R\sin\alpha)}{2} - \left(\frac{\theta}{R}\right)^3\cdot\frac{(R\cos\alpha)}{4}.$$

Putting $R = 1$, this becomes the third-order Taylor Series approximation of the sine function except that we have 4 instead of 6 in the last term.

No doubt the Indian form is far from being the true Taylor series expansion of 4 terms; the fact that it was given more than two centuries before Taylor expansion

was discovered by Gregory (about 1668) is interesting.[58] Parameśvara's supercommentary, called *Siddhānta-dīpikā*, contains several other rules for computations of trigonometric functions.

The *TSV* verses 455–456 (p. 119) contain the series for the square of the sine function equivalent to

$$\sin^2 x = x^2 - \frac{x^4}{\left(2^2 - \frac{2}{2}\right)} + \frac{x^6}{\left(2^2 - \frac{2}{2}\right)\left(3^2 - \frac{3}{2}\right)} - \frac{x^8}{\left(2^2 - \frac{2}{2}\right)\left(3^2 - \frac{3}{2}\right)\left(4^2 - \frac{4}{2}\right)} + \ldots$$

This topic of *jyā-vargānayanam* is also found treated in the sixteenth century Malayalam work *YB* (Part I, pp. 203–206) as has been mentioned elsewhere.[59]

Lastly, it may be pointed out that the possibility of a principle to frame the transcendental number *e* (the base of natural logarithms) in the medieval Indian mathematics has been discussed. More than a decade ago scholars have exemplified the mechanism of the motivational kinetics by examining the possibility that the basic constant *e* of mathematical analysis would have appeared as a definite object at the horizon of mathematical thinking of medieval India.[60]

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Paper 39

Ranjan Roy (1990)

The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha

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ARTICLES

The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha

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1. Introduction

The formula for π mentioned in the title of this article is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \tag{1}$$

One simple and well-known modern proof goes as follows:

$$\begin{aligned}\arctan x &= \int_0^x \frac{1}{1+t^2}\,dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1}\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt.\end{aligned}$$

The last integral tends to zero if $|x| \leqslant 1$, for

$$\left|\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt\right| \leqslant \left|\int_0^x t^{2n+2}\,dt\right| = \frac{|x|^{2n+3}}{2n+3} \to 0 \quad \text{as } n \to \infty.$$

Thus, arctan x has an infinite series representation for $|x| \leqslant 1$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots . \tag{2}$$

The series for $\pi/4$ is obtained by setting $x = 1$ in (2). The series (2) was obtained independently by Gottfried Wilhelm Leibniz (1646–1716), James Gregory (1638–1675) and an Indian mathematician of the fourteenth century or probably the fifteenth century whose identity is not definitely known. Usually ascribed to Nilakantha, the Indian proof of (2) appears to date from the mid-fifteenth century and was a consequence of an effort to rectify the circle. The details of the circumstances and ideas leading to the discovery of the series by Leibniz and Gregory are known. It is interesting to go into these details for several reasons. The infinite series began to play a role in mathematics only in the second half of the seventeenth century. Prior to that, particular cases of the infinite geometric series were the only ones to be used. The arctan series was obtained by Leibniz and Gregory early in their study of infinite series and, in fact, before the methods and algorithms of calculus were fully developed. The history of the arctan series is, therefore, important because it reveals early ideas on series and their relationship with quadrature or the process of finding the area under a curve. In the case of Leibniz, it is possible to see how he used and

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transformed older ideas on quadrature to develop his methods. Leibniz's work, in fact, was primarily concerned with quadrature; the $\pi/4$ series resulted (in 1673) when he applied his method to the circle. Gregory, by comparison, was interested in finding an infinite series representation of any given function and discovered the relationship between this and the successive derivatives of the given function. Gregory's discovery, made in 1671, is none other than the Taylor series; note that Taylor was not born until 1685. The ideas of calculus, such as integration by parts, change of variables, and higher derivatives, were not completely understood in the early 1670s. Some particular cases were known, usually garbed in geometric language. For example, the fundamental theorem of calculus was stated as a geometric theorem in a work of Gregory's written in 1668. Similar examples can also be seen in a book by Isaac Barrow, Newton's mentor, published in 1670. Of course, very soon after this transitional period, Leibniz began to create the techniques, algorithms and notations of calculus as they are now known. He had been preceded by Newton, at least as far as the techniques go, but Newton did not publish anything until considerably later. It is, therefore, possible to see how the work on arctan was at once dependent on earlier concepts and a transitional step toward later ideas.

Finally, although the proofs of (2) by Leibniz, Gregory and Nilakantha are very different in approach and motivation, they all bear a relation to the modern proof given above.

2. Gottfried Wilhelm Leibniz (1646–1716)

Leibniz's mathematical background[1] at the time he found the $\pi/4$ formula can be quickly described. He had earned a doctor's degree in law in February 1667, but had studied mathematics on his own. In 1672, he was a mere amateur in mathematics. That year, he visited Paris and met Christiaan Huygens (1629–1695), the foremost physicist and mathematician in continental Europe. Leibniz told the story of this meeting in a 1679 letter to the mathematician Tschirnhaus, "at that time... I did not know the correct definition of the center of gravity. For, when by chance I spoke of it to Huygens, I let him know that I thought that a straight line drawn through the center of gravity always cut a figure into two equal parts,... Huygens laughed when he heard this, and told me that nothing was further from the truth. So I, excited by this stimulus, began to apply myself to the study of the more intricate geometry, although as a matter of fact I had not at that time really studied the *Elements* [Euclid]... Huygens, who thought me a better geometer than I was, gave me to read the letters of Pascal, published under the name of Dettonville; and from these I gathered the method of indivisibles and centers of gravity, that is to say the well-known methods of Cavalieri and Guldinus."[2]

[1]For further information about Leibniz's mathematical development, the reader may consult: J. E. Hofmann, *Leibniz in Paris 1672–1676* (Cambridge: The Cambridge University Press, 1974) and its review by A. Weil, *Collected Papers* Vol. 3 (New York: Springer-Verlag, 1979). An English translation of Leibniz's own account, *Historia et origo calculi differentialis*, can be found in J. M. Child, *The Early Mathematical Manuscripts of Leibniz* (Chicago: Open Court, 1920). An easily available synopsis of Leibniz's work in calculus is given in C. H. Edwards, Jr., *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979).

[2]*The Early Mathematical Manuscripts*, p. 215.

Bonaventura Cavalieri (1598–1647) published his *Geometria Indivisibilibus* in 1635. This book was very influential in the development of calculus. Cavalieri's work indicated that

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The study of Pascal played an important role in Leibniz's development as a mathematician. It was from Pascal that he learned the ideas of the "characteristic triangle" and "transmutation." In order to understand the concept of transmutation, suppose A and B are two areas (or volumes) which have been divided up into indivisibles usually taken to be infinitesimal rectangles (or prisms). If there is a one-to-one correspondence between the indivisibles of A and B and if these indivisibles have equal areas (or volumes), then B is said to be obtained from A by transmutation and it follows that A and B have equal areas (or volumes). Pascal had also considered infinitesimal triangles and shown their use in finding, among other things, the area of the surface of a sphere. Leibniz was struck by the idea of an infinitesimal triangle and its possibilities. He was able to derive an interesting transmutation formula, which he then applied to the quadrature of a circle and thereby discovered the series for π. To obtain the transmutation formula, consider two neighboring points $P(x, y)$, and $Q(x + dx, y + dy)$ on a curve $y = f(x)$. First Leibniz shows that area $(\Delta OPQ) = (1/2)$ area (rectangle $(ABCD)$). See FIGURE 1. Here PT is tangent to $y = f(x)$ at P and OS is perpendicular to PT. Let p denote the length of OS and z that of $AC = BD$ = ordinate of T.

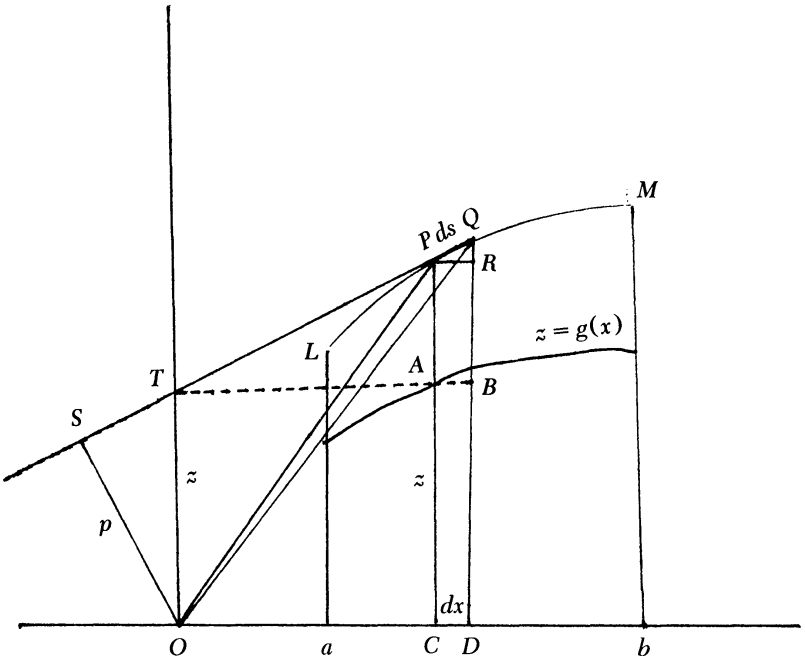

FIGURE 1

$$\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1},$$

when n is a positive integer.

Blaise Pascal (1623–1662) made important and fundamental contributions to projective geometry, probability theory and the development of calculus. The work to which Leibniz refers was published in 1658 and contains the first statement and proof of

$$\int_{\theta_0}^{\theta} \sin\phi\,d\phi = \cos\theta_0 - \cos\theta.$$

This proof is presented in D. J. Struik's *A Source Book in Mathematics 1200–1800* (Cambridge: Harvard University Press, 1969), p. 239.

Paul Guldin (1577–1643), a Swiss mathematician of considerable note, contributed to the development of calculus, and his methods were generally more rigorous than those of Cavalieri.

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Since ΔOST is similar to the characteristic ΔPQR,

$$\frac{dx}{p} = \frac{ds}{z},$$

where ds is the length of PQ. Thus,

$$\text{area } (OPQ) = \frac{1}{2}pds = \frac{1}{2}zdx. \tag{3}$$

Now, observe that for each point P on $y = f(x)$ there is a corresponding point A. Thus, as P moves from L to M, the points A form a curve, say $Z = g(x)$. If sector OLM denotes the closed region formed by $y = f(x)$ and the straight lines OL and OM, then (3) implies that

$$\text{area (sector } OLM) = \frac{1}{2}\int_a^b g(x)\,dx. \tag{4}$$

This is the transmutation formula of Leibniz. From (4), it follows that the area under $y = f(x)$ is

$$\begin{aligned}\int_a^b y\,dx &= \frac{b}{2}f(b) - \frac{a}{2}f(a) + \text{area (sector } OLM)\\ &= \frac{1}{2}\left([xy]_a^b + \int_a^b z\,dx\right).\end{aligned} \tag{5}$$

This is none other than a particular case of the formula for integration by parts. For it is easily seen from FIGURE 1 that

$$z = y - x\frac{dy}{dx}. \tag{6}$$

Substituting this value of z in (5), it follows that

$$\int_a^b y\,dx = [xy]_a^b - \int_{f(a)}^{f(b)} x\,dy,$$

which is what one gets on integration by parts.

Now consider a circle of radius 1 and center $(1, 0)$. Its equation is $y^2 = 2x - x^2$. In this case, (6) implies that

$$z = \sqrt{2x - x^2} - \frac{x(1-x)}{\sqrt{2x - x^2}} = \sqrt{\frac{x}{2-x}} = \frac{x}{y}, \tag{7}$$

so that

$$x = \frac{2z^2}{1 + z^2}. \tag{8}$$

In FIGURE 2, let $A\hat{O}B = 2\theta$. Then the area of the sector $AOB = \theta$ and

$$\theta = \text{area } (\triangle AOB) + \text{area (region between arc } AB \text{ and line } AB). \tag{9}$$

By the transmutation formula (4), the second area is $\frac{1}{2}\int_0^x z\,dt$ where z is given by (7). Now, from FIGURE 3 below it is seen that

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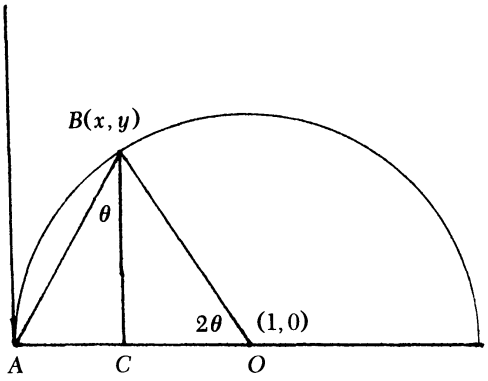

FIGURE 2

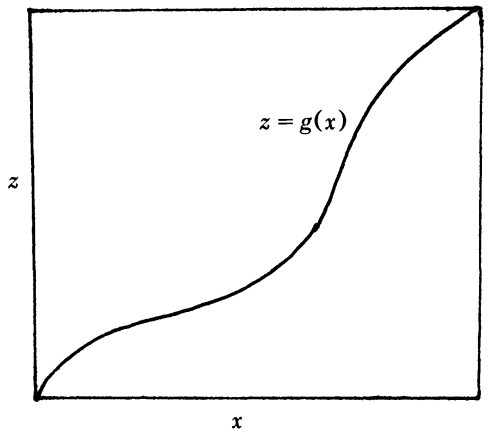

FIGURE 3

$$\frac{1}{2}\int_0^x z\,dt = \frac{1}{2}\left(xz - \int_0^z x\,du\right). \tag{10}$$

Using (8) and (10), it is now possible to rewrite (9) as

$$\begin{aligned}\theta &= \tfrac{1}{2}y + \tfrac{1}{2}xz - \int_0^z \frac{t^2}{1+t^2}\,dt \\ &= \tfrac{1}{2}[z(2-x) + xz] - \int_0^z \frac{t^2}{1+t^2}\,dt \quad (\text{since } y = z(2-x)) \\ &= z - \int_0^z \frac{t^2}{1+t^2}\,dt.\end{aligned}$$

At this point, Leibniz was able to use a technique employed by Nicolaus Mercator (1620–1687). The latter had considered the problem of the quadrature of the hyperbola $y(1+x) = 1$. Since it was already known that

$$\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1},$$

he solved the problem by expanding $1/(1+x)$ as an infinite series and integrating term by term. He simultaneously had the expansion for $\log(1+x)$. Mercator published this result in 1668, though he probably had obtained it a few years earlier. A year later, John Wallis (1616–1703) determined the values of x for which the series is valid. Thus, Leibniz found that

$$\theta = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots. \tag{11}$$

In FIGURE 2, $A\hat{B}C = \theta$ and $z = x/y = \tan\theta$. Therefore, (11) is the series for $\arctan z$.

Of course, Leibniz did not invent the notation for the integral and differential used above until 1675, and his description of the procedures is geometrical but the ideas are the same.

The discovery of the infinite series for π was Leibniz's first great achievement. He communicated his result to Huygens, who congratulated him, saying that this remarkable property of the circle will be celebrated among mathematicians forever. Even Isaac Newton (1642–1727) praised Leibniz's discovery. In a letter of October 24, 1676, to Henry Oldenburg, secretary of the Royal Society of London, he writes,

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"Leibniz's method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else."[3] Of course, for Leibniz this was only a first step to greater things as he himself says in his "Historia et origo calculi differentialis."

3. James Gregory (1638–1675)

Leibniz had been anticipated in the discovery of the series for arctan by the Scottish mathematician, James Gregory, though the latter did not note the particular case for $\pi/4$.[4] Since Gregory did not publish most of his work on infinite series and also because he died early and worked in isolation during the last seven years of his life, his work did not have the influence it deserved. Gregory's early scientific interest was in optics about which he wrote a masterly book at the age of twenty-four. His book, the *Optica Promota*, contains the earliest description of a reflecting telescope. It was in the hope, which ultimately remained unfulfilled, of constructing such an instrument that he travelled to London in 1663 and made the acquaintance of John Collins (1624–1683), an accountant and amateur mathematician. This friendship with Collins was to prove very important for Gregory when the latter was working alone at St. Andrews University in Scotland. Collins kept him abreast of the work of the English mathematicians such as Isaac Newton, John Pell (1611–1685) and others with whom Collins was in contact.[5]

Gregory spent the years 1664–1668 in Italy and came under the influence of the Italian school of geometry founded by Cavalieri. It was from Stefano degli Angeli (1623–1697), a student of Cavalieri, that Gregory learned about the work of Pierre de Fermat (1601–1665), Cavalieri, Evangelista Torricelli (1608–1647) and others. While in Italy, he wrote two books: *Vera Circuli et Hyperbolae Quadratura* in 1667, and *Geometriae Pars Universalis* in 1668. The first book contains some highly original ideas. Gregory attempted to show that the area of a general sector of an ellipse, circle or hyperbola could not be expressed in terms of the areas of the inscribed and circumscribed triangle and quadrilateral using arithmetical operations and root extraction. The attempt failed but Gregory introduced a number of important ideas such as convergence and algebraic and transcendental functions. The second book contains the first published statement and proof of the fundamental theorem of calculus in geometrical form. It is known that Newton had discovered this result not later than 1666, although he did not make it public until later.

Gregory returned to London in the summer of 1668; Collins then informed him of the latest discoveries of mathematicians working in England, including Mercator's recently published proof of

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

[3]See H. W. Turnbull (ed.), *The Correspondence of Isaac Newton* (Cambridge: The University Press, 1960), Vol. 2, p. 130.

[4]Peter Beckmann has persuasively argued that Gregory must have known the series for $\pi/4$ as well. See Beckmann's *A History of Pi* (Boulder, Colorado: The Golem Press, 1977), p. 133.

[5]The reader might find it of interest to consult: H. W. Turnbull (ed.), *James Gregory Tercentenary Memorial Volume* (London: G. Bell, 1939). This volume contains Gregory's scientific correspondence with John Collins and a discussion of the former's life and work.

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Meditation on these discoveries led Gregory to publish his next book, *Exercitationes Geometricae*, in the winter of 1668. This is a sequel to the *Pars Universalis* and is mainly about the logarithmic function and its applications. It contains, for example, the first evaluations of the indefinite integrals of $\sec x$ and $\tan x$.[6] The results are stated in geometric form.

In the autumn of 1668, Gregory was appointed to the chair in St. Andrews and he took up his duties in the winter of 1668/1669. He began regular correspondence with Collins soon after this, communicating to him his latest mathematical discoveries and requesting Collins to keep him informed of the latest activities of the English mathematicians. Thus, in a letter of March 24, 1670, Collins writes, "Mr. Newtone of Cambridge sent the following series for finding the Area of a Zone of a Circle to Mr. Dary, to compare with the said Dary's approaches, putting R the radius and B the parallel distance of a Chord from the Diameter the Area of the space or Zone between them is $=$

$$2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5} - \frac{5B^9}{576R^7} \text{ &c.}"^{7}$$

This is all Collins writes about the series but it is, in fact, the value of the integral $2\int_0^B (R^2 - x^2)^{1/2}\,dx$ after expanding by the binomial theorem and term by term integration. Newton had discovered the binomial expansion for fractional exponents in the winter of 1664/1665, but it was first made public in the aforementioned letter of 1676 to Oldenburg.

There is evidence that Gregory had rediscovered the binomial theorem by 1668.[8] However, it should be noted that the expansion for $(1-x)^{1/2}$ does not necessarily

[6]A proof of the formula

$$\int_0^\theta \sec\phi\, d\phi = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

was of considerable significance and interest to mathematicians in the 1660's due to its connection with a problem in navigation. Gerhard Mercator (1512–1594) published his engraved "Great World Map" in 1569. The construction of the map employed the famous Mercator projection. Edward Wright, a Cambridge professor of mathematics, noted that the ordinate on the Mercator map corresponding to a latitude of θ^0 on the globe is given by $c\int_0^\theta \sec\phi\,\mathrm{d}\phi$, where c is suitably chosen according to the size of the map. In 1599, Wright published this result in his *Certaine Errors in Navigation Corrected*, which also contained a table of latitudes computed by the continued addition of the secants of $1', 2', 3'$, etc. This approximation to $\int_0^\theta \sec\phi\,\mathrm{d}\phi$ was sufficiently exact for the mariner's use. In the early 1640's, Henry Bond observed that the values in Wright's table could be obtained by taking the logarithm of $\tan(\pi/4 + \theta/2)$. This observation was published in 1645 in Richard Norwood's *Epitome of Navigation*. A theoretical proof of this observation was very desirable and Nicolaus Mercator had offered a sum of money for its demonstration in 1666. John Collins, who had himself written a book on navigation, drew Gregory's attention to this problem and, as we noted, Gregory supplied a proof. For more details, one may consult the following two articles by F. Cajori: "On an Integration ante-dating the Integral Calculus," *Bibliotheca Mathematica* Vol. 14 (1913/14), pp. 312–19, and "Algebra In Napier's Day and Alleged Prior Invention of Logarithms," in C. G. Knott (ed.), *Napier Memorial Volume* (London: Longmans, Green & Co., 1915), pp. 93–106. More recently, J. Lohne has established that Thomas Harriot (1560–1621) had evaluated the integral $\int_0^\theta \sec\phi\,\mathrm{d}\phi$ in 1594 by a stereographic projection of a spherical loxodrome from the south pole into a logarithmic spiral. This work was unpublished and remained unknown until Lohne brought it to light. See J. A. Lohne, "Thomas Harriot als Mathematiker," *Centaurus*, Vol. 11, 1965–66, pp. 19–45. Thus it happened that, although $\int \sec\theta\,\mathrm{d}\theta$ is a relatively difficult trigonometric integral, it was the first one to be discovered.

[7]*James Gregory*, p. 89.

[8]See *The Correspondence of Isaac Newton*, Vol. 1, p. 52, note 1.

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imply a knowledge of the binomial theorem. Newton himself had proved the expansion by applying the well-known method for finding square roots of numbers to the algebraic expression $1-x$. Moreover, it appears that the expansion of $(1-x)^{1/2}$ was discovered by Henry Briggs (1556–1630) in the 1620's, while he was constructing the log tables.[9] But there is no indication that Gregory or Newton knew of this. In any case, for reasons unknown, Gregory was unable to make anything of the series—as evidenced by his reply of April 20, "I cannot understand the series you sent me of the circle, if this be the original, I take it to be no series."[10] However, by September 5, 1670, he had discovered the general interpolation formula, now called the Gregory-Newton interpolation formula, and had made from it a number of remarkable deductions. He now knew how "to find the sinus having the arc and to find the number having the logarithm." The latter result is precisely the binomial expansion for arbitrary exponents. For, if we take x as the logarithm of y to the base $1+d$, then $y=(1+d)^x$ and Gregory gives the solution as

$$(1+d)^x = 1 + xd + \frac{x(x-1)}{1\cdot 2}d^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}d^3 + \cdots .$$[11]

It is possible that Newton's series in Collins' letter had set Gregory off on the course of these discoveries, but he did not even at this point see that he could deduce Newton's result. Soon after, he did observe this and wrote on December 19, 1670, "I admire much my own dullness, that in such a considerable time I had not taken notice of this."[12] All this time, he was very eager to learn about Newton's results on series and particularly the methods he had used. Finally on December 24, 1670, Collins sent him Newton's series for $\sin x$, $\cos x$, $\sin^{-1}x$ and $x\cot x$, adding that Newton had a universal method which could be applied to any function. Gregory then made a concentrated effort to discover a general method for himself. He succeeded. In a famous letter of February 15, 1671 to Collins he writes, "As for Mr. Newton's universal method, I imagine I have some knowledge of it, both as to geometrick and mechanick curves, however I thank you for the series ye sent me and send you these following in requital."[13] Gregory then gives the series for $\arctan x$, $\tan x$, $\sec x$, $\log\sec x$, $\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, $\operatorname{arcsec}(\sqrt{2}\,e^x)$, and $2\arctan\tanh x/2$. However, what he had found was not Newton's method but rather the Taylor expansion more than forty years before Brook Taylor (1685–1731). Newton's method consisted of reversion of series, expansion by the binomial theorem, long division by series and term by term integration.[14] Thinking that he had rediscovered Newton's method, Gregory did not

[9] See D. T. Whiteside, "Henry Briggs: The Binomial theorem Anticipated," *The Mathematical Gazette*, Vol. 15, (1962), p. 9. Whiteside shows how the expansion of $(1+x)^{1/2}$ arose out of Brigg's work on logarithms.

[10] *James Gregory*, p. 92.

[11] In their review of the Gregory Memorial Volume, M. Dehn and E. Hellinger explain how the binomial expansion comes out of the interpolation formula. See *The American Mathematical Monthly*, Vol. 50, (1943), p. 149.

[12] *James Gregory*, p. 148.

[13] *Ibid.*, p. 170.

[14] It should be mentioned that Newton himself discovered the Taylor series around 1691. See D. T. Whiteside (ed.), *The Mathematical Papers of Isaac Newton*, Vol. VII (Cambridge: The Cambridge University Press, 1976), p. 19. In fact, Taylor was anticipated by at least five mathematicians. However, the

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publish his results. It is only from notes that he made on the back of a letter from Gedeon Shaw, an Edinburgh stationer, dated January 29, 1671, that it is possible to conclude that Gregory had the idea of the Taylor series. These notes contain the successive derivatives of $\tan x$, $\sec x$, and the other functions whose expansions he sent to Collins. The following extract from the notes gives the successive derivatives of $\tan\theta$; here m is successively y, $\frac{dy}{d\theta}$, $\frac{d^2y}{d\theta^2}$, etc., and $q = r\tan\theta$. Gregory writes[15]:

1st $m = q$

2nd $m = r + \frac{q^2}{r}$

3rd $m = 2q + \frac{2q^3}{r^2}$

4th $m = 2r + \frac{8q^2}{r} + \frac{6q^4}{r^3}$

5th $m = 16q + \frac{40q^3}{r^2} + \frac{24q^5}{r^4}$

6th $m = 16r + \frac{136q^2}{r} + \frac{240q^4}{r^3} + \frac{120q^6}{r^5}$

7th $m = 272q + 987\frac{q^3}{r^2} + 1680\frac{q^5}{r^4} + 720\frac{q^7}{r^6}$

8th $m = 272r + 3233\frac{q^2}{r} + 11361\frac{q^4}{r^3} + 13440\frac{q^6}{r^5} + 5040\frac{q^8}{r^7}.$

It is clear from the form in which the successive derivatives are written that each one is formed by multiplying the derivative with respect to q of the preceding term by $r + \frac{q^2}{r}$. Now writing $a = r\theta$, Gregory gives the series in the letter to Collins as follows:

$$r\tan\theta = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{3233a^9}{181440r^8} + \cdots.$$

The reasons for supposing that these notes were written not much before he wrote to Collins and were used to construct the series are (i) the date of Gedeon Shaw's letter and (ii) Gregory's error in computing the coefficient of $\frac{q^3}{r^2}$ in the 7th m, which should be 1232 instead of 987 and which, in turn, leads to the error in the 8th m, where the coefficient of $\frac{q^2}{r}$ should be 3968 instead of 3233. This error is then repeated in the series showing the origin of the series. Moreover, in the early parts of the notes, Gregory is unsure about how he should write the successive derivatives. For example, he attempts to write the derivative of $\sec\theta$ as a function of $\sec\theta$ but then abandons the idea. He comes back to it later and sees that it is easier to work with m^2 instead of m since the m^2's can be expressed as polynomials in $\tan\theta$. This is, of course, sufficient to give him the series for $\sec\theta$. The series for $\log\sec\theta$ and $\log\tan(\pi/4 + \theta)$ he then obtains by term by term integration of the series for $\tan\theta$ and $\sec\theta$

Taylor series is not unjustly named after Brook Taylor who published it in 1715. He saw the importance of the result and derived several interesting consequences. For a discussion of these matters see: L. Feigenbaum, "Brook Taylor and the Method of Increments," *Archive for History of Exact Sciences*, Vol. 34, (1985), pp. 1–140.

[15] *James Gregory*, p. 352.

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respectively. Naturally, the 3233 error is repeated. He must have obtained the series for arc tan x from the 2nd m which can be written as

$$\frac{da}{dq} = \frac{r^2}{r^2+q^2} = 1 - \frac{q^2}{r^2} + \frac{q^4}{r^4} - \cdots .$$

The arctan series follows after term by term integration. Clearly, Gregory had made great progress in the study of infinite series and the calculus and, had he lived longer and published his work, he might have been classed with Newton and Leibniz as a co-discoverer of the calculus. Unfortunately, he was struck by a sudden illness, accompanied with blindness, as he was showing some students the satellites of Jupiter. He did not recover and died soon after in October, 1675, at the age of thirty-seven.

4. Kerala Gargya Nilakantha (c.1450–c.1550)

Another independent discovery of the series for arctan x and other trigonometric functions was made by mathematicians in South India during the fifteenth century. The series are given in Sanskrit verse in a book by Nilakantha called *Tantrasangraha* and a commentary on this work called *Tantrasangraha-vakhya* of unknown authorship. The theorems are stated without proof but a proof of the arctan, cosine and sine series can be found in a later work called *Yuktibhasa*. This was written in Malayalam, the language spoken in Kerala, the southwest coast of India, by Jyesthadeva (c.1500–c.1610) and is also a commentary on the *Tantrasangraha*. These works were first brought to the notice of the western world by an Englishman named C. M. Whish in 1835. Unfortunately, his paper on the subject had almost no impact and went unnoticed for almost a century when C. Rajagopal[16] and his associates began publishing their findings from a study of these manuscripts. The contributions of medieval Indian mathematicians are now beginning to be recognized and discussed by authorities in the field of the history of mathematics.[17]

It appears from the astronomical data contained in the *Tantrasangraha* that it was composed around the year 1500. The *Yuktibhasa* was written about a century later. It is not completely clear who the discoverer of these series was. In the *Aryabhatiya-bhasya*, a work on astronomy, Nilakantha attributes the series for sine to Madhava. This mathematician lived between the years 1340–1425. It is not known whether

[16]Rajagopal's work may be found in the following papers: (with M. S. Rangachari) "On an Untapped Source of Medieval Keralese Mathematics," *Archive for History of Exact Sciences*, Vol. 18, (1977), pp. 89–102; "On Medieval Kerala Mathematics," *Archive for History of Exact Sciences*, Vol. 35, (1986), pp. 91–99. These papers give the Sanskrit verses of the *Tantrasangrahavakhya* which describe the series for the arctan, sine and cosine. An English translation and commentary is also provided. A commentary on the proof of arctan series given in the *Yuktibhasa* is available in the two papers: "A Neglected Chapter of Hindu Mathematics," *Scripta Magnematica*, Vol. 15, (1949), pp. 201–209; "On the Hindu Proof of Gregory's Series," *Ibid.*, Vol. 17, (1951), pp. 65–74. A commentary on the *Yuktibhasa's* proof of the sine and cosine series is contained in C. Rajagopal and A. Venkataraman, "The sine and cosine power series in Hindu mathematics," *Journal of the Royal Asiatic Society of Bengal, Science*, Vol. 15, (1949), pp. 1–13.

[17]See J. E. Hofmann, "Über eine alt indische Berechnung von π und ihre allgemeine Bedeutung," *Mathematische-Physikalische Semester Berichte*, Bd. 3, H. 3/4, Hamburg (1953). See also D. T. Whiteside, "Patterns of Mathematical Thought in the later Seventeenth Century," *Archive for History of Exact Sciences*, Vol. 1, (1960–1962), pp. 179–388. For a discussion of medieval Indian mathematicians and the *Tantrasangraha* in particular, one might consult: A. P. Jushkevich, *Geschichte der Mathematik in Mittelalter* (German translation Leipzig, 1964, of the Russian original, Moscow, 1961).

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Madhava found the other series as well or whether they are somewhat later discoveries.

Little is known about these mathematicians. Madhava lived near Cochin in the very southern part of India (Kerala) and some of his astronomical work still survives. Nilakantha was a versatile genius who wrote not only on astronomy and mathematics but also on philosophy and grammar. His erudite expositions on the latter subjects were well known and studied until recently. He attracted several gifted students, including Tuncath Ramanujan Ezuthassan, an early and important figure in Kerala literature. About Jyesthadeva, nothing is known except that he was a Brahmin of the house of Parakroda.

In the *Tantrasangraha-vakhya*, the series for arctan, sine and cosine are given in verse which, when converted to mathematical symbols may be written as

$$r \arctan \frac{y}{x} = \frac{1}{1}\cdot\frac{ry}{x} - \frac{1}{3}\cdot\frac{ry^3}{x^3} + \frac{1}{5}\cdot\frac{ry^5}{x^5} - \cdots, \text{ where } \frac{y}{x} \leqslant 1,$$

$$y = s - s\cdot\frac{s^2}{(2^2+2)r^2} + s\cdot\frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2} - \cdots \text{ (sine)}$$

$$r - x = r\cdot\frac{s^2}{(2^2-2)r^2} - r\cdot\frac{s^2}{(2^2-2)r^2}\cdot\frac{s^2}{(4^2-4)r^2} + \cdots \text{ (cosine).}$$

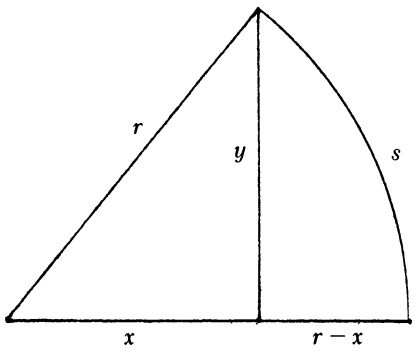

FIGURE 4

There are also some special features in the *Tantrasangraha's* treatment of the series for $\pi/4$ which were not considered by Leibniz and Gregory. Nilakantha states some rational approximations for the error incurred on taking only the first n terms of the series. The expression for the approximation is then used to transform the series for $\pi/4$ into one which converges more rapidly. The errors are given as follows:

$$\frac{\pi}{4} \simeq 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n} \pm f_i(n+1) \qquad i = 1,2,3, \tag{12}$$

where

$$f_1(n) = \frac{1}{2n},\ f_2(n) = \frac{n/2}{n^2+1} \text{ and } f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2}.$$

The transformed series are as follows:

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots \tag{13}$$

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and

$$\frac{\pi}{4} = \frac{4}{1^5 + 4 \cdot 1} - \frac{4}{3^5 + 4 \cdot 3} + \frac{4}{5^5 + 4 \cdot 5} - \cdots .$$

Leibniz's proof of the formula for $\pi/4$ was found by the quadrature of a circle. The proof in Jyesthadeva's book is by a direct rectification of an arc of a circle. In the diagram given below, the arc AC is a quarter circle of radius one with center O and $OABC$ is a square. The side AB is divided into n equal parts of length δ so that $n\delta = 1$, $P_{r-1}P_r = \delta$. EF and $P_{r-1}D$ are perpendicular to OP_r. Now, the triangles OEF and $OP_{r-1}D$ are similar, which gives

$$\frac{EF}{OE} = \frac{P_{r-1}D}{OP_{r-1}}, \quad \text{that is, } EF = \frac{P_{r-1}D}{OP_{r-1}}.$$

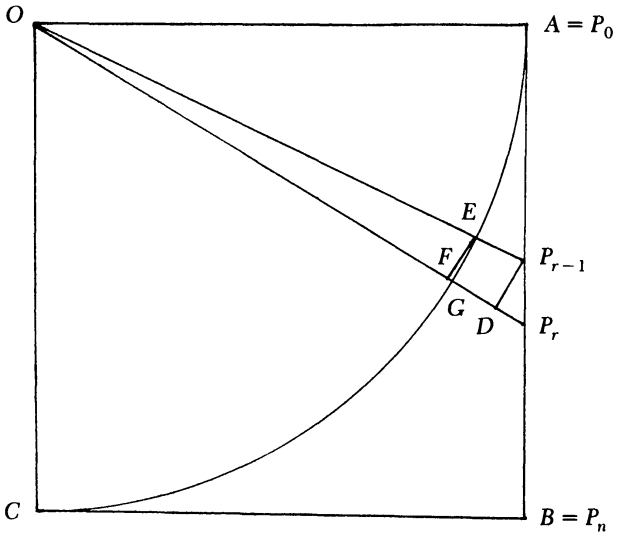

FIGURE 5

The similarity of the Δs $P_{r-1}P_rD$ and OAP_r gives

$$\frac{P_{r-1}P_r}{OP_r} = \frac{P_{r-1}D}{OA} \text{ or } P_{r-1}D = \frac{P_{r-1}P_r}{OP_r}.$$

Thus,

$$EF = \frac{P_{r-1}P_r}{OP_{r-1}OP_r} \simeq \frac{P_{r-1}P_r}{OP_r^2} = \frac{\delta}{1 + AP_r^2} = \frac{\delta}{1 + r^2\delta^2}.$$

Since arc $EG \simeq EF \simeq \dfrac{\delta}{1 + r^2\delta_r^2}$, $\dfrac{1}{8}$ arc of circle is

$$\frac{\pi}{4} = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{\delta}{1 + r^2\delta^2}. \tag{14}$$

Of course, a clear idea of limits did not exist at that time so that the relation was understood in an intuitive sense only. To evaluate the limit, Jyesthadeva uses two lemmas. One is the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

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Jyesthadeva says that the expansion is obtained on iterating the following procedure:

$$\frac{1}{1+x} = 1 - x\left(\frac{1}{1+x}\right) = 1 - x\left(1 - x\left(\frac{1}{1+x}\right)\right).$$

The other result is that

$$S_n^{(p)} \equiv 1^p + 2^p + \cdots + n^p \sim \frac{n^{p+1}}{p+1} \quad \text{for large } n. \tag{15}$$

A sketch of a proof is given by Jyesthadeva. He notes first that

$$nS_n^{(p-1)} = S_n^{(p)} + S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_{n-1}^{(p-1)}. \tag{16}$$

This is easy to verify. Relation (16) is also contained in the work of the tenth century Arab mathematician Alhazen, who gives a geometrical proof in the Greek tradition[18]. He uses it to evaluate $S_n^{(3)}$ and $S_n^{(4)}$ which occur in a problem about the volume of a certain solid of revolution. *Yuktibhasa* shows that for $p = 2, 3$

$$S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_{n-1}^{(p-1)} \sim \frac{S_n^{(p)}}{p}, \tag{17}$$

and then suggests that by induction the result will be true for all values of p. Once this is granted, it follows that if

$$S_n^{(p-1)} \sim \frac{n^p}{p},$$

then by (16) and (17),

$$nS_n^{(p-1)} \sim S_n^{(p)} + \frac{S_n^{(p)}}{p} \quad \text{or} \quad S_n^{(p)} \sim \frac{n^{p+1}}{p+1},$$

and (15) is inductively proved.

We now note that (14) can be rewritten, after expanding $1/(1+r^2\delta^2)$ into a geometric series, as

$$\begin{aligned}\frac{\pi}{4} &= \lim_{n\to\infty}\left[\delta\sum_{r=1}^{n} 1 - \delta^3\sum_{r=1}^{n} r^2 + \delta^5\sum_{r=1}^{n} r^4 - \cdots\right] \\ &= \lim_{n\to\infty}\left[1 - \frac{1}{n^3}\sum_{r=1}^{n} r^2 + \frac{1}{n^5}\sum_{r=1}^{n} r^4 - \cdots\right] \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,\end{aligned}$$

where we have used relation (15) and the fact that $\delta = 1/n$. Now consider the approximation (12) and its application to the transformation of series. Suppose that

$$\sigma_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \pm \frac{1}{n} \mp f(n+1),$$

where $f(n+1)$ is a rational function of n which will make σ_n a better approximation of $\pi/4$ than the nth partial sum S_n. Changing n to $n-2$ we get

[18] See *The Historical Development of the Calculus* (mentioned in footnote 1), p. 84. Alhazen is the latinized form of the name Ibn Al-Haytham (c. 965–1039).

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$$\sigma_{n-2} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n-2} \pm f(n-1).$$

Subtracting the second relation from the first,

$$\pm u_n = \sigma_n - \sigma_{n-2} = \pm \frac{1}{n} \mp f(n+1) \mp f(n-1). \tag{18}$$

Then

$$\begin{aligned}\sigma_n &= \sigma_{n-2} \pm u_n \\ &= \sigma_{n-4} \mp u_{n-2} \pm u_n \\ &= \cdots = \sigma_1 - u_3 + u_5 - u_7 + \cdots \pm u_n \\ &= 1 - f(2) - u_3 + u_5 - u_7 + \cdots u_n.\end{aligned}$$

It is clear that

$$\lim_{n\to\infty} \sigma_n = \frac{\pi}{4}$$

and therefore

$$\frac{\pi}{4} = 1 - f(2) - u_3 + u_5 - u_7 + \cdots. \tag{19}$$

Thus, we have a new series for $\pi/4$ which depends on how the function $f(n)$ is chosen. Naturally, the aim is to choose $f(n)$ in such a way that (19) is more rapidly convergent than (1). This is the idea behind the series (13). Now equation (18) implies that

$$f(n+1) + f(n-1) = \frac{1}{n} - u_n. \tag{20}$$

For (19) to be more rapidly convergent than (1), u_n should be $o(1/n)$, that is, negligible compared to $1/n$. It is reasonable to assume $f(n+1) \simeq f(n-1) \simeq f(n)$. These observations together with (20) imply that $f(n) = 1/2n$ is a possible rational approximation in equation (12). With this $f(n)$, the value of u_n is given by (20) to be

$$u_n = \frac{1}{n} - \frac{1}{2(n+1)} - \frac{1}{2(n-1)} = -\frac{1}{n^3 - n}.$$

Substituting this in (19) gives us (13), which is

$$\frac{\pi}{4} = 1 - \frac{1}{4} + \frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots.$$

The other series

$$\frac{\pi}{4} = \frac{4}{1^5 + 4 \cdot 1} - \frac{4}{3^5 + 4 \cdot 3} + \frac{4}{5^5 + 4 \cdot 5} - \cdots$$

is obtained by taking $f(n) = \dfrac{n/2}{n^2 + 1}$ in (19).

It should be mentioned that Newton was aware of the correction $f_1(n) = 1/2n$. For in the letter to Oldenburg, referred to earlier, he says, "By the series of Leibniz also if half the term in the last place be added and some other like devices be employed, the computation can be carried to many figures." However, he says nothing about transforming the series by means of this correction.

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It appears that Nilakantha was aware of the impossibility of finding a finite series of rational numbers to represent π. In the *Aryabhatiya-bhasya* he writes, "If the diameter, measured using some unit of measure, were commensurable with that unit, then the circumference would not likewise allow itself to be measured by means of the same unit; so likewise in the case where the circumference is measurable by some unit, then the diameter cannot be measured using the same unit."[19]

The *Yuktibhasa* contains a proof of the arctan series also and it is obtained in exactly the same way except that one rectifies only a part of the 1/8 circle.

It can be shown that if $\pi/4 = S_n + f(n)$, where S_n is the nth partial sum, then $f(n)$ has the continued fraction representation

$$f(n) = \frac{1}{2}\left[\frac{1}{n+}\,\frac{1^2}{n+}\,\frac{2^2}{n+}\,\frac{3^2}{n+}\cdots\right]. \tag{21}$$

Moreover, the first three convergents are

$$f_1(n) = \frac{1}{2n}, \qquad f_2(n) = \frac{n/2}{n^2+1} \quad \text{and} \quad f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2},$$

which are the values quoted in (13). Clearly, Nilakantha was using some procedure which gave the successive convergents of the continued fraction (21) but the text contains no suggestion that (20) was actually known to him. This continued fraction implies that

$$\frac{2}{4-\pi} = 2 + \frac{1^2}{2+}\,\frac{2^2}{2+}\,\frac{3^2}{2+}\cdots,$$

which may be compared with the continued fraction of the seventeenth century English mathematician, William Brouncker (1620–1684), who gave the result

$$\frac{4}{\pi} = 1 + \frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots.$$

The third approximation

$$f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2}$$

is very effective in obtaining good numerical values for π without much calculation. For example

$$1 - \frac{1}{3} + \cdots - \frac{1}{19} + f_3(20)$$

gives the value of π correct up to eight decimal places.[20] Nilakantha himself gives 104348/33215 which is correct up to nine places. It is interesting that the Arab mathematician Jamshid-al-Kasi, who also lived in the fifteenth century, had obtained the same approximation by a different method.

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[19] See *Geschichte der Mathematik*, p. 169.

[20] These observations concerning the continued fraction expansion of $f(n)$ and its relation to the Indian work and that of Brouncker, and concerning the decimal places in f (20), are due to D. T. Whiteside. See "On Medieval Kerala Mathematics" of footnote 13.

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5. Independence of these discoveries.

The question naturally arises of the possibility of mutual influence between or among the discoverers of power series, in particular the series for the trigonometric functions. Because of the lively trade relations between the Arabs and the west coast of India over the centuries, it is generally accepted that mathematical ideas were also exchanged. However, there is no evidence in any existing mathematical works of the Arabs that they were aware of the concept of a power series. Therefore, we may grant the Indians priority in the discovery of the series for sine, cosine and arctangent. Moreover, historians of mathematics are in agreement that the European mathematicians were unaware of the Indian discovery of infinite series.[21] Thus, we may conclude that Newton, Gregory and Leibniz made their discoveries independently of the Indian work. In fact, it appears that yet another independent discovery of an infinite series giving the value of π was made by the Japanese mathematician Takebe Kenko (1664–1739) in 1722. His series is

$$\pi^2 = 4\left[1 + \sum_{n=1}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+1)!}\right].$$[22]

This series was not obtained from the arctan series and its discussion is therefore not included. However, the independent discovery of the infinite series by different persons living in different environments and cultures gives us insight into the character of mathematics as a universal discipline.

Acknowledgement. I owe thanks to Phil Straffin for encouraging me to write this paper and to the referees for their suggestions.

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[21] See "Patterns of Mathematical Thought in the later Seventeenth Century" of footnote 17. See also A. Weil, "History of Mathematics: Why and How" in *Collected Papers*, Vol. 3 (New York: Springer-Verlag, 1979), p. 435.

[22] See D. E. Smith and Y. Mikami, *A History of Japanese Mathematics* (Chicago: Open Court, 1914). This series was also obtained by the French missionary Pierre Jartoux (1670–1720) in 1720. He worked in China and was in correspondence with Leibniz, but the present opinion is that Takebe's discovery was independent. Leonhard Euler (1707–1783) rediscovered the same series in 1737. A simple evaluation of it can be given using Clausen's formula for the square of a hypergeometric series.

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Paper 40

David Gold and David Pingree (1991)

A Hitherto Unknown Sanskrit Work Concerning Mādhava's Derivation of the Power Series for Sine and cosine

Historia Scientiarum: International Journal of the History of Science Society of Japan, Vol. 42, (30 March 1991), pp. 49–65.

Notes

This paper may be read along with a paper titled "*Mahājyānayanaprakāraḥ*: Infinite Series for the Sine and Cosine Functions in the Kerala Works" authored by G. Raja Rajeswari and M. S. Sriram (see p. 750).

David Pingree

David Edwin Pingree (2 January 1933–11 November 2005) was an American historian of mathematics in the ancient world. He was Professor of History of Mathematics and Classics at Brown University.

> "Pingree devoted himself to the study of the exact sciences, such as mathematics, mathematical astronomy and astral omens. He was also acutely interested in the transmission of those sciences across cultural and linguistic boundaries. His interest in the transmission of the exact sciences came from two fronts or, perhaps more correctly, his interest represents two sides of the same coin. On the one hand, he was concerned with how one culture might appropriate, and so alter, the science of another (earlier) culture in order to make that earlier scientific knowledge more accessible to the recipient culture. On the other hand, Pingree was also interested in how scientific texts surviving from a later culture might be used to reconstruct or cast light on our fragmentary records of earlier sciences. In this quest, Pingree would, with equal facility use ancient Greek works to clarify Babylonian texts on divination, turn to Arabic treatises to illuminate early Greek astronomical and astrological texts, seek Sanskrit texts to explain Arabic astronomy, or track the appearance of Indian astronomy in medieval Europe." (Wikipedia)

David Gold

David Gold was a collaborator of David Pingree.

Source: National Diet Library Digital Collection, Japan

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A Hitherto Unknown Saṁskṛit Work concerning Mādhava's Derivation of the Power Series for Sine and Cosine

David Gold and David Pingree*

Preface

Scholars since the nineteenth century have been familiar with the trigonometrical work of the Indian mathematician Mādhava[1] only through the writings of his disciples.[2] It is known that one of his major achievements was the discovery of a method for computing Sines and Cosines,[3] which is equivalent to Newton's power series, but which was devised well over two hundred years earlier and without the benefit of the infinitesimal calculus.

The text which follows is an edition of a Saṁskṛit text on ff. 12–16 of a manuscript, Burnell 17c, which is in the India Office Library in London. It was identified by K. V. Sarma[4] as having the title "Mahājyānayanaprakāra" ("the Method for the Computation of the Great Sine" or, perhaps, "the Great Method for the Computation of the Sine"). Sarma attributed it to Mādhava. It appeared, on that basis, to be the only surviving work on this subject by the mathematician himself.[5]

However, upon examination it soon became apparent that the text is a commentary by an anonymous author who seems extremely unlikely to be Mādhava. He is, though, from Mādhava's school, and the text is of interest because it expresses methods for computing the Sines in ways which differ somewhat from those utilized by other followers of Mādhava.

* Brown University, Providence, R. I., U.S.A.

[1] Concerning Mādhava, see D. Pingree, *Census of the Exact Sciences in Sanskrit* (hereafter *CESS*), series A, vols. 1–4, Philadelphia 1971–1981, A4, 414b–415a, and vol. 5 (in preparation).

[2] See D. Pingree, *Jyotiḥśāstra*, Wiesbaden 1981, pp. 49–51, 65–66 and 128.

[3] Nīlakaṇṭha (b. 1444; see *CESS* A3, 175b–177b, and A4, 142a–142b) says in his *Tantrasaṅgraha* 2, 15c–d: *iti yāvāṁścaiṣā kāryaṁ prabhavaṁ mādhavoditam*.

[4] K. V. Sarma, *A History of the Kerala School of Hindu Astronomy*, *VJS* 55, 1972, p. 149.

[5] His other surviving works are listed in *CESS*: *Sphuṭacandrāpti*, *Veṇvāroha* (epoch 1403), *Ayanāṅgaprakāra* (1418), *Candravākyāni*, *Laghuprakaraṇa*, *Madhyamānayanaprakāra*(?), and *Golasāra*(?).

Introduction

Mādhava and all those who knew of his work before the eighteenth century,[6] lived and worked in large family compounds called *illams* within a small area on the Western coast of south India, in the modern state of Kerala. He himself was from an *illam* called Bakulavihāra in the town of Saṅgamagrāma,[7] a few miles north of Cochin. His family was Brāhmaṇa, but of a group called Emprāntiri, considered socially inferior to the dominant Brāhmaṇa community, the Nampūtiri. He was both a mathematician and an astronomer, and his dated works were composed in 1403 and 1418.

Since none of Mādhava's own compositions on power series survive, what we know of his work in this field we know through his successors. The first of these was a Nampūtiri Brāhmaṇa called Parameśvara.[8] He was a prominent member of the most famous center of astronomical learning in fourteenth and fifteenth century Kerala, an *illam* called Vaṭaśreṇi, located about 35 miles northeast of Bakulavihāra in the town of Aśvattagrāma.[9] Parameśvara composed a variety of mathematical and astronomical works between 1408 and 1443 and was the teacher of his son Dāmodara.[10]

Two important mathematicians from nearby *illams*, Nīlakaṇṭha[11] and Jyeṣṭhadeva,[12] both Nampūtiris, were trained by Dāmodara at Vaṭaśreṇi. Nīlakaṇṭha wrote his *Tantrasaṅgraha* in 1501 and his *Āryabhaṭīyabhāṣya* soon afterwards. Jyeṣṭhadeva's *Yuktibhāṣā* is a Malayālam work related to Nīlakaṇṭha's *Tantrasaṅgraha*, which is in Sanskrit. Śaṅkara,[13] a temple-sweeper by caste, studied under both these men and was the last known pupil in this line of *gurus* descending from Mādhava. He wrote between 1529 and 1556. His *Yuktidīpikā* is a modified Sanskrit version of Jyeṣṭhadeva's *Yuktibhāṣā* and his *Kriyākramakarī* is a commentary on the well-known *Līlāvatī* by Bhāskara.[14]

This completes the list of the major authors and works through which Mādhava's achievements are preserved. The power series for Sines and Cosines, as they are described by his successors, are often attributed to Mādhava, who is sometimes quoted. Upon the death of Śaṅkara, or soon afterwards, the transmission of these ideas was effectively terminated, and the power series were for all practical purposes lost to India. In the West they were re-invented by Newton and Gregory in the second half of the seventeenth century.

The manuscript edited below begins with the quotation of two verses, which are also found, in a slightly modified form, in Śaṅkara's *Kriyākramakarī*[15] and *Yuktidīpikā*.[16] This connection alone would place our author in the school of Mādhava. The verse teaches the method for computing the Sine, but only in an extremely condensed manner, intended rather as a reminder for those formally trained in the school than as a full explanation. The rest of our text is a commentary, elucidating the verse, and explaining its mathematical significance in four different ways. In relation to its commentary, the verse quoted at the beginning is the *mūla* (the "root"), while our anonymous author refers to his commentary as a *yukti* (the "reasoning" behind or the "proof" of the *mūla*).[17]

The appearance of the *mūla* in the *Kriyākramakarī* suggests that our author was a writer later than Śaṅkara. If so, this may be the latest extant text from the direct line of Mādhava's school, and probably dates from the second half of the sixteenth century.

The *mūla* makes reference to another verse, which begins "*vidvān*". This verse, which is later quoted in full by our author, is cited by Śaṅkara in his commentary

[6] The sole eighteenth century author to know of Mādhava's work was an anonymous author from the Putumana *illam* (see *CESS* A4, 206b–209b) in Śivapura who wrote a *Karaṇapaddhati*, apparently in 1733. In this he describes Mādhava's series in adhyāya 6. He incorporates into this chapter, without acknowledgement, one of Mādhava's verses (also cited by our author anonymously) as verse 14:

vidvāṃs tunnabalaḥ kapīśanicayas sarvārthaśīlaḥ sthiro
nirviddhāṅganarendraruṅ nigaditeṣv eṣu kramāt pañcasu//
ādhastyād guṇitād abhīṣṭadhanuṣaḥ kṛtyā vihṛtyāntimasy-
āptaṃ śodhyam upary upary atha ghanenaivaṃ dhanuṣy antataḥ//

The variants from our version are underlined.

In the nineteenth century, Mādhava's series were described by Śaṅkaravarman in his *Sadratnamālā*, composed in 1823. The first English exposition was by C. M. Whish, "On the Hindu Quadrature of the Circle and the Infinite Series of the Proportion of the Circumference to the Diameter in the Four Sastras, The Tantra Sangraham, Yucti Bhasha, Carana Padhati, and Sadratnamala," *TRAS* 3, 1830, 509–523.

[7] Mādhava's identification verse (*Veṇvāroha* 13) is:

bakulādhiṣṭhitatvena vihāro yo viśeṣyate//
gṛhanāmani so 'yaṃ syān nijanāmani mādhavaḥ//

[8] See *CESS* A4, 187b–192b.

[9] This is known from the verse (*Dṛggaṇita* 1 2, 8):

grāmād aśvatthākhyād asmād dhṛtiyojane kila prācyām//
samarekhātra tu palabhā sadhṛtivyaṅgulāṅguladvitayam//

[10] Thus the colophon to the gaṇitapāda of Nīlakaṇṭha's *Āryabhaṭīyabhāṣya* (p. 180) begins: *iti śrīkuṇḍagrāmajena gārgyagotreṇāśvalāyanena bhāṭṭena keralasadgrāmagrhasthena śrīśvetāraṇyanāthaparameśvarakaruṇādhikaraṇabhūtavigrahena jātavedaḥputreṇa śaṅkarāgrajena jātavedomātulena drgganitanirmāpakaparameśvaraputradāmodarāttajyotiṣām ayanena ravita āttavedāntaśāstreṇa subrahmaṇyasuhṛdayena nīlakaṇṭhena somasutā viracita...*

[11] See note 3.

[12] *CESS* A3, 76b–77a, and A4, 100a.

[13] See D. Pingree, *Jyotiḥśāstra*, pp. 50, 62, 66 and 128. The article on him in *CESS* will appear in vol. 6.

[14] *CESS* A4, 299a–326a.

[15] *Kriyākramakarī* on *Līlāvatī* 199 (p. 390):

nihatya cāpavargeṇa cāpaṃ tattatphalāni ca//
haret samūlayugvargair vyāsavargahataiḥ kramāt//
cāpaṃ phalāni cādho 'dho nyasyopary upari tyajet//

[16] *Yuktidīpikā* 2, 440–441 (p. 118) agrees entirely with the above, except that it coincides with our text in reading in the second lines: "*vargais trijyā*", and that it includes the fourth line:

jīvāptyaisaṅgraho 'syaiva vidvān ityādinā kṛtaḥ//

A. K. Bag, "Mādhava's Sine and Cosine Series," *IJHS* 11, 1976, 54–55, quotes four and a half verses beginning with these two as coming from the *Tantrasaṅgraha* itself, while in a note to the quotation he refers to Jyeṣṭhadeva's *Yuktibhāṣā*. In fact, he used Śaṅkara's *Yuktidīpikā*.

[17] For the genre of texts called *yuktis*, see K. V. Sarma, *Gaṇitayuktayaḥ*, *PUIS* 24, Hoshiarpur 1979.

on Nīlakaṇṭha's *Tantrasaṅgraha*,[18] as well as by Putumana in his *Karaṇapaddhati*,[19] but without attribution. Nīlakaṇṭha himself quotes it in his *Āryabhaṭīyabhāṣya*, where he ascribes it to Mādhava.[20] This provides confirmation of the commentator's connection to Mādhava, but no further evidence for dating the text.

There is one other quotation in the text, but it is only a reference to a verse which is not quoted in full:

> The accomplishment of the result is indicated by that verse mentioned in the first act of writing of that one, which begins, "After that by that and by that" (*tatparas tena tena ca*).

Perhaps "that one" is Mādhava. But it is only possible to speculate.

We have made our edition from a microfilm copy, since the manuscript itself is in London. According to A. B. Keith, who catalogued it,[21] the manuscript is on 8 3/8″ × 1 5/8″ *talipat* leaves, and the script is nineteenth century Malayālam. It is written on 9 sides of five folios. Each side has 8 or 9 lines of text, each between about 30 and 40 *akṣaras* in length.

The edition below is in the Roman alphabet and is divided into paragraphs according to what seemed most appropriate and most convenient. The manuscript has no such divisions. All emendations are noted in the apparatus criticus, with the exception of those of two types: where the *sandhi* in the manuscript is correct, but no longer standard, we have changed it silently; numerous other *sandhi* changes are, however, noted. We have also added *daṇḍas*, silently, where it seemed appropriate, although the manuscript only rarely uses them. *Sandhi* changes which resulted from the addition of *daṇḍas* are noted in the apparatus.

[18] *Yuktidīpikā* 2, 437 (p. 117) agrees entirely with out text's version.

[19] See note 6.

[20] *Āryabhaṭīyabhāṣya* on gaṇitapāda 17a (p. 113): *ata evāha mādhavaḥ*. The verse itself agrees with our version save that in the first line Nīlakaṇṭha, like the author of the *Karaṇapaddhati*, reads *kapīśa*° instead of *kavīśa*°.

[21] A. B. Keith, *Catalogue of the Sanskrit and Prakrit Manuscripts in the Library of the India Office*, vol. 2, Oxford 1935, pp. 774–775 (no. 6301).

Edited Text

hari//
nihatya cāpavargeṇa cāpaṃ tattatphalāni ca[1]/
haret samūlayugvargais[2] trijyāvargahataiḥ kramāt//
cāpaṃ phalāni cādho 'dho nyasyopary upari tyajet/
jīvānte[3] saṅgraho 'syaiva vidvān ityādinā kṛtaḥ//
iṣṭacāpavargeṇa iṣṭacāpaṃ tattatphalāni[4] ca nihatya/ samūlayugvargaiḥ samūlair mūlasahitaiḥ/ yujāṃ dvyādiyugmasaṃkhyānām/ vargair haret/ punaś cāpaṃ hṛtaphalāni cādho 'dho nyasyopary upari tyajet[5]/ ante labdhajā jīvā bhavati/ etad uktaṃ bhavati/

(1) prathamam iṣṭacāpam iṣṭacāpavargeṇa nihatya/ dvisaṁkhyayā/ samūlayugvarga⟨:⟩ṣaṭsaṁkhyaḥ/ tadguṇitena[6] trijyāvargeṇa haret/ tatra labdhaṃ prathamaphalam[7]/ iṣṭacāpasyādho nyasyet/ punas tat prathamaphalam iṣṭacāpavargeṇaiva nihatya/ samūlacaturvargeṇa viṃśatisaṃkhyena trijyāvargaṃ hatvā tena haret/ labdhaṃ dvitīyaphalaṃ prathamaphalasyādho nyaset/ punar dvitīyaphalam iṣṭacāpavargeṇa nihatya samūlaṣaḍvargeṇa dvicatvāriṃśatsaṃkhyena trijyāvargaṃ hatvā tena[8] haret/ labdhaṃ tṛtīyaphalam/ evam uttarottaraphalāny ānīya kramād adho 'dho vinyasya/ sarveṣām adha⟨:⟩-sthitaṃ svordhvasthitāt tyajet/ evam upary[9] upari kṛte 'nte[10] cāpād viśodhitasya śeṣam iṣṭajīvā bhavati/ phalāvasānaṃ tūttarottaraphalanyūnatvād[11] avaśyaṃ bhavati/ etad eva jīvānayanam etanmūlena prakārāntareṇa proktam ity āha/
saṃgraho 'syaiva vidvān ityādinā kṛta⟨:⟩
iti/

(2) asyaiva saṃgraho vidvān[12] ityādinā ślokena[13] racitaḥ[14]/
vidvāṃs tunnabalaḥ kavīśanicayaḥ sarvārthaśīlasthiro
nirviddhāṅganarendraruṅ nigaditeṣv eṣu kramāt pañcasu/
ādhastyād[15] guṇitād abhīṣṭadhanuṣaḥ[16] kṛtyā vihṛtyāntimasy-[17]
āptaṃ śoddhyam upary upary atha ghanenaivaṃ[18] dhanuṣy antataḥ//
iti/ kathaṃ tasya jīvānayanasya etanmūlatvaṃ pravartate[19]/ trirāśicāpasya phalānayane ye hārā[20] iṣṭacāpaphalānayane[21] 'pi ta eva hārāḥ/ guṇakānāṃ guṇyapade 'das trirāśicāpaghanaṃ[22] ṣaḍguṇitena[23] trijyāvargeṇa hṛtvā prathamaphalaṃ nītam[24]/ atha prathamaphalam iṣṭacāpaghanena[25] hatvā trirāśicāpaghanena hṛta[26] iṣṭacāpasya prathamaphalaṃ bhavati/ dvitīyaphalaṃ tu punar iṣṭacāpavargeṇa hataṃ hareṇa[27] vibhājitam[28]/ at⟨o⟩ dvitīyaphalam iṣṭacāpaghaneneṣṭacāpavargeṇa[29] ca hatvā trirāśicāpasya ghanena vargeṇa ca vibhakta[30]

Apparatus. (1) °phalānicca (2) °vargair (3) jīvājye (4) iṣṭacāpatat° (5) niyasya tyajet upary upa° (6) tatguṇi° (7) °phalaṃ (8) hatvā tvo (9) evalupary° (10) kṛte ante (11) °phalānyu° (12) viṭān (13) śyokena (14) racitāṃ (15) °styāt (16) abhiṣṭadadhanuṣaḥ (17) °tyāntama° (18) uparyuparyaghanainaiva (19) pradartyate (20) hārāḥ (21) °phalaḥāna° (22) ḍaḥtri° (23) ṣaṭguṇi° (24) °phalanitaṃ (25) °cāpavargeṇa ghanena (26) °ghaṭanana hṛte (27) °vargeṇa hatvā hāreṇa (28) vibhajitaṃ (29) °ghanena iṣṭa° (30) vibhakte

iṣṭacāpasya dvitīyaphalaṃ bhavati/ tṛtīyaphalaṃ tu ghanena [ghanena] sakṛdvargeṇa dvi⟨dhā⟩ guṇaharaṇe kṛta āneyam[31]/ evaṃ pañcamaphalasya ghanena sakṛdvargeṇa caturdhā[32] guṇaharaṇe kārye/ tat punaś caturthaphalāc choddhyam[33]/ caturthasya pañcamāt tv anyūnatayaiva[34] viśeṣaḥ/

(3) ataḥ pañcamaphalam iṣṭacāpavargeṇa guṇaharaṇe[35] sakṛn nihatya trirāśicāpavargeṇa śiṣṭakarmaṇā caturthaphale na sāmyaṃ bhavati/ atas tadā⟨nī⟩m eva caturthaphalāc[36] choddhyaṃ tadūnacaturthaphalāt tṛtīyaphalasyādhikyam asti[37]/ ataś caturthaphalaśeṣam iṣṭacāpavargeṇa sakṛn nihatya trirāśicāpavargeṇa vibhajya tṛtīyaphalāc choddhyam/ evam uttarottaram api guṇaharaṇe kārye[38]/ tasya sakṛ⟨n⟩nyūnatvāt tṛtīyaśeṣam[39] ⟨dvitīyaphalāc chodhyam/ tasmāl⟩ labdhaṃ prathamāc choddhyam[40]/ prathamaphalaśeṣaṃ punar iṣṭacāpaghanena[41] nihatya trirāśicāpaghanena vibhaktam[42] iṣṭacāp⟨āj⟩ jīvānayanāya śoddhya⟨ṃ⟩ phalaṃ labhyate/ ata⟨ḥ⟩/

ādhastyād[43] guṇitād abhīṣṭadhanuṣaḥ kṛtyā vihṛtyāntimasyāptam[44]
śodhyam upary u[paryu]pary atha ghanenaivaṃ dhanuṣy antataḥ/

ity uktam/ evaṃ[45] vidvān ityādinā kṛtam[46] etanmūlatvam/

(4) nihatya cāpavargeṇetyādineṣṭacāpasya jyānayane[47] kīdṛśy upapattiḥ/ tatpradarśanāya vṛttam ālikhya mātṛpitṛrekhāś ca kuryāt/ tatra pūrvadiśi rekhāgrāt pravṛttaṃ dakṣiṇadiggāminam iṣṭacāpaṃ saṅkalpya[48] cāpāgre binduṃ kṛtvā tasya cāpasya tulyāntarālatvena kānicit khaṇḍāni kalpayet/ tadānīṃ teṣv[49] iṣṭakhaṇḍamadhyāt pravṛttām uttaradiggāminīṃ bhujāṃ[50] paścimadiggāminīṃ koṭiṃ kendragāminīṃ trijyāṃ ca likhet/ tadānīṃ tasya cāpakhaṇḍasya samastajyāṃ bhujākoṭibhyāṃ pṛthaṅ nihatya trijyayā dve 'pi[51] vibhajya labdhe koṭikhaṇḍaṃ bhujākhaṇḍaṃ ca bhavataḥ/ tataḥ[52] kṣetradvayasya tulyākāratvān[53] nirṇayaḥ[54]/ kṣetradvaye tv ekasya vṛtte kṣetrasya bhujaiva[55] bhujā tatkoṭir eva koṭiḥ/ trijyākhaṇḍaḥ [koṭikhaṇḍaṃ] koṭiḥ/ anayos tulyākāra[sa]tvaṃ tu parasparaṃ bhujākoṭikhaṇḍānāṃ viparītadiktvād[56] avagantavyam[57]/ etam anyeṣv api khaṇḍeṣu khaṇḍamadhyaprasṛtābhyāṃ[58] bhujākoṭibhyāṃ khaṇḍasamastajyāṃ pṛthaṅ nihatya trijyayā dve 'pi[59] vibhajya tatra koṭijyākhaṇḍaṃ bhujājyākhaṇḍaṃ ca labhya⟨n⟩te/ atha[60] yāvat khaṇḍite khaṇḍe[61] samastajyāyāḥ khaṇḍacāpasya ca bhedābhāvas[62] tāvad antarālasāmyeneṣṭacāpasya[63] khaṇḍanaṃ kurvāmaḥ/ khaṇḍamadhyaprasṛtānāṃ[64] bhujānāṃ sarveṣāṃ[65] yogena khaṇḍaṃ nihatya trijyayā dve 'pi[66] vibhajya labdham iṣṭacāpaśaro[67] bhavati/ evaṃ khaṇḍamadhyaprasṛtānāṃ koṭīnāṃ[68] yogena khaṇḍaṃ nihatya trijyayā ⟨dve 'pi⟩ vibhajya labdham

(31) kṛtvāneyaṃ (32) catuṣutvo (33) choddhyaṃ (34) pāñcamā vargeṇa guṇaṃ haraṇe satvanyūnataiva (35) guṇaṃ haraṇe (36) tatarttha (37) °phalasyatatṛtīyaphalāt vargeṇa guṇaharaṇe sakṛdādhikyam asti (38) cchoddhyaṃ evam uparautaram api guṇaṃ re° (39) °śeṣāt (40) choddhyaṃ (41) iṣṭātnacapaghanena (42) vibhakte (43) °styāt (44) °syāstaṃ (45) ity uktaṃ evaṃ (46) kṛtasya (47) °tyādinā śaṣṭacāpasya jyānakane (48) °gāminaṃ iṣṭacāpasaṃkalpā (49) °nyeṣu (50) bhujā (51) dve api (52) tatat (53) °kārāścā (54) nirṇṇayaṃ (55) tu etasya vṛttakṣetrabhujaiva (56) viparitandi° (57) °tavyaṃ (58) °sṛtānāṃ (59) dve api (60) labhyate atva (61) khaṇḍaṃ (62) bhedābhāvaḥ (63) antarātu sāmyena iṣṭa° (64) kutvāmakhaṇḍamadhyānmāpra° (65) sarvasāṃ (66) dve api (67) °cāpataro (68) koṭijyānāṃ

iṣṭacāpajīvā[69] bhavati/ khaṇḍamadhyaprasṛtānāṃ bhujānām yogānayanasya[70] duṣṭaratvāt taccāpayogānayanam eva prakriyate[71] tajjanitaṃ dūṣaṇaṃ[72] parihriyate/ khaṇḍamadhyādi[73] vā khaṇḍādi vā khaṇḍottara⟨m⟩ iṣṭacāpāntaraṃ yat saṃkalitaṃ tasmin khaṇḍaguṇite satīṣṭacāpād dhīnaṃ[74] saṃpadyate/ atra yat kiṃci⟨n⟩ nyūnātiriktaṃ[75] dṛśyate + savarṇa⟨ṃ⟩ syāc ca tv e⟨vaṃ⟩ kṛte manḍūka hiśranāṃ lāṁdralapalūpyamānā tasminn eva nimajjati: + evam uttarottaram[76] api punaḥ punaḥ khaṇḍaguṇitānāṃ saṃkalitasaṃkalitādīnām ānayanāyeṣṭacāpaguṇanaṃ[77] ⟨tri⟩jyādibhir haraṇaṃ[78] ca kartavyam[79]/ uttarottarasaṃkalitānām ānayane nātropayogaṃ ca bhaviṣyati/ saṃkalitānāṃ na yuktis taddurvacanatvād[80] granthavistarabhayāc cātra na pradṛśyate/ ekādyekottarā aṅkā vyastā⟨s tri⟩jyābhājyāḥ[81]/ pṛthaksthita eva para⟨s ta⟩t pūrveṇa[82] saṃguṇyam/[83]

tatparas tena tena[84] ca
ity anenaitasyādilekhakriyoktena ślokena phalasiddhiḥ sūcitā[85]/

(5) anena mārgeṇeṣṭa[86]-cāpavargam[87] eva trijyayā ⟨dve 'pi⟩ vibhajya labdham iṣṭacāpaśaraḥ prāk kalpanīyaḥ/ jīvānayane[88] tu ⟨kha⟩ṇḍamadhyaprasṛtābhiḥ koṭibhiḥ khaṇḍasya guṇane kartavye ⟨'⟩pi koṭijyāntarabhūtaiḥ śarakhaṇḍam[89] hatvā trijyayā[90] vibhajya labdham iṣṭacāpād viśoddhyam[91]/ śiṣṭam iṣṭacāpajyā bhavati/ atha vā śaraikyena khaṇḍaṃ guṇanīyam/[92] saṃkalitāṁśabhūtaśarāṇām aikyam[93] tatsaṃkalitasaṃkalitam eva/ ataḥ[94] saṃkalitasaṃkalitajñānāya[95] trijyayā vibhaktam iṣṭacāpārdham iṣṭacāpavargeṇa[96] nihatya trisaṃkhyena vibhajya punas trijyayā [ca] haret/ labdhaṃ jyānayanāya cāpāc choddhyaṃ ca bhavati/ pūrvaṃ śarānayanāya khaṇḍamadhyaprasṛtānāṃ jīvānāṃ yogena khaṇḍe [na] guṇanīye 'py antaratvāj jātalaghucāpānām[97] eva yogḥ[98] kṛtaḥ/ idānīṃ jyānayanāya[99] cāpād iyac choddhyam it⟨o⟩ jātam/[100] atas teṣām aikyam apy āneyam/[101] jātāni jyācāpāntarāṇi tu saṃkalitaikyāṁśībhūtāni[102]/ teṣāṃ yogasyānayanāya jñātajyācāpāntaram[103] iṣṭacāpena nihatya catussaṃkhyena hartavyam/ tat punas trijyayā[104] vibhajya pūrvānītāc charāc choddhyatvam[105] ca/ puna⟨r⟩ jyāśodhanāya śarāc chodhitasya saṃkalitaṃ kāryam/[106] tadartham ānīya tṛtīyaphalam[107] iṣṭacāpena hatvā pañcasaṃkhyena vibhajya trijyayā hartavyam/[108] labdhasya pūrvaṃ cāpād viśodhitatvāt phalasya pratiśodhyatvam/[109] ata eva śarasya[110] pratikṣepyatvam/[111] evam uttarottaraphalāni neyāni/etad uktaṃ bhavati/

iṣṭacāpajīvāyā iṣṭacāpaśarasya ca[112]/

(69) iṣṭe jīvācāpati (70) bhujagājyānāyogāṃna° (71) prākriyate (72) duṣaṇaṃ (73) khaṇḍāddhyadi (74) sati iṣṭacāpavargāddhi (75) °tiriko (76) °uttarātta° (77) °tādināṃ ānayānāya iṣṭa° (78) gādibhir haṇaraṇa° (79) kattavyaṃ (80) tadardvacatvāt (81) aṅkapyastābhājyā (82) pṛthakistate eva caratpūrveṇa (83) saṃguṇyaṃ (84) nena eta (85) phalasiddhisūcitvā (86) mārgeṇa iṣṭa° (87) °vargām (88) jīvanayane (89) °bhūtai[.]śara° (90) trijyāyā (91) °cāpāviśoddhyaṃ (92) °niyaṃ (93) ekyan (94) ataṃ (95) °saṃkalittajā° (96) iṣṭacāpena (97) guṇanīye cintāramajjāttatvāntuccāpa° (98) yāgaḥ (99) °nayānāya (100) choddhyam iti jātaṃ (101) āneyaṃ (102) °taikyāṁśatabhū° (103) °cāpāntaraṃ (104) harttavyaṃ tatpunaḥ tri° (105) pūrvānīyaccharāt śoddhyatvaṃ (106) kāryaṃ (107) °phalaṃ (108) hattavyaṃ (109) cāpatviśodhitaṃ phalaṃ pratiśoddhyatvaṃ (110) śaraṃ (111) °kṣepyatvaṃ (112) iṣṭaśāpaśarasya cā

yuga⟨pa⟩dā[va]nayanāc ca cāpaṃ cāpena nihatya dviguṇayā[113] trijyayā vibhajya prathamaphalam ānayet/ punas tat prathamaphalaṃ cāpena nihatya triguṇayā trijyayā hṛt⟨v⟩ā dvitīyaphalam ānayet/ puna⟨r⟩ dvitīyaphalaṃ cāpena nihatya caturguṇi[ya]tayā trijyayā vibhajya tṛtīyaphalam ānayet/ evaṃ muhur api phalānām iṣṭacāpaguṇanam[114] ekaikādhikasaṃkhyāguṇitayā trijyayā haraṇam ca kṛtvā phalāny ānīya teṣv ojāni śarānayanāyaikatrādho 'dho nyaset/ yugmāni jyānayanāyānyatra cādho 'dho nyaset/ evaṃ vinyasyāpy ubhayatrāpy adha⟨:⟩sthitam[115] upray upari śodhayet/ tadānīm ojapadebhyo labdhaṃ śaro bhavati/ yugmapadebhyo labdhaṃ cāpād viśoddhya⟨ṃ⟩ jīvā ca bhavati/ śarānayanam upekṣya jyānayane kartavye yugmaphalānām evāneyatve cāpavargeṇa guṇanaṃ dvyādiyugmasaṃkhyāyās tadūrdhvagataviṣamasaṃkhyāyāś ca ghātena trijyāvargaṃ nihatya tena haraṇaṃ ca kāryam/ samasaṃkhyāyās tadūrdhvagataviṣamasaṃkhyāyāś ca ghātaḥ/[116] samasaṃkhyāyāḥ ⟨---⟩ samūlayug[117]-vargam[118] eva/ ata uktaṃ nihatya cāpavargeṇa ityādinā/

ity atra yuktiḥ saṃkṣepāt[119] kathaṃcil[120] likhitā mayā/
ūnātirekatā śoddhyā vicārya viduṣāṃ gaṇaiḥ//

(113) viguṇayā (114) thalānam iṣṭacāpaguṇanā (115) °sthitaṃ (116) °yājya ghātas (117) °saṃkhyāyās sa° (118) vargaṃ (119) yuktis sakṣepāt (120) °cit

Translation

"Having multiplied the arc and the results of each (multiplication) by the square of the arc, one ought to divide by the squares of the even (numbers) together with the (square) roots, multiplied by the square of the Radius, in order. Having put down the arc and the results one below the other, one ought to subtract going upwards. At the end is the Sine. A summary of this is made by (the verse) beginning, '*vidvān*'."

Having multiplied the given arc and the results of each (multiplication) by the square of the given arc.

"The squares of the even (numbers) together with the (square) roots". "Together with the (square) roots" means "put together with the (square) roots"; "of the even (numbers)" means "of the even numbers beginning with two"; By the squares one ought to divide.

Moreover, having put down the arc and the results of the division one below another, one ought to subtract going upwards. The Sine is produced by what is obtained in the end. This is what was said.

(1) Having multiplied the first given arc by the square of the given arc, by the number two—the square of the even (number two) together with the (square) root is the number six—one ought to divide by the square of the Radius multiplied by that. The quotient there is the first result. One ought to put it down below the given arc. Moreover, having multiplied that first result by just the square of the given arc, (and) having multiplied the square of the Radius by the square of four together with the (square) root, (i.e.) by the number twenty, one ought to divide (the first product) by that (second product). The quotient is the second result. One ought to put (it) down below the first result. Moreover, having multiplied the second result by the square of the given arc, (and) having multiplied the square of the Radius by the square of six together with the (square) root, (i.e.) by the number forty-two, one ought to divide (the first product) by that (second product). The quotient is the third result. Having thus computed more and more results, (and) having put them down in order one below the other, one ought to subtract the one below from the one above it for all (of the results). When an end is made (of the operations) thus going upwards, the remainder of (that) subtracted from the arc is the Sine of the given (arc). But it is necessary to limit (the number of) the results because of the minuteness of further results. (Thinking,) "this very computation of the Sine is taught by a different method which is its origin," he said, "a summary of this is made by (the verse) beginning '*vidvān*'."

(2) Of just this a summary is made by the verse beginning "*vidvān*":

"The wise ruler whose army has been struck down gathers together the best

of advisors and remains firm in his conduct in all matters; then he shatters the king whose army has not been destroyed.[1]

When these five (numbers) have been spoken in order, (starting) from the one which is at the bottom multiplied by the square of the given arc, when one has divided (by the square of the arc of 90 degrees), the quotient (of each) is to be subtracted, proceeding upwards, (but the quotient) of the last (is multiplied and divided) by the cube; (the quotient is to be subtracted) from the end on the arc."

How does this computation of the Sine come to be the origin of that one? Those divisors which (are used) in the computation of the results for an arc of 90 degrees (are) just those divisors (that are used) also in the computation of the results for the given arc. When one has divided the cube of the arc of 90 degrees (which is) in the multiplicand place of the multipliers, by the square of the Radius multiplied by 6, a "first result" has been computed. Then, when one has multiplied (that) "first result" by the cube of the given arc, when (the result) has been divided by the cube of the arc of 90 degrees, the first result of the given arc comes to be. But, however, the "second result" (is) multiplied by the square of the given arc, (and it is) then divided by the divisor. Therefore, when one has multiplied (this) "second result" by the cube of the given arc and by the square of the given arc, when (the product) has been divided by the cube of the arc of 90 degrees and by (its) square, the second result of the given arc comes to be. And when multiplication and division have been made, once by the cube (and) two times by the square, the third result must be computed. In the same way, for the fifth result, multiplication and division are to be made once by the cube (and) four times by the square. Moreover, that is to be subtracted from the fourth result. But, because the fourth is not less than the fifth, there is a remainder.

(3) Then, when one has multiplied once, in multiplication and division by the prescribed operation, the "fifth result" by the square of the given arc and by the square of the arc of 90 degrees, there is not equality with the "fourth result". Now (it) must be subtracted from the "fourth result". There is excess of the "third result" over the "fourth result" diminished by that ("fifth result"). Therefore when one has multiplied once the remainder of the "fourth result" by the square of the given arc, (and) when one has divided by the square of the arc of 90 degrees, (it is) to be subtracted from the "third result". Thus even (all the way) upwards multiplication and division are to be done. Because it is less, the remainder of the "third (result)" is to be subtracted from the "second result". What is obtained from that is to be subtracted from the "first (result)". Moreover, when one has multiplied the remainder of the "first result" by the cube of the given arc, when (the product) has been divided by the cube of the arc of 90 degrees, for the computing of the Sine, the result to be

[1] Or, this verse can be read, according to the *kaṭapayādi* system, as "2220, 39, 40—273, 57, 47—16, 05, 41—33, 06—44".

subtracted from the given arc is obtained. Therefore:

"(Starting) from the one which is at the bottom multiplied by the square of the given arc, when one has divided (by the square of the arc of 90 degrees), the quotient (of each) is to be substracted, proceeding upwards, (but the quotient) of the last (is multiplied and divided) by the cube; (the quotient is to be subtracted) from the end on the arc."

So it was said. In this way the origin of this was made (by the verse) beginning "*vidvān*".

(4) What sort of a demonstration is there for the computation of the Sine of the given arc by the verse beginning "*nihatya cāpavargeṇa*"? For an explanation of this, when one has drawn a circle, one ought to make the mother and father lines (i.e. perpendicular diameters[2]). When one has created the given arc on it, produced from the tip of a (mother or father) line in the east (and) going in a southern direction, (and) when one has made a dot at that tip of the arc, one ought to create some segments with their having equal intervals of this arc. Then one ought to draw a *bhujā* produced from the middle of a given segment among these (and) going in a northern direction, (and) a *koṭi* going in a western direction going to the center (of the circle), and a Radius. Then when one has multiplied the Chord of this segment of the arc by the *bhujā* and the *koṭi* separately, (and) when one has also divided twice by the Radius, the (two) quotients are the *koṭikhaṇḍa* and the *bhujākhaṇḍa*. Then the investigation (proceeds) from the quality in the shape of the two figures. But in the two figures, the *bhujā* alone of one figure, that in the circle, is the *bhujā*, and its *koṭi* alone is the *koṭi*; (and) the *koṭi* is a segment of a Radius. But the mutual equality in form of these two must be understood from the fact that the *bhujākhaṇḍa* and the *koṭikhaṇḍa* are in opposite directions. In this way, with respect to the other segments also, when one has multiplied separately the Chord of the segment by the *bhujā* and the *koṭi* that extend from the middle of the segment, (and) when one has also divided twice by the Radius, in this the *koṭijyākhaṇḍa* and the *bhujājyākhaṇḍa* are obtained. Until a segment is segmented, (so that) there is the absence of a difference between the Chord and the arc of the segment, so long ought we make a sectioning of the given arc by an equality of intervals. When one has multiplied the segment by the sum of all the *bhujās* extending from the middles of the segments, (and) when one has also divided the result by the Radius twice, the quotient is the Versine of the given arc. In the same way, when one has multiplied the segment by the sum of the *koṭis* extending from the middles of the segments, (and) when one has divided by the Radius twice, the quotient is the Sine of the given arc. On account of the difficulty of the computation of the sum of the *bhujās* extending from the middles of the segments, only the computation of the sum of its arcs is done. The

[2] Or so we suppose.

error produced by this is then removed. Whatever is the difference of the given arc which is in excess of a segment, whether it begins from the middle of the segment or from its beginning, is summed when this is multiplied by the (number of) segments, (and the product) is subtracted from the given arc. Whatever is seen to be less or in excess in this...[3]

In this way, in order to compute the sums of the sums multiplied by the segments again and again to higher and higher places, multiplication by the given arc and division by the Radius and so on are to be done. There will be no usefulness here in the computation of higher and higher sums. No employment of the sums is seen (here), because it is difficult to express and because of fear of expanding (the length of) the book. The Versed (Sines) are the numbers beginning with one (and) with a difference of one (in a series) divided by the Radius. When this has been put down separately, afterwards, it is to be multiplied by the previous (number).

The accomplishment of the result is indicated by that verse mentioned in the first act of writing of that one, which begins,

"After that by that and by that."

(5) By this method, when one has divided the square of the given arc by the radius ⟨twice⟩, the quotient is the previously computed Versine of the given arc. But in the computation of the Sine, in which multiplication of the segment by the *koṭis* extending from the middles of the segments is also to be done, when one has multiplied the Versine segment by the intervals of the *koṭijyās* that have resulted, (and) when one has divided by the Radius, the quotient is to be subtracted from the given arc. The remainder is the Sine of the given arc. Or else the segment is to be multiplied by the sum of the Versines. That sum of the Versines which are parts of the sum is the sum of the sums. Then in order to know the sum of the sums, when one has multiplied half of the given arc divided by the Radius by the square of the given arc, (and) when one has divided by the number three, again one ought to divide by the Radius. And for the computation of the Sine, the quotient is to be subtracted from the arc. Previously, for the computation of the Versine, in which the segment is to be multiplied by the sum of the Sines extending from the middles of the segments, the sum is made of the little arcs which result from there being intervals. Now for the computation of the Sine, as much as is to be subtracted from the arc is produced from that. Therefore their sum is to be obtained. But the intervals between the Sines and the arcs are produced, which have become parts of the sum of the sums. For the computation of the sum of those, when one has multiplied the intervals produced between the Sines and the arcs, by the given arc, (the product) is to be divided by the number four. When one has again divided that by the Radius, (there results) the state of being subtractable from the previously computed Versine. Moreover, for the subtraction of the Sine, the sum of what has been subtracted from the Versines

[3] This small section is corrupt.

(i.e. the sum of the differences) is to be made. When one has computed (it) for that purpose, and when one has multiplied the THIRD RESULT by the given arc, (and) when one has divided (the product) by the number five, (the quotient) is to be divided by the Radius. The result's state of being counter-subtractable is on account of the quotient's having been subtracted from the arc previously. For this very reason the Versine is counter-addable. In this way further and further results are to be computed. This is what was said. On account of the simultaneous computation of the Sine of the given arc and of the Versine of the given arc, when one has multiplied the arc by the arc, (and) when one has divided (the product) by the Radius multiplied by two, one ought to compute the FIRST RESULT. Moreover, when one has multiplied that FIRST RESULT by the arc, (and) when one has divided (the product) by the Radius multiplied by three, one ought to compute the SECOND RESULT. Moreover, when one has multiplied the SECOND RESULT by the arc, (and) when one has divided (the product) by the Radius multiplied by four, one ought to compute the THIRD RESULT. In this way, when one has continuously made multiplication of the results by the given arc and division (of the product) by the Radius multiplied by the number one increased by one (each successive time), (and) when one has computed the result, among these one ought to put down the odd (numbers), one below another, in one place for the computation of the Versine. And one ought to put down the even (numbers), one below another, in another place for the computation of the Sine. When one has put (them) down in this way, one ought to subtract in both places (the one) situated below, (proceeding) upwards. Then the result from (those) in the place of the odd (numbers) is the Versine; the result of (those) in the place of the even (numbers) is to be subtracted from the arc, and the Sine comes to be. With regard to the computation of the Versine, in which the computation of the Sine is to be done, when the even results are to be computed, (there is) multiplication by the square of the arc. When one has multiplied the square of the Radius by the product of an even number beginning with two and the odd number coming above it, division is to be done by that (product). "The product of an even number and the odd number coming above it"; "of the even number"; ⟨---⟩ "the square of the even numbers together with their roots."[4] Thus was it said by the verse beginning, "*nihatya cāpavargeṇa*."

"Thus here is this *yukti*, somehow written summarily by me. Deficiency and excess are to be removed by multitudes of wise men studying (it)."

Commentary

In the summer of 1669, it was proved by Isaac Newton that, for any arc θ,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots$$

[4] This is a quotation from the *mūla*.

(although he did not express it in exactly this notation). Expressed in similar modern notation, it was Mādhava's discovery, in the early fifteenth century, that

$$\text{Sin}\,\theta=\theta-\frac{\theta^3}{3!R^2}+\frac{\theta^5}{5!R^4}-\frac{\theta^7}{7!R^6}+\frac{\theta^9}{9!R^8}\cdots$$

where the capital "S" in "Sin" indicates that the radius is not assumed to be equal to 1 but R is the radius. For Mādhava, it was standard that

$$R=3437;44,48\simeq\frac{21600}{2\pi}.$$

The present *yukti* presents four ways of demonstrating the truth of this formula. All four are discussed by Śaṅkara.

The verse with which the text begins (the *mūla*) gives a method for the computation of the terms in the above formula and for computing the Sine from them. See Śaṅkara in *Yuktidīpikā* p. 118 and in *Kriyākramakarī* p. 390.

(1) The commentator explains the method described in the verse.

$$\text{first result}\quad=\frac{\theta^3}{R^2(2^2+2)},$$

$$\text{second result}=\frac{\theta^3}{R^2(2^2+2)}\times\frac{\theta^2}{R^2(4^2+4)},$$

$$\text{third result}\quad=\left(\frac{\theta^3}{R^2(2^2+2)}\times\frac{\theta^2}{R^2(4^2+4)}\right)\times\frac{\theta^2}{R^2(6^2+6)},\quad\text{etc.}$$

$$\text{Sin}\,\theta=\theta-(\text{first result}-(\text{second result}-(\text{third result}$$
$$-(\text{fourth result}-(\text{fifth result}))))\cdots$$

(2) See Śaṅkara in *Yuktidīpikā* pp. 117–119. The commentator quotes a verse which lists 5 sexagesimal numbers,

2220; 39,40
273; 57,47
16; 5,41
0; 33, 6
0; 0,44

which may be used in another, related method for computing the Sine. The numbers are equal to the first result, second result, third result, fourth result and fifth result (above) if θ = the arc of 90° and R = 3437; 44, 48. The commentator uses the words, 'first result', 'second result', etc. to refer to both these numbers and to the terms he describes in the preceding section. For convenience, quotation marks (in the translation only) indicate when he uses these terms to refer to the numbers listed by the verse.

Where R = 3437; 44, 48, the arc of 90° = 5400 .

$$\text{"first result"}\quad=\frac{5400^3}{3!R^2}=2220;39,40$$

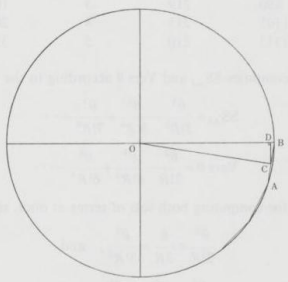
$$\text{first result}\quad=\frac{2220;39,40\times\theta^3}{5400^3}$$

$$\text{"second result"}=\frac{5400^5}{5!R^4}=273;57,47$$

$$\text{second result}\quad=\frac{273;57,47\times\theta^5}{5400^5}$$

(3) $$\text{Sin}\,\theta=\theta-\left(\frac{\theta}{5400}\right)^3\left[\frac{5400^3}{3!R^2}-\left(\frac{\theta}{5400}\right)^2\left[\frac{5400^5}{5!R^4}-\left(\frac{\theta}{5400}\right)^2\left[\frac{5400^7}{7!R^6}\right.\right.\right.$$
$$\left.\left.\left.-\left(\frac{\theta}{5400}\right)^2\left[\frac{5400^9}{9!R^8}-\left(\frac{\theta}{5400}\right)^2\frac{5400^{11}}{11!R^{10}}\right]\right]\right]$$

(4) See Śaṅkara in *Yuktidīpikā* pp. 120–121 and in *Kriyākramakarī* pp. 358–360.



segment = AB = chord; segment/2 = CB; radius = OB; koṭi = OC; bhujā = CD = bhujākhaṇḍa; koṭikhaṇḍa = DB; △OCD is similar to △CDB.

The commentator syas that

$$\text{Vers}\,\theta = \frac{\text{CB} \times \text{S(bhujās)}}{R},$$

where S(bhujās) means the sum of the *bhujās*. We have divided the given arc into so many equal segments that the chord of each segment is approximately equal to the arc. Therefore, C bisects both the chord and the segment, and $\text{OC} \simeq R$.

(5) See Śaṅkara in *Yuktidīpikā* pp. 112–117. One final method is described, as set out by an unidentified verse, which our commentator does not quote in full. It was known in India that

$$\text{Sin}\,\theta = \theta - \text{SS}_{\Delta\Delta}, \quad \text{and}$$
$$\text{Cos}\,\theta = R - \text{Vers}\,\theta,$$

where $\text{SS}_{\Delta\Delta}$ is the sum of the sums of the second differences of the Sines. In illustration of this, we give here the beginning of the Indian Sine table, with increments of 3; 45° of arc:

θ	Sin θ	Δ Sin	$\Delta\Delta$ Sin	$\text{SS}_{\Delta\Delta}$	$\theta - \text{SS}_{\Delta\Delta}$
225′	225	225	—	—	—
450	449	224	1	1	449
675	671	222	2	4	671
900	890	219	3	10	890
1125	1105	215	4	20	1105
1350	1315	210	5	35	1315

The commentator computes $\text{SS}_{\Delta\Delta}$ and Vers θ according to the following formulas:

$$\text{SS}_{\Delta\Delta} = \frac{\theta^3}{3!R^2} - \frac{\theta^5}{5!R^4} + \frac{\theta^7}{7!R^6} - \cdots$$
$$\text{Vers}\,\theta = \frac{\theta^2}{2!R} - \frac{\theta^4}{4!R^3} + \frac{\theta^6}{6!R^5} - \cdots$$

He uses a method for computing both sets of terms at once, since

$$\frac{\theta^2}{2!R} \times \frac{\theta}{3R} = \frac{\theta^3}{3!R^2}, \quad \text{and}$$
$$\frac{\theta^3}{3!R^2} \times \frac{\theta}{4R} = \frac{\theta^4}{4!R^3}, \quad \text{etc.}$$

The terms, 'first result', 'second result', etc., are used in yet a third way in this section of the text. To avoid confusion, we have printed them in CAPITAL LETTERS in the translation when they have this third meaning.

$$\text{FIRST RESULT} = \frac{\theta^2}{2!R},$$

$$\text{SECOND RESULT} = \frac{\theta^3}{3!R^2},$$

$$\text{THIRD RESULT} = \frac{\theta^4}{4!R^3}, \quad \text{etc.}$$

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Paper 41

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K. V. Sarma

For a biographical note on K. V. Sarma see p. 79.

S. Hariharan

No biographical details of S. Hariharan were available online at the time of compiling this collection,

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YUKTIBHĀṢĀ OF JYEṢṬHADEVA
A Book of Rationales in Indian Mathematics and Astronomy
An Analytical Appraisal

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During medieval times there had occurred in Kerala a spurt of activity in the mathematical and astronomical studies. What is particularly significant herein is that some of the works produced related to *Yukti-s* and *Upapatti-s*, i.e. rationales and proofs. For this reason, these texts flash a torch, as it were, in the dark areas of Indian mathematics and astronomy and enable us to understand how these scientists designed methodologies and derived formulae, the application of which produced reliable results.

One of the important texts of this type is the *Yuktibhāṣā of* Jyeṣṭhadeva (1500-1610). Since the work has been composed in the local language of Kerala, *viz.*, Malayalam, it has, by and large, remained beyond the ken of scholars not knowing that language, with the result that there have been but sporadic studies and that too only on certain topics dealt with therein. An attempt is made in this paper to give an idea of the contents of the work, drawing attention to certain peculiarities therein, towards making a detailed study thereof. The question of its authorship and date have also been tackled.

The work, which when printed, would cover about 300 pages, is divided into two parts, with 17 chapters in all. The First Part is related to the depiction of the rationales of Arithmetic, Algebra, Geometry and Trignometry, and the Second Part to Astronomy. The seven chapters of Part I deal with: I. Logistics, II. Ten Algebraic problems, III. Fractions, IV. Rule of Three, V. *Kuṭṭākāra*, VI. Circle, and VII. Disquisition on R-Sine. Part II deals with: Ch. VIII. Planetary motion, IX. Celestial sphere, X. Declination and Right Ascension, XI. Fifteen Problems in Right Ascension, Declination etc., XII. Directions and Shadow, XIII. Ten Problems on Spherical Triangles, XIV. *Lagna* and *Kālalagna*, XV. Eclipses and Parallax Corrections, XVI. *vyatīpāta*, and XVII. *Dṛkkarma* or Reduction to observation, and the phases of the Moon.

Introduction

During medieval times, there had been in Kerala, in the southern-most part of India, an effusion of mathematical and astronomical investigations, at least a substantial part of which has been documented in original works, commentaries and minor tracts devoted solely or primarily to the elucidation of theories and rationales of computation.[1] An important work among these is the *Yuktibhāṣā* (*YB*), 'Rationale in the Malayalam language', whose sole aim is to rationalise the theories involved in the constants and computations occurring in the *Tantrasaṅgraha*, an important astronomical work of Nīlakaṇṭha Somayāji (A.D. 1443-1560).[2] Thus, after the benedictory verses, the work commences with the statement:

> "aviṭe naṭe trantrasaṅgrahatte anusariccu grahagatiyiṅkal upayogamuḷḷa gaṇitaṅṅale muzhuvanāyi colluvān tuṭaṅṅunnēṭattu..."

"Here, commencing an elucidation in full of the rationales of planetary computations according to the *Tantrasaṅgraha*..."

The work finds its first reference in modern writings in an article by C.M. Whish in 1835,[3] where it is referred to towards verifying the date of the author of *Tantrasaṅgraha*.[4] Whish had contemplated "a farther (*sic*) account of the Yucti-Bhasha...will be given in a separate paper,"[5] which, however, does not appear to have been written or published.

Yuktibhāṣā, known also as *Gaṇitanyāyasaṅgraha* 'Compendium of astronomical rationale', has been a popular text in Kerala for more than four hundred years, since its composition towards A.D. 1530. Several manuscripts of the work are known.[6] However, since the work is couched in the Malayalam language which is spoken only in Kerala it has remained, practically, beyond the purview of scholars who did not know the language, in spite of its having been published.[7] And the few articles on this important work relate to only certain individual topics treated herein.[8] It is therefore necessary that a critical appraisal of the nature and contents of this work as a whole is made, so as to enable scholars take up the same for further study.

Authorship Of Yuktibhāṣā

The introductory verses of the *YB* do not mention the name of its author, not do its manuscripts indicate his name at their closing colophons. However, one of its manuscripts preserved in the Sanskrit College Library, Tripunithura which had been used by Rama Varma Maru Thampuran for his edition of the work, had at its close the verse:

> alekhi yuktibhāṣā vipreṇa Brahmadattasaṃjñenal
> 'ye golapathasthās syuḥ' kalirahitāḥ śodhayantas tel

Taking the word *alekhi* in the verse to mean 'composed' instead of its natural meaning 'written, copied', the Introduction to the said edition took Brahmadatta mentioned in the verse as the author, and the date given by the Kali chronogram *ye golapathasthās syuḥ* corresponding to A.D. 1750, as the date of its composition.[9]

There are, however, evidences which point to the correct name of the author of *yuktibhāṣā* as Jyeṣṭhadeva and his date to be A.D. 1500-1610. Thus, an old palmleaf manuscript, No.755, of the Kerala University Manuscripts Library, entitled *Gaṇitayuktayaḥ* contains many astronomical tracts, in one of which, dealing with the precession of the equinoxes (*ayanacalana*), occurs the statement:

atha vikṣiptacalanasyānītau pūrvaśūribhiḥ|
proktā ye matabhedās tān yakṣye tattvabubhutsayā||

* * * * * * *

jūkakriyādike pāte svarṇam tatsādhane vidhau|
ity uktā kṣepacalanasyānītis *tantrasaṅgrahe*|
jyeṣṭhadevo 'pi *bhāṣāyam* nādhikam kiñcid uktavān||

The *Tantrasaṅgraha* referred to here is,obviously, the work of Nīlakaṇṭha Somayāji, and the *Bhāṣā* specified as the work of Jyeṣṭhadeva is the *Yuktibhāṣā*, which, as indicated above, seeks to set out and elucidate the theories and practices involved in the *Tantrasaṅgraha*.

There are more clear evidences which point to the correct name of the author of *Yuktibhāṣā* as Jyeṣṭhadeva and his date to be A.D.1500-1610. Thus, an astronomical chronology (*granthavari*) in the Malayalam language found as a post-colophonic statement in an old palmleaf manuscript of a Malayalam commentary on *Sūryasiddhānta* preserved in the Oriental Institute, Baroda, Ms. No. 9886, contains in it the statements:

"parameśvaran vaṭaśśeri nampūri, nilāyāḥ saumyatīrasthaḥ parameśvaraḥ...asya tanayo dāmodaraḥ, āsya śiṣyo nīlakaṇṭha-somayāji, iddeham *tantrasaṅgraham, āryabhaṭīyabhāṣyam* mutalāya granthaṅṅaḷku karttāvākunnu. 'lakṣmīśanihitadhyānaiḥ' iti asya kalinā kālanirṇayaḥ.

"purvokta-dāmodarasya śiṣyaḥ jyeṣṭhadevaḥ, iddeham paraṅṅoṭṭu nampūriyākunnu *yuktibhāṣa*-granthatte uṇṭākkiyatum iddeham canne

"jyeṣṭhadevante śiṣyan tṛkkaṇṭiyūru acyuta-piṣāraṭi, iddeham *sphuṭanirṇayam, goladīpikā* mutalāya grantha (ṅṅaḷkku) karttāvākunnu.

"acyuta-piṣāraṭiyute śiṣyan nelputtūru nārāyaṇa bhaṭṭatiri, iddeham *nārāyaṇāyam, (prakriyā) sarvasvam* mutalāya granthaṅṅaḷkku kartā. 'āyur-ārogyasaukhyam' ityādi-kalinā kālanirṇayaḥ. ...."

'Parameśvara was a Nampūri from vaṭaśśeri (family). He resided on the northern bank of the nilā(river)...His son was Dāmodara. Nīlakaṇṭha Somayāji was his pupil. He, (the latter), is the author of the *Tantrasaṅgraha, Āryabhaṭīyabhāṣya* and other works. His date is determined by the Kali days 16, 80, 553 (A.D. 1500)

'Jyeṣṭhadeva was the pupil of the above Dāmodara. He was a Nampūri from Paraṅṅoṭṭu (family).[10] He is also the author of the work *Yuktibhāṣā*. 'Acyuta Piṣāraṭi of Tṛkkaṇṭiyūr was the pupil of Jyeṣṭhadeva. He is the author of the *Sphuṭanirṇaya, Goladīpikā* and other works.

'Melputtūr Nārāyaṇa Bhaṭṭatiri was the pupil of Acyuta Piṣāraṭi. He is the author of the *Nārāyaṇīya*, (*Prakriyā*) *sarvasva* and other works. His date is determined by the Kali days 17, 12, 210 (A.D.1587).'

Here, it is specifically stated that Jyeṣṭhadeva is the author of *Yuktibhāṣā* and the teacher-pupil succession is: Parameśvara (A.D.1360-1455) — son, Damodara — pupils, Nīlakaṇṭha Somayāji (1443-1560) and Jyeṣṭhadeva (1500-1610 *c.*) — pupil, Acyuta Piṣāraṭi (1550-1621) — pupil, Nārāyaṇa Bhaṭṭatiri (A.D.1587).

That Jyeṣṭhadeva was the teacher of Acyuta Piṣāraṭi is stated by Piṣāraṭi himself in the concluding verse of his work on the computation of eclipses, entitled *Uparāgakriyākrama*:

> 'proktaḥ pravayaso dhyānāt' jyeṣṭhadevasya sadguroḥ|
> vicyutāśeṣadoṣenetyacyutena kriyākramaḥ||

'Thus has been stated the (*Uparāga*)-*Kriyākramaḥ* by Ācyuta of clear thought through his contemplation of (the teachings of) his aged benigh teacher Jyeṣṭhadeva.' The Malayalam commentary to *Uparāgakriyākrama* explains that the expression *proktaḥ pravayaso dhyānāt* serves also a chronogram to give the date of completion of the work. This chronogram works out to A.D. 1592, when Acyuta Piṣāraṭi, pupil of jyeṣṭhadeva, composed the work.[11]

Whish, in his article referred to above, records a tradition that the author of the *Yuktibhāṣā* wrote also a work called *Dṛkaraṇa*.[12] The *Dṛkkaraṇa* in question, an astronomical manual in Malayalam verse, is available in manuscript from (no.C.7-C of the Kerala Univ. Mss. Library) but does not give anywhere the name of its author. However, it gives its date of composition in its final verse through the Kali chronogram *kolambe barhisūnau* (Kali days 7, 83, 391) which would be A.D.1608. In view of the tradition recorded by Whish and this date being not far from 1592 mentioned by Acyuta Piṣāraṭi, we might take the *Dṛkkaraṇa* to be a work of Jyeṣṭhadeva and that he lived up to about 1610.

In view of the facts that Dāmodara (c. 1400-1500) was a teacher both of Nīlakaṇṭha Somāyāji and Jyeṣṭhadeva, and that Jyeṣṭhadeva wrote his *Yuktibhāṣā* in the wake of

Nīlakaṇṭha's *Tantrasaṅgraha*, he must be a younger contemporary of Nīlakaṇṭha. He is remembered in 1592 by his pupil Acyuta Piṣāraṭi as *pravayas* ('very old'). His *Dṛkkaraṇa* is dated in 1608. Jyeṣṭhadeva should, therefore, have been long-lived, his date being *c.* 1500-1610. His family house Paraṅṅoṭṭu (Skt. Parakroḍa) still exists in the vicinity of Ālattūr and Trakkaṇṭiyūr where well known astronomers like Parameśvara, Nīlakaṇṭha and Acyuta Piṣāraṭi flourished about those times.

Scope And Extent Of Yuktibhāṣā

The entire text of *Yuktibhāṣā* occurs as one continuum, without any internal or closing colophons to mark off the subjects treated in the work. However, towards the middle of the work, where the treatment of mathematics ends and that of astronomy commences, occurs a general benedictory statement which reads: *Śrīr astu, hariḥ śrī-gaṇapataye namaḥ, avighnam astu*. This would naturally mean that the author had conceived his work as consisting of two parts, devoted respectively to mathematics and astronomy. Since the work deals with several main subjects and a number of topics under each, the needed subject and topic divisions shall have to be made editorially with suitable indication. Demarkating the work thus, the main subjects treated in Part I: Mathematics are: *Parikarma* (Logistics), *Daśapraśna* (Ten problems involving logistics), *Bhinnagaṇita* (Fractions), *Trairāśika* (Rule of three), *Kuṭṭākāra* (pulverisation), *Paridhi-vyāsa* (Relation between circumference and diameter) and *Jyānayana* (Derivation of Rsines). The subjects treated in Part II. Astronomy are: *Grahagati* (Planetary motion), *Bhagola* (Sphere of the zodiac), *Madhyagraha* (Mean Planets), *Sūryasphuṭa* (True Sun), *Grahasphuṭa* (True Planets), *Bhū-Vāyu-Bha-gola* (Spheres of the Earth, Atmosphere and Asterisms), *Ayanacalana* (Precession of the Equinoxes), *Pañcadasapraśna* (Fifteen problems relating to spherical triangles), *Digjñāna* (Orientation), *Chāyāgaṇita* (Shadow computations), *Lagna* (Rising point of the Ecliptic), *Nati-Lambana* (Parallaxes of Latitude and Longitude), *Grahaṇa* (Eclipse), *Vyatīpāta*, and *Śṛngonnati* (Elevation of the Moon's horns).

Kriyākramakarī, Yuktidīpikā And Yuktibhāṣā

There are two extensive commentaries, both by Śaṅkara Vāriyar of Tṛkkuṭaveli family (A.D. 1500-1660), being *Kriyākramakarī* and *Yuktidīpikā*, the former on the *Līlāvatī* of Bhāskara II,[13] and the latter on the *Tantrasaṅgraha* of Nīlakaṇṭha Somayāji.[14] Interestingly, there is a close affinity between the *Yuktibhāṣa* and the above-said two commentaries. Even more, there is some sequence of arguments and verbal correspondences amongst them in the treatment of identical topics. From this similitude it has been suggested that the *Yuktibhāṣā* is just a rendering into Malayalam of certain passage from that Sanskrit. It is further suggested that for this reason, there is not much that is original in the *Yuktibhāṣā*.[15] But the fact is just the other way round, namely that the Sanskrit versions are adaptations and paraphrases of the relevant portions from the *Yuktibhāṣā*. This is confirmed by Śaṅkara, the author of both the commentaries, when he states specifically in the colophonic verses of his commentary

Yuktidīpikā on the *Tantrasaṅgraha*, that what he had done in the commentary was only 'the setting out of the material elucidated in the work of the Brāhmaṇa of Parakroḍa (viz. Jyeṣṭhadeva) (author of the *Yuktibhāṣā*). *Cf., for instance one such colophonic verse:*

ity eṣa parakroḍāvāsa-dvijavara-samīrito yo 'rthaḥ|
sa tu tantrasaṅgrahasya prathame 'dhyāye mayā kathitaḥ|| (Edn., p. 77)

Yuktibhāṣā And Gaṇita-Yuktibhāṣā

There is a work entitled *Gaṇita-Yuktibhāṣā* (Ms. No. R.4382 of the Govt. or. Mss. Library, Madras) in Sanskrit and it has been suggested that it might be the source of the Malayalam *Yuktibhāṣā*. However, a detailed comparison of the two shows that the *Gaṇita-Yuktibhāṣā* is but a rough and ready translation into Sanskrit of the Malayalam original by one who lacked not only the ability of writing idiomatic Sanskrit but also an adequate knowledge of the subject. Moreover, at places there occur haplographical omissions in the Sanskrit version of passages available in the Malayalam work, which fact too confirms that the Sanskrit version is the derived form.

Presentation Of The Rationale

The mathematical and astronomical rationale presented in the *Yuktibhāṣā* relate to several aspects, to wit, concepts, theories, constants, computations, demonstration by diagrammatic representation and the like. The procedure of depiction is logical, going step by step, first presenting the fundamentals and gradually building up the argument. It is, if one might say so, 'intimate' in that it inculcates the rationales and elucidates the steps even as a teacher does to a student. The work aims at understanding and conviction by the reader. A passage might be cited to illustrate the point. The selected passage relates to the concept of the motion of the planets:

Grahaṅṅalḷuṭe gatiprakāram

Iviṭe grahaṅṅaḷellām oru vṛttamārgēṇa gamikkum, Divasattil vṛttattinte itra aṃśam gamikkum ennu niyatam tānum, Aviṭe divasattil itra yojana gamikkum ennuḷḷa yojanāgati ellā grahattinum samam, Aviṭe ceriya vṛttattiṅkal gamikkunnavattinnu kuraññoru kālam koṇṭu baṭṭam kūṭum, Valiya vṛttattiṅkal gamikkunnavattinnu perikekkālam Kūṭiyē vaṭṭam tikayū, Enniṭṭu cantrannu irupattēzhu divasam koṇṭu pantraṇṭu rāśiyiṅkalum gamiccu kūṭum, Muppattiraṇṭāṇṭu kūṭiyē śani naṭannu kūṭū, Vṛttattinte valippattinnu takkavaṇṇam kālattinte perukkam, Atāṭu graham tante vṛttattiṅkal oru vaṭṭam gamiccu kūṭunnatinu bhagaṇam ennu pēr, Caturyyugattiṅkal etra āvṛtti gamikkum tante tante vṛttattiṅkal atu tante tante yugabhagaṇam ākunnatu.

Iviṭe candrane oru nāḷ oru nakṣatrattōṭu kūṭe kaṇṭāl pittennāḷ atinte kizhakke nakṣatrattōṭu kūṭe kāṇām, Itinekkoṇṭu candranu gatiyuṇṭennum kizhakkōṭṭu gatiyennum kalpikkām, Kizhakkōṭṭu rāśikramam ennum kalpikkām.

Ī vṛttaṅṅaḷkku ellāttinum kūṭi oru pradeśatte ādiyennu kalpikkumāruṇṭu, Aviṭattinnu meṣarāśiyuṭe ādiyennu pēr, Ī goḷattiṅkal kalpikkunna vṛttaṅṅaḷe ellāttēyum irupattōrāyirattarunūru khaṇḍamāyi vibhajikkumāruṇṭu, Itil ōrō khaṇḍam iliyākunnatu! Iva valiya vṛttattiṅkal valutu, ceriya vṛttattiṅkal cerutu, Saṅkhya ellāttinkalum okkum, Atatu graham tante tante vṛttattiṅkal itra īli gamikkum ōrō divasam ennu niyatam, Graham gamikkunna vṛttattinte kendrttiṅkal irunnu nōkkum draṣṭāvu eṅkil nityavum gati okkum ī grahattinnu ennu tōnnum, Bhūmadhyattiṅkēnnu oṭṭu cellū grahavṛttakendrzm, Bhūmīṅkalu draṣṭāvu, Ī draṣṭāviṅkal kendram āyiṭṭu oru vṛttam kalpippū grahattōṭu sparśikkumātu, Ā vṛttattiṅkal etra cenniṭṭirikkuṁ graham atra cennū meṣādiyiṅkal ninnu graham ennu tōnnum ī draṣṭāvinu, Itariyum prakāram sphuṭakriyayākunnatu."

Planetary Motion

'Now, all planets move along circular (orbits). The number of minutes which each planet moves in its orbit in the course of a day is fixed.

There, the number of *yojana-s* moved per day is the same for all planets. For planets which move along smaller orbits the circle would be completed in a shorter time. For those which move along bigger orbits the circle would be completed only in a longer period. For instance, the Moon would have completely moved through the twelve signs in 27 days, while Saturn will complete it only in 30 years. The length of time taken is proportional to the size of the orbit. The completion of one motion of a planet in its orbit is called a *bhagaṇa* (of that planet). The number of times that a planet moves in its orbit during a *catur-yuga* is called its *yuga-bhagaṇa*. (revolutions per aeon).

'Now, if the Moon is seen with an asterism on a particular day, it will be seen the next day with the asterism to the east of (the first one). From this it might be understood that the Moon has eastward motion. The sequence of the signs can also be understood as eastward. For all these orbits, a particular point is taken as the commencing point. This point is termed as the First point of Aries. All the circles in a sphere are divided into 21,600 equal parts. Each part is a minute. They are larger (in length) in bigger circles and smaller in smaller circles, though the number of parts is the same in all. The number of minutes that a planet will move along its orbit during the course of a day is fixed. If one perceives the said motion placing himself at the centre of the orbit of a planet, the motion of the planet would appear equal every day. The centre of the planetary orbit is the centre of the earth. The perceiver is, however, situated on the Earth's surface. Conceive a circle with the perceiver as centre and touching the planet. The perceiver would see the planet as much advanced from the First point of Aries as the planet has advanced in the said circle. The method by which the perceiver ascertains this (advanced) point is called 'Computation of the True planet'.

Analytic Contents Of The Yuktibhāṣā

As mentioned earlier, Pt. I of the *Yuktibhāṣā*, dealing with Mathematics, can be

divided into seven chapters.

Ch. I on Logistics (*Parikarma*) deals with the Nature of numbers (pp. 1-2),[17] Multiplication as explained from several standpoints including the use of diagrams (pp. 2-9), Multiplication by easy methods (pp. 9-13), Divison (p. 15), Squaring in its several methods (pp. 15-29), and Roots of sums of squares and difference of squares (pp. 29-31).

Ch. II. *Daśapraśnottara* (Ten algebraic problems and their solutions) relates to the finding of two numbers when two from among the five, viz., their sum, difference, product, sum of squares and difference of squares, are given. (pp. 32-35).

Ch. III deals with Fractions (pp. 36-44). Highly analytical in presentation, the topics dealt with are: Nature of fractions, their addition, subtraction, multiplication and divison.

Ch. IV is entirely devoted to the Rule of three, where both its general and inverse rules are set out. (pp. 45-49).

In Ch. V (pp. 50-71), the importance of the Rule of three in mathematical and astronomical computations is indicated with the computing of the current Kali day.

The Mean Planets are divided therefrom. *Apavartana* (Reduction) and *Kuṭṭākāra* (Pulverisation) are introduced here and their purpose specified. An elaborate rationalisation of the principles involved in it and the practices followed in both Reduction and Pulverisation is made and explained with examples. It is also indicated how the process helps in arriving at the *Bhājya-s* (Divisers) and *Bhājaka-s* (Multiplicands) in Pulverisation by means of the *Vallī* (Series of divisons), towards computing the planets more and more accurately.

In Ch. VI (pp. 72-142), the *Yuktibhāṣā* gives several formulae for determining the circumference of a circle of a given diameter. Some of the methods involve the properties of right angled triangles, towards which the properties of right angled triangles are demonstrated graphically. For the method which involve different summations of series, the derivation of those series is also demonstrated. Among the latter are the summations of consecutive numbers, the summation of squares, the summation of cubes and higher powers, and the summation of summations. In the case of certain formulae, *Yuktibhāṣā* enunciates further rules towards making the derived results more and more accurate (pp. 120ff.). Attention of scholars might be drawn here to a series of papers wherein Prof. C.T. Rajagopal, lately Director, Ramanujan Institute of mathematics, University of Madras, and his associates have worked out, in terms of modern mathematics, the above-said series and the different formulae enunciated in the *Yuktibhāṣā*. They have also shown that these are independent of the discoveries made more than a century later by the Western scientists, James Gregory (1671), G.W. Leibniz (1673) and Isaac Newton (1670).[17a]

Ch. VII (pp. 143-290) forms a long disquision of rationales relating to Rsines, their modifications and allied subjects. The chapter commences with an elaborate exposition of the geometrical derivation of the 24 Rsines for 3° 45′ each (pp. 143-50), and explanations of allied terms like Rcosine, Rversed sine, ʰarc, *bhujākhaṇḍa, koṭikhaṇḍa, jīvākhaṇḍa* and *khaṇḍajyā*, and their mutual relationship (pp. 150-60). Some of the other topics elucidated herein are: Accurate determination of the Rsines (pp. 160-65), Computation of the Rsines of any given arc (pp. 165-73), Computation of the arc of any given Rsine (pp. 173-75), Summation of Rsine differences and Rversed sine differences (pp. 176-98). Accurate determination of the circumference of a circle making use of the said summations (pp.198-206), Addition of Rsines of two angles (pp. 206-24, 237-44), Cyclic quadrilaterals and their properties (pp. 224-37, 247-64), Square of the area of a circle (pp. 264-72), Derivation of Rversed sines (pp. 272-75), Derivation of Rsine shadow (pp. 275-78), Surface area of a sphere (pp. 278-82), and Volume of a sphere (pp. 282-90). The author of the *Yuktibhāṣa* elucidates the rationales of the several items either geometrically or algebraically or using both means as the occasion demands.

Part II of *Yuktibhāṣā* contains many interesting formulae which are scarcely found in other works. For the sake of convenience, this Part is amenable to be divided into ten chapters, VIII too XVII, in consideration of the subject dealt with in them.

Ch. VIII deals with the Planetary Theory. At the outset, the concepts of the Sun, Moon and other planets, their linear velocities and angular motions are described. As in other Hindu astronomical treatises, the assumption is made that the linear velocity of all planets is the same but the angular motion varies depending on the dimension of their circular orbits. The Epicyclic Theory for the sun and the moon and for the other planets and the interaction of the *manda* and *śīghra* epicycles are explained. There is a new concept in the treatment of the epicycles as compared to the *Āryabhaṭīya* and other works.

In a nut-shell the planetary theory broadly is like this. The earth is the centre and the sun and the moon go round the earth. As for other planets, with earth as centre the *śīghra* goes round the earth with the mean motion of the sun. The mean planet moves on a circle with *śīghra* as centre. The true planet is on the *mandocca* circle with mean planet as its centre. Alternatively, the last two circles can be interchanged. This theory is advocated byNilakaṇṭhain his commentary on *Āryabhaṭīya* and practically all later Kerala authors have followed suit. In factNilakaṇṭhatries to say that it was the view of Āryabhaṭa also. If *śīghra* is identified with the sun itself then this agrees broadly with the modern theory with the positions of earth and sun reversed. In fact the western astronomer Tycho Brahe (1546-1601) appears to have adopted a similar theory.

The rationale for adopting three or four stages of operations to find the geocentric longitude of a planet is also explained. This particular aspect has been baffling the scholars as the rationale for this is not to be found in any work. It is generally believed

that the Hindu astronomers were not aware that the true geocentric longitude is to be obtained in two steps by first applying the *manda* correction, and using the corrected planet the *śīghra* equation is to obtained and applied to the once corrected planet; therefore they adopted different methods involving three or four stages in different ways.[18] However later Kerala works like *Sphuṭanirṇaya* have clearly explained the procedure in two stages correctly. This is not surprising if the longitude is arrived at on the basis of the planetary theory described above. Nīlakaṇṭha also in his commentary on Āryabhaṭīya has given a similar method. Then how and why these three or four stages method got in. Some explanation has been given in some papers.[19] But in the *Yuktibhāṣā* the rationale has been expounded beautifully. In short it is like this. In arriving at the *śīghra* correction the mean *mandakarṇa* is taken instead of true *mandakarṇa*. This is because the tables of *śīghra-jyās* can be constructed only on the basis of mean *mandakarṇa*. Hence a correction becomes necessary for the *śīghra-phala*. To achieve this a correction is effected in the true *manda* planet in such a way that this correction together with its effect on the tabular *śīghra-phala* will compensate for the error in *śīghra-phala* due to the difference in *mandakarṇa*. The longer commentary on *Tantrasaṅgraha* also gives this rationale as also the planetary theory. There are many small tracts[20] in Kerala wherein various alternate methods are advocated to give effect to this correction. It is a moot point whether Āryabhaṭa and other astronomers were aware of this rationale or they just hit at these methods by trial and error.

The treatment of latitude with regard to the planets in the *YB* is satisfactory. The theory is that the *śīghra* circle is always on the ecliptic plane and only the plane of the planet's path gets deflected. This accords with facts and therefore the resulting helio and geocentric latitudes also represent the correct position as also the distance between the earth and the planet making allowance for the latitude.

Ch. IX deals with Celestial Sphere. This chapter deals with the celestial sphere and the related great circles such as meridian, horizon, equator, ecliptic and their secondaries, and the small circles such as day parallels etc. The poles of great circles and their relation with mutually perpendicular great circles, the equinoctial and solstitial points are explained. First the celestial sphere for an observer on the terrestrial equator is described and then changes are explained as the observer moves to a northern latitude. Then the effects of *Ayana-calana* or the backward motion of the first point of Aries on the equator, ecliptic etc., are considered. Lastly the procedure for construction of an armillary sphere is described.

Ch. X is on Declination, Right Ascension and related problems. In the first place the declination (*krānti*) of the sun (an object of the ecliptic) is considered. The formula for this sin (declination) = sin (longitude) x sin (obliquity), is got by using the properties of similar triangles and by proportion. In addition, two more concepts which are not found in other works are introduced. These are: Draw a secondary to the equator

through the first point of aries and a great circle through the solstitial points and the sun. The arcs between their point of intersection and the sun and the first point of aries are called *Krāntikoṭi* and *Nata*, respectively. (*Krāntikoṭi* is not cos *krānti* which is known as *dyujyā*). We may call them as 'inverse declination' and 'inverse R.A.'. These concepts are used to determine various other formulae which are described in the next chapter. Then the method of arriving at the R.A. or *Kālajyā* is described.

But the most important derivations in this chapter are the exact formulae for declination and R.A. of a star which is not on the ecliptic and therefore has a latitude, not necessarily of a small magnitude. This problem has not been satisfactorily solved by Indian astronomers till then and only approximate solutions have been given which are valid only if the latitude is small.[21] The formulae derived here accord with the modern formulae.

Ch. XI has tackled 15 problems. Taking the sun and the *krānti* triangle and the *krāntikoṭi* triangle there are six elements viz., longitude, R.A., declination, obliquity of the ecliptic, *nata* and *krāntikoṭi*. The problem is, given two of these six the other four are to be found out. There will be in all 15 cases which will arise. All these cases are examined and the methods for finding the other 4 elements are discussed exhaustively from fundamentals. This is a very interesting and instructived exercise to understand how the contemporary astronomer's mind worked in solving problems on spherical trigonometry.

Ch. XII deals with Direction and Shadow. This chapter starts with the method for finding accurately the east-west and north-south lines or directions. The method adopted is the familiar one, marking off the points where the shadow of a gnomon touches the circumference of a circle with centre at the foot of the gnomon, forenoon and afternoon, and joining them to get the east-west line. Since there will be a change in the declination of the sun between the forenoon and afternoon this line will not accurately depict the east-west line and a correction is provided to rectify the error. Then *Kujyā* and *Carajyā* are defined and the formulae derived. The method for arriving at the shadow for given time is next considered and the standard method is adopted. But in this two corrections are made. The sun is not a point but a sphere and the umbra is to be taken as the shadow and a correction is needed for this. Secondly the correction due to the parallax of the sun is also explained. The converse problem of finding time from shadow is tackled by the standard method after giving effect to the above two corrections. Then the problems connected with noon shadow, *samaśaṅku*, corner shadow etc., are dealt with in the usual manner.

Ch. XIII is again an interesting section concerned with shadow problems. We have five elements of the spherical triangle joining the sun, zenith and the north pole. The

five elements are the three sides and two angles of the triangle. That is, zenith distance, co-declination, co-latitude which are the three sides and, the azimuth and the hour angle the two angles.(Actually the text would use either these elements or their complements). Out of these five, if any three are known, the problem is to find the other two. There will be ten cases to be solved. All these cases are taken up and the solutions are derived methodically on the basis of the properties of spherical triangles.

This subject is dealt with in *Tantrasaṅgraha* also as it should be and based on this R.C. Gupta has presented a detailed paper.[22] He has however remarked that the rationales of the rules are not given in the work (*Tantrasaṅgraha*) and his paper verifies the rules by the modern formulae. Well! the rationales are fully documented here in *Yuktibhāṣā*. In fact the rationales are also given in the longer commentary on *Tantrasaṅgraha*, that is, *Yuktidīpikā*.

Ch. XIV is on Lagna and Kālalagna. The equator cuts the horizon of any place at two fixed points, east and west. The ecliptic also cuts the horizon at two points but since the position of the ecliptic varies every moment these points also very moment to moment, oscillating on either side of the east and west points. These two points at east and west are called *Udayalagna* and *Aṣṭalagna* or the rising and setting points of the ecliptic. The distance from the first point of the aries to the east point and *Udayalagna* are called the *Kālagna* and *Lagna*, respectively. This chapter deals with the method of calculating these two longitudes. Arriving at the lagna is a standard problem in all texts. The procedure followed generally is to find the rising times of the twelve signs for the local place and from the longitude of the sun obtain by interpolation the longitude of the rising point or *lagna* for the desired time. This is bound to be approximate. However, in *Yuktibhāṣā* a direct method is adopted. First the zenith distance of nonagesimal (*dṛkkṣepa*) is obtained and from that the *lagna* is arrived at. This method perhaps is again one of the Kerala methods.

Ch. XV deals with Eclipses and Parallax Corrections. In the treatment of the circumstances of an eclipse there is not much to comment. However we deal with two matters, one the second correction for the moon and the other the effect of parallax in longitude and latitude. The second correction for the moon which takes the place of the modern Evection plus the deficit in the equation of centre of the moon was known in India at least from the 10th century. There is a detailed paper by K.S. Shukla on this subject[23] wherein he has dealt with this subject as contained *Tantrasaṅgraha* also. As can be expected *Yuktibhāṣā* follows *Tantrasaṅgraha*, more or less, and in addition gives the basis for this correction. This basis or theory is the same as explained in Shukla's paper. However a refinement has been made in this work and that is when the moon has a latitude its distance from earth should be calculated taking the latitude also into account.

Now coming to parallax the idea is to claculate the effect of parallax in the longitude and the latitude known as *lambana* and *nati* respectively. The usual formula for these as is in vogue in Kerala texts are derived on the basis that the object is on ecliptic, that is, there is no latitude. While other texts deal with this problem differently giving in some cases only approximate results, the method followed here is exact. Further other texts do not take into account the latitude on the plea that during an eclipse the latitude of the moon is negligible. But here necessary formulae are derived to take into account the latitude also.

Then to derermine the angle of the sun's or the moon's disc at which the eclipse starts or ends and to graphically depict an eclipse the *valanas* or deflections are required to be calculated. Generally the texts deal with two deflections *ayanavalana* and *aksavalana* being the deflections due to obliquity of the ecliptic and the latitude of the place respectively. In *YB* in addition, a deflection due to latitude of the moon is also derived. This deflection is generally found in other Kerala works also.

Ch. XVI is on Vyatīpāta. *Vyatīpāta* occurs when the sun and the moon have equal declination and they are in the same *ayanabhāga* but in different quadrants or in different *ayanas* but in the same type of quardant, odd or even. This is considered very inauspicious, particularly in Kerala, and the days on which *vyatīpāta* falls are discarded for the performance of good *karmas* All Kerala works on astronomy devote a chapter for determining *vyatīpāta*. Not only this, there are full fledged works which deal with exclusively *vyatīpāta* and as in the case of eclipses the beginning, middle and ending of *vyatīpāta* are defined and the methods to arrive at these moments are also explained. From the definition it will be seen that the declinations of the sun and the moon are required to be calculated for this purpose. The declination of the sun presents no difficulty. In the case of the moon allowance has to be made for the latitude. The method of arriving at the exact declination of the moon has already been explained in Chapter IX.

In this Chapter an alternate method is given with its rationale. The method adopted is like this. The angle between the moon's path and equator is not constant but varies between 19.5 to 28.5 degs. depending upon the position of its node. The exact figure for the moment is first obtained. Then the distance between the point of intersection of the moon's path and the equator, and the moon is got by marking a correction to the node's longitude.

This correction is called *Vikṣepa-calana*. The method of arriving at these two elements, with rationale, is explained. With these two elements the declination of the moon is obtained in the same way as for the sun. This is a very interesting procedure and

appears to be peculiar to Kerala.

In chapter XVII, *Dṛkkarma* or Reduction to observation is explained with rationale as also *Candraśṛṅgonnati* relating to the phases of the moon.

As is his wont, throughout the work, Jyeṣṭhadeva, the author of *Yuktibhāṣā*, elucidates the rationales of the several items, either geometrically or algebraically or using both means as the occasion demands. Towards a full evaluation of the work, it is necessary to deal in detail, in terms of modern mathematics, with the numerous rationales and theories, some of them new, anumbrated in the work. It may be added that this work has been taken up by the present writers who intend to translate with elucidation the entire text of the *Yuktibhāṣā*, which is in the Malayalam language, and also edit the *Gaṇita-Yuktibhāṣā* which is a Sanskrit rendering of the Malayalam original.

Model Demonstrations From Yuktibhāṣā

Here we demonstrate some of the derivations as explained in the *Yuktibhāṣā*. We generally follow the logic as found therein but with some modification here and there for the sake of convenience and ease of understanding.

The *Yuktibhāṣā* uses the radian measure as the radius for the sines of all angles. As a matter of fact, the angles as such are not at all used, and all angles are measured in arcs only. Thus, the full circle of 360 degrees or 21600 minutes (minutes are the usual unit of measure with sub-divisions of seconds etc., wherever needed) is taken as the circumference and, on this basis, the radius is measured as 3438 minutes which is sin 90°, and, accordingly, the sines of all angles are measured and used for various calculations. However, in out modern conventions as this will not affect the arguments used in the derivations or rationales.

Only the sine function is used though the cosine function is also well known. Whenever the cosine function is to be used the sine of the co-angle (90° minus the angle) is used or some times $\sqrt{1-\sin^2}$ is used. Sometimes the angle and its sine are used as synonyms and from the context one has to decide whether it refers to the angle or sine.

No general formulae have been derived for the solution of spherical triangles. But the properties of small and great circles and the relations between mutually perpendicular great circles and their poles, and the properties of similar triangles are used with imagination. The presentations are quite long and detailed and verbose. Anybody who reads this will get an impression that, while writing this, the author had before him a model of the celestial sphere and expects the readers also to have one before them. It may be mentioned that the procedure for constructing such an armillary sphere has also been explained in the text.

We shall discuss the solutions with their rationale to the following four problems.

i. Declination and R.A. of the Sun.
ii. One of the fifteen problems.
iii. Declination and R.A. of a star with a latitude.
iv. One of the ten problems.

I. *Declination and R.A. of the Sun*

The Hindu method of solving this problem is well known, but still for the sake of completeness we start with this.

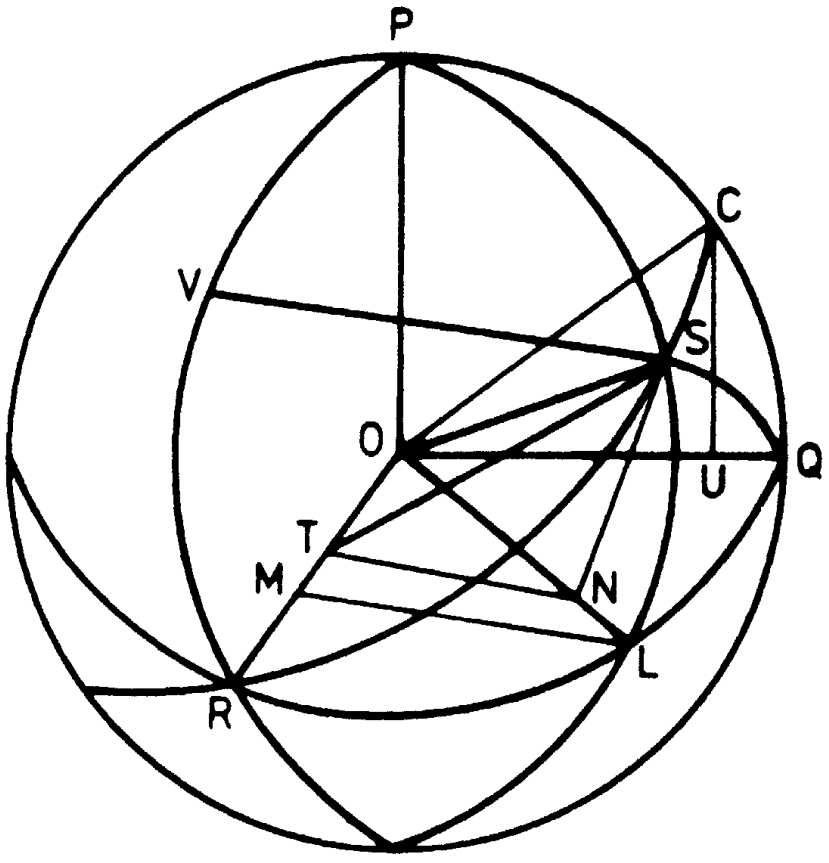

Fig. 1

In the above figure:

P Pole of Equator RLQ.
R First Point of Aries.
S Sun of Ecliptic RSC

Draw CU and SN $\perp$ to the plane of Equator ROQ.
Draw NT and LM $\perp$ OR on the plane of Equator.

It will be seen that

CQ or $\angle$ COQ Obliquity of the Ecliptic (w) and CU its sine.
RS or $\angle$ ROS Longitude of the Sun (l) and ST its sine.
SL or $\angle$ SOL Declination of the Sun (d) and SN its sine.
RL or $\angle$ ROL R.A. of the Sun (a) and ML its sine.

Since the two triangles COU and STN are similar we have, SN/ST = CU/CO or SN = ST × CU as CO = 1 being the Radius. So we get the declination formula

$$\sin d = \sin l \times \sin w$$

The radius of the diurnal circle of the Sun is ON and TN is the sine R.A. on this circle.

$$TN = \cos d \times \sin a \qquad \text{or } \sin a = TN/\cos d$$

Also from the two similar triangles considered above we have

$$TN/TS = OU/OC \text{ or } TN = TS \times OU \text{ or } TN = \text{Sin } l \times \cos w$$

Thus we have,

$$\sin a = \sin l \times \cos w/\cos d$$

which is the formula for R.A.

In the subsequent demonstrations we shall take into account the results of these derivations without again going into the details. This will also apply to spherical triangles in similiar situations of the type SRL considered here.

We had referred to two more elements in the main paper in this connection. They are shown in Fig.1 as SV and VR.

SV *Krāntikoṭi* or inverse declination (d_1)
VR *Nata* or inverse R.A. (a_1)

These can be calculated from triangle RSV in a similar way keeping in mind that ∠ SRV is $90° - w$. Thus

$$d_1 = \sin l \times \cos w \qquad \text{and } a_1 = \sin l \times \sin w/\cos d_1$$

It will be seen that

$$\sin^2 d + \sin^2 d_1 = \sin^2 l$$

II. *One of the fifteen problems*

As stated in the main paper these 15 problems are, given two out of the six elements viz., w, d, d_1, l, a, a_1 to find the other four. We shall take the case where d and a are given.

In Fig. 1 considering the triangles PSC and PLQ we have

$$\sin SC = \sin PS \times \sin LQ$$

or

$$\sin (90° - l) = \sin (90° - d) \times \sin (90° - a)$$

(This is the same as the present cosine formula)

From this l can be calculated. Then by the declination formula, w can be calculated and the rest as in I above.

Alternatively, considering triangles PVS and PRL we have

$$\sin SV = \sin PS \times \sin RL$$

Or

$$\sin d_1 = \sin (90° - d) \times \sin a$$

So d_1 is known. Obtain l from the relation

$\sin^2 l = \sin^2 d + \sin^2 d_1$. Others as before.

III. *Declination and R.A. of a star with a Latitude*

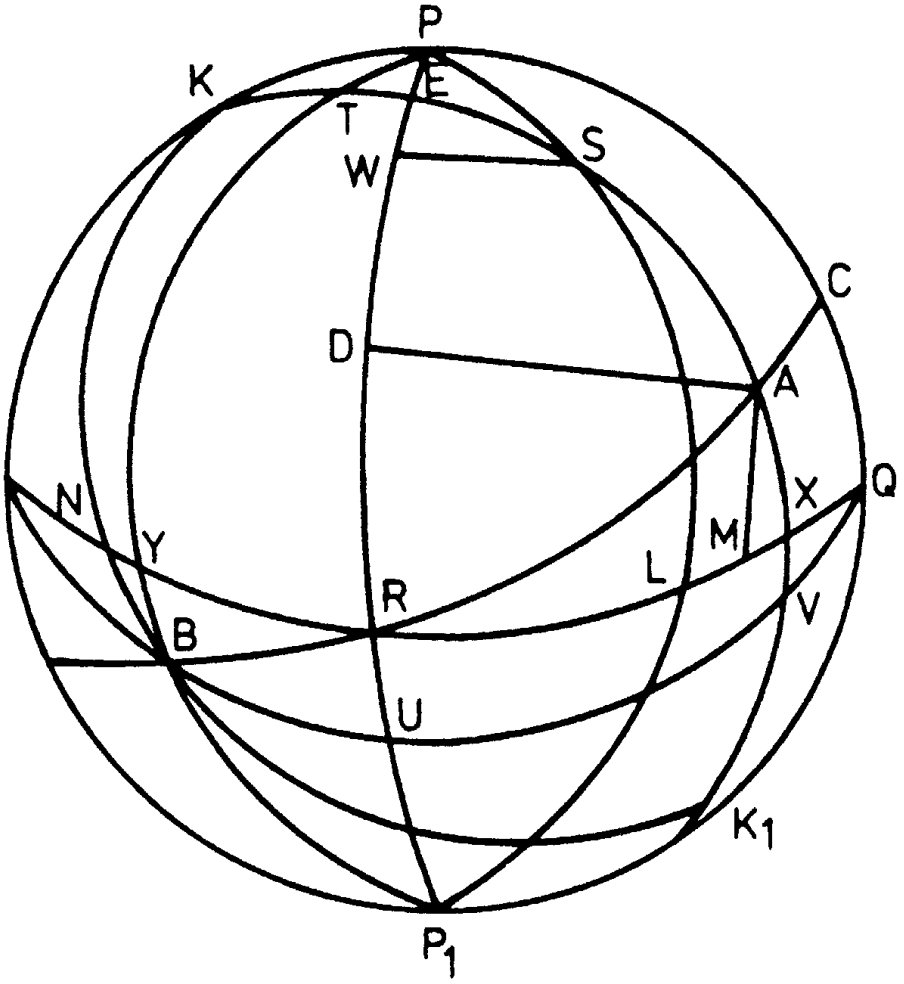

Fig. 2

In the Fig. 2

P, P_1	Poles of the Equator.
K, K_1	Poles of the Ecliptic.
RQ and RC	Equator and Ecliptic.
R	First point of Aries.
S	A star away from ecliptic.
PSL and KSAX	Secondaries to Equator and Ecliptic through S.

Take a point B on the ecliptic 90° behind A. Draw PYB and KNB secondaries through B to equator and ecliptic and AM to the Equator. Draw AD and SW secondaries to PRP_1.

Given RA the longitude (l) and SA the latitude (v) find SL, the declination (d) and RL the R.A. (a).

AM is the declination of the point A on the Ecliptic, say d'
and so $\sin d' = \sin l \times \sin w$

BK = BA = 90°. So B is the pole of AK. Therefore BT and XT are mutually perpendicular as also XY and YT. Hence X is the pole of BP. Now consider the triangles XSL and XTY. XT and XY are equal to 90°. SL is perpendicular to LQ. Therefore

$$\sin SL = \sin SX \times \sin TY \text{ or } \sin d = \sin (v + AX) \times \sin TY \text{ or}$$
$$\sin d = \sin v \times \cos AX \times \sin TY + \cos v \times \sin AX \times \sin TY$$

From XAM we get $\sin AX \times \sin TY = \sin AM$. But AM = d'.

So the second term for sin d is $\cos v \times \sin d'$ where both v and d' are known. As for the first term

$$\cos^2 AX \times \sin^2 TY = \sin^2 TY - \sin^2 AX \times \sin^2 TY$$

$$\begin{aligned} \text{L.H.S.} &= (1 - \sin^2 BY) - \sin^2 AM \\ &= 1 - \sin^2 w \times \cos^2 l - \sin^2 w \times \sin^2 l \\ &= 1 - \sin^2 w = \cos^2 w \end{aligned}$$

Hence $\cos AX \times \sin TY = \cos w$. Hence the expression for d is

$$\sin d = \sin v \times \cos w + \cos v \times \sin d'$$

which is the final form given in the book.

Putting $\sin d' = \sin l \times \sin w$ we get the modern form:

$$\sin d = \sin v \times \cos w + \cos v \times \sin l \times \sin w$$

For the R.A. a similar method is adopted. It can be seen that

$$\sin WS = \sin ES \times \sin UV = \sin (AE - v) \times \sin UV$$

Then proceeding on the same lines as for declination we get

$$\sin WS = \sin l \times \cos w \times \cos v - \sin v \times \sin w$$

But from PWS and PRL we get $\sin WS = \sin PS \times \sin RL$
and since PS = 90° − d and RL = a, we get the expression for a,

$$\sin a = (\sin l \times \cos w \times \cos v - \sin v \times \sin w)/\cos d$$

This expression can be obtained from the modernformulae

cos w × sin l = sin w × tan v + cos l × tan a, and
cos a/cos v = cos l/cos d

IV. *One of the ten problems*

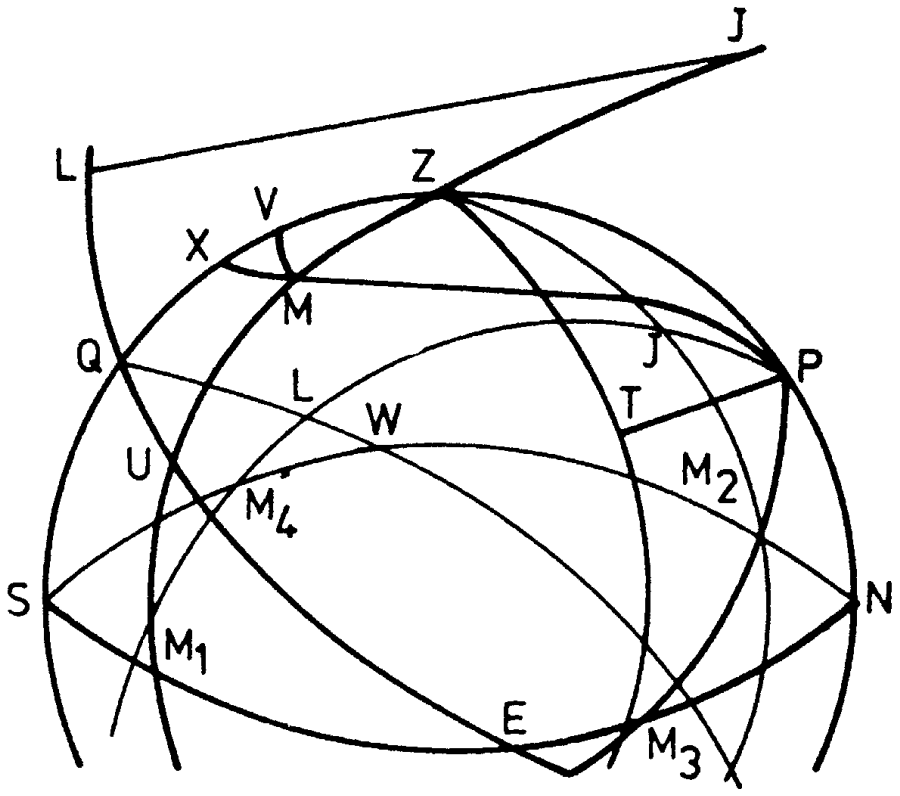

Fig. 3

In Fig. 3 SENW is the horizon, S, E, N and W being the south, east, north and west points. P is the pole, Z zenith, and M the object, and EQW equator.

Draw ZMM_1 vertical through M. Take M_3 on the horizon 90° from M_1 and LPM_3 secondary to the equator through M_3. Draw MV a small circle paralles to equator and MX perpendicular to ZQ. VQ is therefore equal to the declination of M.

In the triangle ZMP, given		
Declination or its supplement	:	PM (90° − d)
Latitude of the place	:	PN φ
Azimuth (A) or A − 90°	:	M_3N_2 B

To get Altitude M_1M = a, and Hour angle $\angle$ MPZ = H,

We shall get ZM the zenith distance z, and hence a = 90° − z

Now z = ZM = ZU − MU. We shall get ZU and MU.

M_3 is the pole of M_1MZ. Therefore UM_3 = 90° and, U being on the equator, UP = 90°. Hence U is the pole of M_3PJL. Therefore UJ = UL = 90°. Now, from the triangles UJL and UZQ we get,

sin QZ = sin ZU × sin LJ, and so sin ZU = sin QZ / sin LJ
QZ = φ, LJ is to be got. Now, M_3J = PL = 90°. So, LJ = PM_3.

From M_3ZN and TZP we have $\sin^2 PT + \sin^2 PN = \sin^2 PM_3$,

But from M_3TP and M_3PN we get sin PT = sin ZP * sin M_3N
or sin PT = cos φ × sin B.

Hence the previous equation becomes $\cos^2 \varphi \times \sin^2 B + \sin^2 \varphi = \sin^2 PM_3$

$$\text{or } \sin M_3P = \sqrt{\sin^2 \varphi + \sin^2 B \times \cos^2 \varphi} = D, \text{ say,}$$

The expression in R.H.S. is called the "Divisor" in *Tantrasaṅgraha* and denoted by D in Gupta's paper.

Substituting this value we get

sin UZ = sin φ / D

Now, to get UM,consider ZQU sin UM / sin UZ = sin QV / sin QZ

or sin UM = (sin φ / D) × sin d / sin φ

or sin UM = sin d / D

we get $ZM = ZU - MU = \sin^{-1}(\sin \varphi/D) - \sin^{-1}((\sin d)/D)$

From this the altitude a is got by a = 90° − ZM.

To get the Hour Angle H,
From ZMX, sin MX = sin z × cos B and
From PMX, sin MX = sin H × cos d Hence we get
sin H = sin z × cos B / cos d

Alternate expression for (a):

$$\sin a = \cos z = \cos (\sin^{-1} ((\sin \varphi)/D) - \sin^{-1} ((\sin d)/D))$$

Expanding R.H.S. and simplifying we get

$$[\surd (D^2 - \sin^2 \varphi) \times \surd (D^2 - \sin^2 d) - \sin \varphi \times \sin d]/D^2,$$

This is the expression given in *Tantrasaṅgraha* and quoted by Gupta in his paper.

As noted by Gupta, this expression can be got by the modern method by solving the following equation for sin a got by the modern cosine rule by transforming it into a quadratic equation.

$$\sin d = \sin \varphi \times \sin a + \cos \varphi \times \cos a \times \cos A.$$

It will be seen the method of *Yuktibhāṣā* is entirely different and does not involve any quadratic equation.

Notes And References

[1]From amongst texts of this type, might be mentioned: *Ganitayuktayaḥ* (Vishveshvaranand Inst., Hoshiarpur, 1979), *Rāśigolasphuṭānīti* Hoshiarpur, 1977), *Grahaṇamaṇḍana* of Parameśvara (Hoshiarpur, 1965), *Grahaṇa-nyāya-dīpikā* of Parameśvara (Hoshiarpur, 1965), *Kriyākramakarī* of Śaṅkara Com. on the *Līlāvatī* of Bhāskara II (Hoshiarpur, 1975), *Sphuṭacandrāpti* of Mādhava (Hoshiarpur, 1973), *Jyotirmīmāmsā* of Nīlakaṇṭha (Hoshiarpur, 1977), and *Yuktidīpikā* of Śaṅkara, Com. on the *Tantrasaṅgraha* (Hoshiarpur, 1977), all edited critically by K.V. Sarma. A large number of tracts, big and small, containing astronomical rationale, have also been identified by him.

[2]*Tantrasaṅgraha* of Nīlakaṇṭha Somayāji, ed. with two commentaries, *Yuktidīpikā* and *Laghuvivṛti*, both by Śaṅkara, ed. by K.V. Sarma, Hoshiarpur, 1977.

[3]C.M. Whish, 'On the Hindu quadrature of the circle and the infinite series of the proportion of the circumference to the diameter exhibited in the four *Sastras*, the *Tantra Sahgraham, Yucti Bhasha, Carana Padhati* (sic) and Sadratnamala, 'Transactions of the Royal Asiatic Society of Great Britain and Ireland, III.iii (1835), 509-23.

[4]Cf, the statement: "The testimonies as to the author (of Tantrasaṅgraha) and the period in which he lived, are the following...the mention made of him...by his commentator, the author of the *Yucti-Bhasha*, Cellalura Nambutiri", p.522. However, it has to be noted, incidentally, that Whish has made wrong identifications, when he states here: (i) that the author of Tantrasangraha is "Talaculattura Nambutiri" (p. 522), for it is correctly Gārgya Kerala Nīlakaṇṭha Somayāji; (ii) that *YB* is a commentary on the *Tantrasaṅgraha*, which it is not; and (iii) that the author of *YB* is Cellalura (i.e. Kelallur) Nambutiri (p.522), for it is actually Paraṅṅoṭṭu Nampūtiri, as would be shown below.

[5]See *op. cit.*,p.523.

[6]Several manuscripts of the work are preserved under the two titles in the Kerala University Mss. Library, Trivandrum, the Sanskrit College Library, Tripunithura, and in private possession.

[7]There is a fine edition of Pt. I of the work, now out of print, by Rama varma Maru Thampuran and A.R. Akhileswara Iyer, Mangalodayam Press, Trichur,1948 and an extremely unsastisfactory and error-ridden edition of the whole work issued by the Govt. Oriental Mass. Library, Madras, 1954.

[8]Cf., S.N. Sen, *A Bibliography of Sanskrit Works on Astronomy and Mathematics*, INSA, New Delhi, 1966, p.74; C.T. Rajagopal and M.S. Rangachari, 'On an untapped source of medieval Keralese

mathematics', *Archive for History of Exact Science*, 18 (1978) 89-101; C.T. Rajagopal and M.S. Rangachari, 'On mediaeval Kerala Mathematics', *ib.* 35 (1986) 91-99.

[9]*Op. Cit.*, Introduction, p.5, where the A.D. date is wrongly given as 1639; Ulloor, *History of Kerala literature*, Trivendrum, vol.III, 1955 p. 439.

[10]For an independent tradition that the author of the *Yuktibhāṣā* belonged to the Paraṅṅoṭṭu faimly (Sanskritised as Parakroḍa), situated in the Ālattūr village in Malabar, see *Nampūtirimār* (Mal.), by parayil Raman Namputiri, Trichur, Kollam era 1093, (A.D. 1918), p.55.

[11]See Ms. C. 628-B, end of the Kerala Uni. Mss. Library.

[12]Whish, *loc. cit.*, p. 523.

[13]Cr. edn. with Introduction by K.V. Sarma, Vishveshvaranand Institute, Hoshiarpur, 1975.

[14]Cr. edn. with detailed Introduction by K.V. Sarma, Vishveshvaranand Institute, Hoshiarpur, 1977. The identity of the authorship of the two commentaries has been dealt with in detail herein, Intro,. pp. li-vi.

[15]See, P. Sridhara Menon, Introduction to Maru Thampuran's edn. of the work, p. 6.

[16]*Loc. cit.*, pp. 5-6

[17]The page reference are to the edition of *YB* by Rama Varma Maru Thampuran.

[17a] (i) K. Mukunda Marar and C.T. Rajagopal, 'On the Hindu quadrature of the circle', *J Bombay Branch of Royal Asiatic Soc.*, NS 20 65-82 (1944).

(ii) C.T. Rajagopas and A. Venkataraman, 'The Sine and Cosine power series in Hindu mathematics' *J Royal Asiatic Soc. of Bengal*, Sc., 15 1-13 (1949).

(iii) C.T. Rajagopal and M.S. Rangachari, 'On an untapped source of medieval Keralese mathematics', *Archive for Hist. of Exact sciences*, 18 (1978)

(iv) C.T. Rajagopal and M.S. Rangachari, 'On medieval Keralese mathematics' *Archeve for Hist. of Exact Sciences*, 35, 91-99 (1986).

[18]See, for instance, E. Burgess, *Translation of the Sūryasiddhānta*, p.86, lines 3ff.; P.C. Sengupta, *Khaṇḍakhādyaka*, p.58, last para.

[19]O. Neugebauer, 'The transmission of planetary theories in ancient and medieval astronomy', *Scripta Mathematica*, New York, 1956, App. 'Hindu planetary theory', p. 12ff.

S.N. Sen, 'Epicyclic eccentric planetary theories in ancient and medieval Indian astronomy', *Indian Jl. Hist. Science*, 9 pp. 107-21 (1974).

[20]See for instance, K.V. Sarma, *Rationales of Hindu astronomy*, Pt.I, Hoshiarpur, 1979, Tracts 24, 25 and 26.

[21]See P.C. Sengupta, *Khaṇḍakhādyaka*,pp. 189-91, Problem viii.

[22]R.C. Gupta, 'Solution of a astronomical triangle as found in the *Tantrasaṅgraha* (1500)', *Indian Jl. of Hist. of Science*, 9, 86-99 (1974).

[23]K.S. Shukla, 'The evection and the deficit of the equation of the centre of the Moon in Hindu astronomy', *Proceedings of the Banaras Mathematical Society*, NS, 7.ii (Dec. 1945) 9-28.

Paper 42

T. Hayashi, T. Kusuba and M. Yano (1991)

The Correction of the Mādhava Series for the Circumference of a Circle

Centaurus, Vol. 33 (1991), pp. 149–174

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The Correction of the Mādhava Series for the Circumference of a Circle

by

T. HAYASHI*, T. KUSUBA**, AND M. YANO***

The aims of this paper are to examine two theories proposed so far on the derivation of the correction of Mādhava's series for the circumference of a circle, and to give a new one.

1. Introduction

Mādhava (fl. A.D. 1400) is one of the most ingenious mathematicians India has ever produced. He is also famous for his innovation of the Vākya system of astronomy.[1] Mādhava's own works on mathematics proper are not available now, but his theory about the circle can be reconstructed to a certain extent from the writings of his successors. Nīlakaṇṭha (b. A.D. 1444) and Śaṅkara (fl. A.D. 1556), for example, attribute not a few stanzas to Mādhava himself, and those stanzas reveal that one of the most remarkable results Mādhava obtained is the following formula for calculating the circumference, $C(n)$, of a circle whose diameter is d.

$$C(n) = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \ldots + (-1)^{n-1}\frac{4d}{2n-1} + (-1)^n \cdot 4d \cdot F(n) \ , \tag{1}$$

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Centaurus 1990: vol. 33: pp. 149–174

where the last term is a correction, and three types of $F(n)$ are known:

$$F_1(n) = \frac{1}{4n} . \tag{2}$$

$$F_2(n) = \frac{n}{4n^2+1} . \tag{3}$$

$$F_3(n) = \frac{n^2+1}{4n^3+5n} . \tag{4}$$

The correction of the second type, $F_2(n)$, is stated together with the Mādhava series (1) itself in three and a half stanzas attributed by Śaṅkara to "himself", which undoubtedly refers to Mādhava himself, because his verses giving another rule for the circumference of a circle are cited by Śaṅkara on the preceding pages of the same work.[2] The first type, $F_1(n)$, is obtained in the course of Śaṅkara's derivation of $F_2(n)$, and according to Śaṅkara the derivation goes back to "the teacher" (*ācārya*, see n. 14), which also most probably refers to Mādhava, although his direct preceptor (*guru*) was Nīlakaṇṭha [Pingree 1981, 128]. The third type, $F_3(n)$, is stated by Śaṅkara in a stanza among other stanzas giving various formulas derived from (1). For the context where $F_3(n)$ occurs, see the last few paragraphs of the next section.

With regard to formula (1), the following two points should be noted. (*i*) Formula (1) without the correction term, when continued to infinity, is equivalent to the so-called Leibniz's series for π,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{n-1}\frac{1}{2n-1} + \ldots , \tag{1a}$$

which is in turn a special case of Gregory's series for arctan (A.D. 1671),

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \ldots . \tag{1b}$$

This series was also stated by Śaṅkara in about A.D. 1530.[3]

(*ii*) Formula (1) without the correction term, or the series (1a), is itself very simple and has a mathematical beauty, but Mādhava was not content with it, and sought for better approximations and practical utility as well. When $d = 1$, for example, formula (1) without a correction will give:

$$C(19) = 3.194...$$
$$C(20) = 3.091...$$

With the third correction $F_3(n)$, on the other hand, it will give:[4]

$$C(19) = 3.1415926529...\ \text{(correct to 9 decimal places)}$$
$$C(20) = 3.1415926540...\ \text{(correct to 9 decimal places)}$$

These examples clearly show that the corrections are very effective and make the Mādhava series quite useful for practical purposes.

For the derivation of the Mādhava series itself, see, for example, Rajagopal & Aiyar, 1951, Saraswathi, 1963, 328–334, and Sarasvati, 1979, 159–167. For rationales of the correction term from the view point of the modern analysis, see Marar & Rajagopal, 1944, 69–72 and Rajagopal & Rangachari, 1978, 89–90. For the trigonometrical series treated by Mādhava and his successors, see Gupta, 1976 and Pingree, 1981/82 in addition to the articles just mentioned.

As for the historical derivation of the correction term, two theories have so far been known: one attributed to "the teacher" by the medieval Indian commentator Śaṅkara, and the other proposed by the modern Russian historian of mathematics Youschkevitch. We shall examine them one by one in the following sections.

2. *Derivation attributed by Śaṅkara to "the teacher"*

In his commentaries on Bhāskara's *Līlāvatī* and on Nīlakaṇṭha's *Tantrasaṃgraha*, Śaṅkara explains how $F_1(n)$ and $F_2(n)$ were obtained, and attributes the derivation to "the teacher" (possibly Mādhava). Śaṅkara's explanations in the two works are almost the same (see note 5), but the commentary (ca. A.D. 1550) on the *Līlāvatī*,

which is written in prose, is easier to follow than the other (ca. A.D. 1530), which is written in verse. So we shall cite from the former unless otherwise indicated.

The derivation begins with a quite ingenious idea, which might remind one of the ε–δ method of the modern analysis, but the process for arriving at F_2 contains dubious statements, which give us the impression that, in case "the teacher" is Mādhava, he first obtained the three types of correction by a certain (perhaps inductive) method (see Section 4 below) and then tried to give them a deductive proof.

Śaṅkara says that if $C(N)$ is equal to $C(N+1)$ for a certain N then the correction (*saṃskāra*) is exact, because $C(n)$ is equal to $C(n+1)$ for any n greater than N.[5] That is to say, if the correction $F(n)$ of formula (1) be ideal then $C(n)$ must be equal to $C(n+1)$ when n is large enough:

$$C(n) = C(n+1), \quad \text{or} \quad F(n) = \frac{1}{2n+1} - F(n+1) \ .$$

His purpose is, therefore, to find such a function $F(n)$ that satisfies the condition,[6]

$$F(n)+F(n+1) = \frac{1}{2n+1} \ . \tag{5}$$

Śaṅkara continues: If we were allowed to set

$$F(n) = F(n+1) = \frac{1}{2(2n+1)} \ ,$$

then the condition (5) would be satisfied, but $F(n)$ and $F(n+1)$ cannot be the same. So, says he, we should define an approximate function $F_1(n)$ as

$$F_1(n) = \frac{1}{2(2n+1)-2} = \frac{1}{4n} \ , \tag{6}$$

or

$$F_1(n) = \frac{1}{2m-2} ,$$

where $m = 2n+1$. Then we have

$$F_1(n+1) = \frac{1}{2m+2} , \text{ and } \quad \frac{1}{2n+1} = \frac{1}{m} .$$

Hence,[7]

$$F_1(n)+F_1(n+1) = \frac{4m^2}{4m^3-4m} ,$$

$$\frac{1}{2n+1} = \frac{4m^2-4}{4m^3-4m} .$$

The difference between $C(n)$ and $C(n+1)$ in this case is:

$$E = /\, C(n)-C(n+1)\,/ = 4d \cdot \frac{4}{4(m^3-m)} = \frac{4d}{m^3-m} . \qquad (7)$$

Śaṅkara calls this value (E) "the error" (*sthaulya*), and says:[8]

> [The teacher], who was not satisfied when seeing this error, sought for another (better) correction. In this case, a bit of excess over the desired value has been produced in the quotient obtained by means of the denominator of the correction. So, in order to remove it, he added unity to both denominators.

That is, according to Śaṅkara, "the teacher" defined $F_{1a}(n)$ by way of trial as,

$$F_{1a}(n) = \frac{1}{4n+1} = \frac{1}{2m-1} , \qquad (8)$$

where $m = 2n+1$ as before. Then we would have,

$$F_{1a}(n+1) = \frac{1}{2m+3}, \quad \text{and} \quad \frac{1}{2n+1} = \frac{1}{m}.$$

Hence,

$$F_{1a}(n)+F_{1a}(n+1) = \frac{4m^2+2m}{m(2m-1)(2m+3)},$$
$$\frac{1}{2n+1} = \frac{(4m^2-4)+(4m+1)}{m(2m-1)(2m+3)}.$$

But Śaṅkara says:[9]

> Therefore, the sum of [the numerators for] the denominators of the correction is greater [than that of the previous case] by twice "the quantity" (*rāśi*) (i.e., *m*), while the numerator of the odd number ($2n+1$, i.e., *m*) [in the present case] is [greater than that of the previous case] by four times "the quantity" (*m*) plus one.

Here, Śaṅkara is trying to evaluate $(F_{1a}(n)+F_{1a}(n+1))$ and $1/(2n+1)$ by comparing their numerators (say, A_{1a} and B_{1a}) with those of the previous case, i.e., the numerators (say, A_1 and B_1) of $(F_1(n)+F_1(n+1))$ and $1/(2n+1)$, after the reduction to the same denominator in either case. That is:

For $F_1(n)$: $B_1 = A_1 - 4$.
For $F_{1a}(n)$: $A_{1a} = A_1 + 2m$, and $B_{1a} = B_1 + (4m+1)$.

Hence, we would have:

$$B_{1a} = A_1 + (4m-3) = A_{1a} + (2m-3),$$

although Śaṅkara does not explicitly state this last relationship.

This method of comparison appears to have been the most convenient for Śaṅkara (and his teacher). He certainly had an ingenious method for expressing rational functions of a single variable. For example, he expressed the sum of $F_1(n)$ and $F_1(n+1)$,

$$\frac{4m^2}{4m^3-4m} \quad \text{as} \quad \begin{array}{|c|c|c|}\hline 4 & 0 & 0 \\ \hline 4 & 0 & 4^0 \\ \hline \end{array} \; 0 \; {}^{10}.$$

But this method was not applicable to such cases where either more than one term of the same power, or an expression consisting of factors, occurs. Thus, for example, he could not express

$$\frac{(4m^2-4)+(4m+1)}{m(2m-1)(2m+3)}$$

as such. Śaṅkara continues:[11]

> Thus, the error in the previous case occurred only in the place of the constant (*rūpa*), but that of the present case occurs in the place of "the quantity" (*rāśi*) also. Therefore, the entire unity should not be added to the denominators of the corrections.

Then Śaṅkara defines $F_{1b}(n)$ as,

$$F_{1b}(n) = \frac{1}{4n+\dfrac{1}{4n}} = \frac{1}{(2m-2)+\dfrac{1}{2m-2}}, \tag{9}$$

with an enigmatic expression that in this case "the excess is its half, which is equal to unity, because the denominators of the corrections are nearly twice the quantity $(2m)$".[12] He does not explain why he "increases the denominator $(4n)$ by unity divided by itself."

Śaṅkara tries to evaluate $(F_{1b}(n)+F_{1b}(n+1))$ and $1/(2n+1)$ in the same manner as above, but this time only approximately. Let A_{1b} and B_{1b} be their numerators after the reduction to the same denominator. Since $(2m \pm 2) \simeq 2m$ when m is large enough, we have,

$$\text{both } F_{1b}(n) \text{ and } F_{1b}(n+1) \simeq \frac{1}{2m+\dfrac{1}{2m}} = \frac{2m}{4m^2+1}.$$

Hence,

$$F_{1b}(n)+F_{1b}(n+1) \simeq \frac{4m^2}{m(4m^2+1)} ,$$

$$\frac{1}{2n+1} = \frac{1}{m} = \frac{4m^2+1}{m(4m^2+1)} .$$

Therefore,

$$B_{1b} \simeq A_{1b}+1 .$$

Śaṅkara says:[13]

> In this case the difference is only unity, but the difference to be obtained is four because the numerator of the odd number is less than [that of] the sum of the others by four (i.e., $B_1 = A_1-4$). Therefore, both denominators of the corrections (F_{1b}) supposed before should be increased by the four divided by themselves.

That is, $F_2(n)$ is defined as,

$$F_2(n) = \frac{1}{4n+\dfrac{4}{4n}} = \frac{1}{(2m-2)+\dfrac{4}{2m-2}} . \qquad (10)$$

Certainly with this definition the approximate relationship desired by him,

$$B_2 \simeq A_2+4 ,$$

is obtained as will be seen below. It is, however, of no logical reason to utilize the approximate relationship, $B_{1b} \simeq A_{1b}+1$, for the definition of $F_2(n)$. Strictly speaking, we have,

$$F_{1b}(n) = \frac{m(2m-2)(4m^2+8m+5)}{D} ,$$

$$F_{1b}(n+1) = \frac{m(2m+2)(4m^2-8m+5)}{D} ,$$

$$\frac{1}{2n+1} = \frac{1}{m} = \frac{(4m^2+8m+5)(4m^2-8m+5)}{D} ,$$

where

$$D = m(4m^2+8m+5)(4m^2-8m+5) .$$

Hence we have,

$$A_{1b} = 16m^4-12m^2 , \quad B_{1b} = 16m^4-24m^2+25 ,$$

from which we obtain the exact relationship,

$$B_{1b} = A_{1b}-(12m^2-25) .$$

This means that $B_{1b} < A_{1b}$ for $m > 1$, which is contrary to Śaṅkara's expectation, $B_{1b} \simeq A_{1b}+1$.

Śaṅkara does not give the details of his approximate computation of "the error" of $F_2(n)$, but we may reconstruct them analogously to the foregoing cases as follows. Since $(2m \pm 2) \simeq 2m$ as in the case of F_{1b}, we have,

$$\text{both } F_2(n) \quad \text{and} \quad F_2(n+1) \simeq \frac{1}{2m+\dfrac{4}{2m}} = \frac{2m}{4m^2+4} .$$

Hence,

$$F_2(n)+F_2(n+1) \simeq \frac{4m^2}{m(4m^2+4)} ,$$

$$\frac{1}{2n+1} = \frac{1}{m} = \frac{4m^2+4}{m(4m^2+4)} .$$

Immediately after the previous citation Śaṅkara continues:[14]

> Then, the numerator for the odd number becomes greater [than B_1] by almost eight, while the sum of the numerators of the others becomes greater [than B_1] by four only. So, having considered that there is an almost complete exactness in this case, *the teacher* made "the addition of the four divided by itself" [to the denominator of $F_1(n)$].

That is to say, according to Śaṅkara,

$$A_2 \simeq B_1+4, \text{ and } B_2 \simeq B_1+8 ,$$

where A_2 and B_2 are the numerators of $(F_2(n)+F_2(n+1))$ and $1/(2n+1)$ respectively, after the reduction to the same denominator. Therefrom he obtains,

$$B_2 \simeq A_2+4 .$$

This relationship is, according to Śaṅkara, what has been desired, because "the difference to be obtained is four". This last statement of Śaṅkara is, in turn, based on the relationship:

$$B_1 = A_1-4 .$$

But it is difficult to see any significance in Śaṅkara's (or "the teacher's") reasoning based on these two relationships, although we certainly have the exact relationship, $B_2 = A_2+4$, as will be demonstrated below. In fact, what is actually desired must be the equality,

$$B_2 = A_2 ,$$

and not the equality, $B_2 = A_2+4$. It seems to us, therefore, that Śaṅkara's (or "the teacher's") attempt to give a deductive proof to the correction term, which had presumably been obtained inductively, was not successful.

In his commentary on the *Tantrasaṃgraha*, Śaṅkara explains the same process, from F_{1b} to F_2, somewhat differently as follows.[15]

> Now, when the unity divided by itself (i.e., the denominator) is added to the denominators of the corrections, one should first make the reduction to the same denominator,

> and then calculate the error. The excess over what has been told before concerning the odd number is two. Since the denominators of the corrections are nearly equal to "two quantities" ($2m \pm 2 \simeq 2m$), the excess was said to be unity when the numerators of the corrections were added together ($B_{1b} \simeq A_{1b}+1$). Since the difference was [nearly] equal to "two quantities" ($2m$) when the entire unity was added [to the denominators], the difference is the entire unity when it is divided by "two quantities". Since the shortage in the odd number was four ($B_1 = A_1 - 4$), three should be now added to the denominators of the corrections. [What should be added is then] "the four divided by itself", and there will be an almost complete exactness.

This passage, written in verse, is not completely clear to us. The number three is stated to be added to "the denominator", i.e., the numerator of $1/4n$, so that one can obtain $4/4n\,(=1/n)$, but we have not as yet found a logical reason of this procedure.

Śaṅkara then proceeds to the computation of the exact value of "the error" (*sthaulya*) of $F_2(n)$.[16]

$$F_2(n) = \frac{m-1}{2m^2-4m+4},$$

$$F_2(n+1) = \frac{m+1}{2m^2+4m+4}.$$

Hence,

$$F_2(n)+F_2(n+1) = \frac{4m^4}{4m^5+16m},$$

while

$$\frac{1}{2n+1} = \frac{1}{m} = \frac{4m^4+16}{4m^5+16m}.$$

Therefore,

$$\frac{1}{2n+1} - \{F_2(n)+F_2(n+1)\} = \frac{16}{4m^5+16m} = \frac{4}{m^5+4m}. \quad (11)$$

Then Śaṅkara gives, without the details of the derivations, five formulas for the circumference of a circle. They are stated in several stanzas, which are found also in his versified commentary on Nīlakaṇṭha's *Tantrasaṃgraha*.

$$C(n) = \frac{16d}{1^5+4\cdot 1} - \frac{16d}{3^5+4\cdot 3} + \ldots \tag{12}$$
$$+ (-1)^{n-1}\frac{16d}{(2n-1)^5+4(2n-1)} .$$

$$C(n) = 3d + \frac{4d}{3^3-3} - \frac{4d}{5^3-5} + \ldots \tag{13}$$
$$+ (-1)^n \frac{4d}{(2n-1)^3-(2n-1)} . \tag{14}$$

$$C(n) = \frac{8d}{2^2-1} + \frac{8d}{6^2-1} + \ldots + \frac{8d}{(4n-2)^2-1} . \tag{15}$$

$$C(n) = 4d - \left(\frac{8d}{4^2-1} + \ldots + \frac{8d}{(4n)^2-1}\right) .$$

$$C(n) = 2d + \frac{4d}{2^2-1} - \frac{4d}{4^2-1} + \ldots \tag{16}$$
$$+ (-1)^{n-1}\frac{4d}{(2n)^2-1} + (-1)^n \frac{4d}{2\{(2n+1)^2+2\}} .$$

Formulas (12) and (13) can be obtained by means of "the errors", (11) and (7), respectively. Formulas (14) and (15) can be arrived at by rewriting formula (1) itself without the correction term. Formula (16) is half of the sum of (14) and (15) with a correction term, which may have been obtained in the same manner as Śaṅkara used for obtaining $F_1(n)$ for formula (1).

Then Śaṅkara gives, without any explanation, the correction $F_3(n)$ for formula (1) in a stanza, which is also found in his commentary on the *Tantrasaṃgraha*.[17] And he concludes this section (his comment on *Līlāvatī* 199) with another method of correction of an already

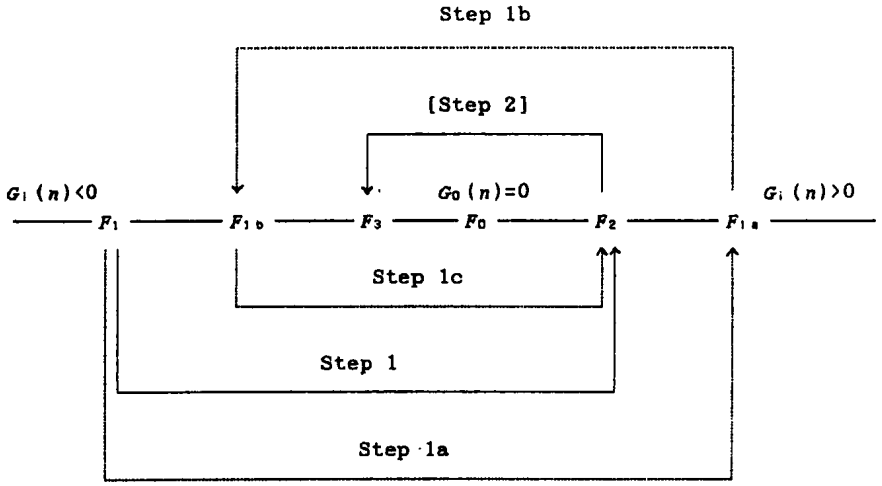
calculated circumference of a circle. No elucidation being given by him, the method is yet to be deciphered, but it is evidently based on a totally different idea.

We summarize in Figure 1 the derivation of the correction term attributed by Śaṅkara to "the teacher".

3. Derivation suggested by Youschkevitch

Youschkevitch [Juschkewitsch 1964, 171–172] explains the derivations of the corrections, $F_1(n)$, $F_2(n)$, and $F_3(n)$, as follows. Let

$$S(n) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^{n-1}}{2n-1}\,, \tag{17}$$



F_0 = Ideal correction. $F_1(n) = 1/4n$.
$G_i(n) = 1/(2n+1) - [F_i(n) + F_i(n+1)]$.
Step 1a: Add unity to the denominator ($4n$).
Step 1b: Divide the unity by $4n$.
Step 1c: Replace the unity by 4 (in *Kriyākramakarī*).
Add 3 to the unity (in *Yuktidīpikā*).
Step 1: Increase the denominator ($4n$) by "4 divided by itself".
Step 2: (not stated): Increase the denominator ($4n$) of 4 by 16 divided by itself (?)

Figure 1. Derivation of the correction term by Śaṅkara's "teacher".

then from (1) we have,

$$F(n) = /\, S(n) - \frac{C(n)}{4d} /\, = /\, S(n) - \frac{\pi}{4} /\ , \tag{18}$$

if $C(n)$ is exact. Youschkevitch thinks that Mādhava, on supposing that $\pi = 377/120$, began by actually calculating $F(n)$ for the first several n's according to (18).

$$F(1) = S(1) - \frac{\pi}{4} = \frac{103}{480} = \cfrac{1}{4+\cfrac{1}{1+\frac{35}{68}}}\ ,$$

$$F(2) = \frac{\pi}{4} - S(2) = \frac{19}{160} = \cfrac{1}{8+\cfrac{1}{2+\frac{3}{8}}}\ ,$$

$$F(3) = S(3) - \frac{\pi}{4} = \frac{13}{160} = \cfrac{1}{12+\cfrac{1}{3+\frac{1}{4}}}\ ,$$

$$F(4) = \frac{\pi}{4} - S(4) = \frac{69}{1120} = \cfrac{1}{16+\cfrac{1}{4+\frac{5}{16}}}\ ,$$

and so on. From these results the following approximations arise.

$$F(1) \simeq \tfrac{1}{4} \quad \text{or} \quad \tfrac{1}{5}\ , \qquad F(2) \simeq \tfrac{1}{8} \quad \text{or} \quad \tfrac{2}{17}\ ,$$
$$F(3) \simeq \tfrac{1}{12} \quad \text{or} \quad \tfrac{3}{37}\ , \qquad F(4) \simeq \tfrac{1}{16} \quad \text{or} \quad \tfrac{4}{65}\ ,$$

and so on. Youschkevitch conjectures that from these examples were first inferred $F_1(n)$ and $F_2(n)$.

$$F_1(n) = \frac{1}{4n}\ , \quad F_2(n) = \frac{n}{4n^2+1}\ .$$

And then, according to Youschkevitch, $F_2(n)$ was rewritten as,

$$F_2(n) = \frac{n^2}{n(4n^2+1)} ,$$

and $F_3(n)$ was obtained by taking the sum of the numerators, and that of the denominators, of $F_1(n)$ and $F_2(n)$ separately:

$$F_3(n) = \frac{[\text{numerator of } F_1] + [\text{numerator of } F_2]}{[\text{denominator of } F_1] + [\text{denominator of } F_2]} = \frac{n^2+1}{n(4n^2+5)} . \tag{19}$$

Youschkevitch's conjecture has been skilfully made, but it has the following two difficulties.

(*i*) It has not been proved that the value, 377/120, of π was used in ancient and medieval India for computations of the circle. Youschkevitch was probably misled by the statement of Tropfke [1940, 275]:

> Bei einem anderen indischen Werte, $\pi = 17/120$, [This must be a misprint of 3 17/120.] ist die Entlehnung aus griechischen Quellen besser aufzudecken, da dieselbe Zahl sich auch in der *Μαθηματικὴ σύνταξις* des Claudius Ptolemäus (zwischen 125 und 151 n. Chr. in Alexandria) vorfindet.

Tropfke refers to two Indian works for this value, the *Āryabhaṭīya* and the *Līlāvatī*, but both references are quite dubious.

Although the value of π given in the *Āryabhaṭīya*, 62832/20000 = 3.1416, is sometimes mentioned in connection with the Greek value, 3;8,30 = 377/120 = 3.14166..., we cannot simply ignore the difference between the two values. If we apply the latter to the diameter, 20000, then we will obtain the circumference,

$$C = 20000 \cdot \tfrac{377}{120} = 62833.33... \simeq 62833 ,$$

as the nearest whole number, and not 62832.

Līlāvatī 211, to which Tropfke refers, is a mathematical example

which requires the lengths of the chords corresponding to various arcs of a circle whose radius is 120. At the beginning of the solution, its circumference is calculated by using Bhāskara's value, 3927/1250, of π, which is equivalent to Āryabhaṭa's value.

$$C = 240 \cdot \tfrac{3927}{1250} = 753\tfrac{1230}{1250} \simeq 754 \ .$$

The resulting ratio, $C:d$, happens to be: 754:240, i.e., 377:120. But, as Bhāskara himself says ("Here, for the sake of easy treatment of numerals, we regard 754 as an accurate circumference by adding 20/1250 [to the result]"), we should not forget that 754 is the result of the approximation, and we do not know Indian examples where this ratio is used for computations.

(*ii*) The process for arriving at $F_3(n)$ appears to be too artificial, and it is difficult to understand how Mādhava came to the idea of rewriting $F_2(n)$ before he took the sums of the numerators and of the denominators. He may as well have taken their sums in their original forms,

$$F_3'(n) = \frac{1+n}{4n+(4n^2+1)} = \frac{n+1}{(2n+1)^2} \ . \tag{20}$$

We, therefore, cannot accept Youschkevitch's conjecture as such, but his idea itself is very much suggestive. So, following the same idea, we would like to improve his conjecture in the next section.

4. A new hypothesis

It has been known that the value, 355/113, of π was known in India at least from the ninth century onward [Hayashi et al. 1989]. We think that Mādhava used this value, instead of 377/120, in his computations. The following values of $F(n)$ for the first few n's are obtained from (18) with $\pi = 355/113$,

$$F(1) = \frac{97}{452} = \cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\frac{31}{33}}}}.$$

$$F(2) = \frac{161}{1356} = \cfrac{1}{8+\cfrac{1}{2+\cfrac{1}{2+\frac{18}{25}}}}$$

$$F(3) = \frac{551}{6780} = \cfrac{1}{12+\cfrac{1}{3+\cfrac{1}{3+\frac{27}{47}}}}$$

$$F(4) = \frac{2923}{47460} = \cfrac{1}{16+\cfrac{1}{4+\cfrac{1}{4+\frac{72}{155}}}}$$

$$F(5) = \frac{21153}{427140} = \cfrac{1}{20+\cfrac{1}{5+\cfrac{1}{5+\frac{315}{753}}}}.$$

$$F(6) = \frac{194457}{4698540} = \cfrac{1}{24+\cfrac{1}{6+\cfrac{1}{6+\frac{1422}{5025}}}}.$$

$$F(7) = \frac{2170599}{61081020} = \cfrac{1}{28+\cfrac{1}{7+\cfrac{1}{7+\frac{18207}{40863}}}}.$$

From these examples would be naturally inferred the general ex-

pression:

$$F(n) = \cfrac{1}{4n + \cfrac{1}{n + \cfrac{1}{n + t_n}}}, \tag{21}$$

where t_n is a fraction. It is remarkable that all the three corrections, $F_1(n)$, $F_2(n)$, and $F_3(n)$, can be obtained from this expression. That is, $F_1(n)$ is obtained by ignoring $\cfrac{1}{n + \cfrac{1}{n+t_n}}$; $F_2(n)$ is obtained by ignoring $\cfrac{1}{n+t_n}$; and $F_3(n)$ is obtained by ignoring t_n.

The above is a modern presentation of the matter. The procedure actually followed by Mādhava was presumably as follows. He first calculated $F(n)$ according to (18) with $\pi = 355/113$, and obtained the series of the fractions:

$$\frac{97}{452}, \frac{161}{1356}, \frac{551}{6780}, \frac{2923}{47460}, \frac{21153}{427140}, \frac{194457}{4698540}, \frac{2170599}{61081020}, \cdots$$

As it was difficult to see a mathematical law in this series, Mādhava calculated the nearest unit-fraction of each term in order to evaluate their approximate values.

$$\frac{97}{452} = \frac{1}{4 + \frac{64}{97}} \simeq \frac{1}{4} \text{ or } \frac{1}{5}.$$

$$\frac{161}{1356} = \frac{1}{8 + \frac{68}{161}} \simeq \frac{1}{8}.$$

$$\frac{551}{6780} = \frac{1}{12 + \frac{168}{551}} \simeq \frac{1}{12}.$$

$$\frac{2923}{47460} = \frac{1}{16 + \frac{692}{2923}} \simeq \frac{1}{16}.$$

$$\frac{21153}{427140} = \frac{1}{20+\frac{4080}{21153}} \simeq \frac{1}{20} .$$
$$\frac{194457}{4698540} = \frac{1}{24+\frac{31572}{194457}} \simeq \frac{1}{24} .$$
$$\frac{2170599}{61081020} = \frac{1}{28+\frac{304248}{2170599}} \simeq \frac{1}{28} .$$

From these computations he inferred the first approximation:

$$F_1(n) = \frac{1}{4n} .$$

Then he sought for a rule also in the fractions neglected in the above procedure, exactly in the same manner as above.

$$\frac{64}{97} = \frac{1}{1+\frac{33}{64}} \simeq \frac{1}{1} \text{ or } \frac{1}{2} .$$
$$\frac{68}{161} = \frac{1}{2+\frac{25}{68}} \simeq \frac{1}{2} .$$
$$\frac{168}{551} = \frac{1}{3+\frac{47}{168}} \simeq \frac{1}{3} .$$
$$\frac{692}{2923} = \frac{1}{4+\frac{155}{692}} \simeq \frac{1}{4} .$$
$$\frac{4080}{21153} = \frac{1}{5+\frac{753}{4080}} \simeq \frac{1}{5} .$$
$$\frac{31572}{194457} = \frac{1}{6+\frac{5025}{31572}} \simeq \frac{1}{6} .$$
$$\frac{304248}{2170599} = \frac{1}{7+\frac{40863}{304248}} \simeq \frac{1}{7} .$$

From these computations he came to know that, when he obtained

$F_1(n)$, he ignored about $1/n$ in the denominator of $F(n)$. So he added this amount to each denominator, and arrived at the second approximation:

$$F_2(n) = \frac{1}{4n + \dfrac{1}{n}} = \frac{n}{4n^2+1} .$$

Then he repeated again the same procedure.

$$\frac{33}{64} = \frac{1}{1+\frac{31}{33}} \simeq \frac{1}{1} \text{ or } \frac{1}{2} .$$

$$\frac{25}{68} = \frac{1}{2+\frac{18}{25}} \simeq \frac{1}{2} \text{ or } \frac{1}{3} .$$

$$\frac{47}{168} = \frac{1}{3+\frac{27}{47}} \simeq \frac{1}{3} \text{ or } \frac{1}{4} .$$

$$\frac{155}{692} = \frac{1}{4+\frac{72}{155}} \simeq \frac{1}{4} .$$

$$\frac{753}{4080} = \frac{1}{5+\frac{315}{753}} \simeq \frac{1}{5} .$$

$$\frac{5025}{31572} = \frac{1}{6+\frac{1422}{5025}} \simeq \frac{1}{6} .$$

$$\frac{40863}{304248} = \frac{1}{7+\frac{18207}{40863}} \simeq \frac{1}{7} .$$

He again realized that he ignored about $1/n$ in each denominator when he obtained $F_2(n)$. So he added it to obtain the third approximation:

$$F_3(n) = \cfrac{1}{4n + \cfrac{1}{n + \cfrac{1}{n}}} = \frac{n^2+1}{4n^3+5n} .$$

Presumably again he applied the same procedure to the neglected fractions.

$$\frac{31}{33} = \frac{1}{1+\frac{2}{31}} \simeq \frac{1}{1}.$$

$$\frac{18}{25} = \frac{1}{1+\frac{7}{18}} \simeq \frac{1}{1}.$$

$$\frac{27}{47} = \frac{1}{1+\frac{20}{27}} \simeq \frac{1}{1} \text{ or } \frac{1}{2}.$$

$$\frac{72}{155} = \frac{1}{2+\frac{11}{72}} \simeq \frac{1}{2}.$$

$$\frac{315}{753} = \frac{1}{2+\frac{123}{315}} \simeq \frac{1}{2}.$$

$$\frac{1422}{5025} = \frac{1}{3+\frac{759}{1422}} \simeq \frac{1}{3} \text{ or } \frac{1}{4}.$$

$$\frac{18207}{40863} = \frac{1}{2+\frac{4449}{18207}} \simeq \frac{1}{2}.$$

But this time he could not find any rule governing the fractions, and stopped here.

5. *Conclusion*

The above is the hypothetical process which presumably led Mādhava to the discovery of the three types of correction, (2), (3), and (4). The idea underlying this method, which is equivalent to the so-called Euclidean algorithm of division, had been known to Indian mathematicians since at least the time of Āryabhaṭa. They used it in their solutions of indeterminate equations of the first degree (called *kuṭṭaka*) [Datta & Singh 1962, II, 87–125], and in the reduction of astronomical parameters expressed in large terms into smaller ones [Somayaji 1985, 158–162]. In addition, this hypothesis

resolves the two difficulties we pointed out in Youschkevitch's conjecture.

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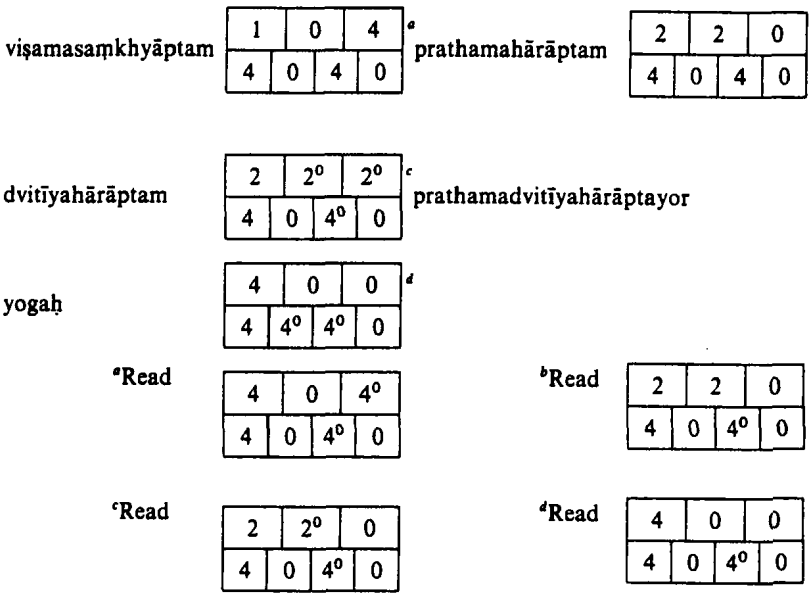
NOTES

1. The word *vākya* means "a sentence". In the *Vākya* system of Indian astronomy, astronomical tables are given by means of versified sentences which, expressed in an alphabetical system of numerals (called *kaṭapayādi*), not only have numerical values, but also narrate a story such as the life of Kṛṣṇa. See, for example, Pingree, 1981, 47–49.
2. The context in the *Kriyākramakarī* where the words "Mādhava" and "himself" occur is as follows.
   "On this point Mādhava has told [thus] (*atrāha mādhavaḥ*)." (p. 377, line 22)
   +[Four stanzas cited]+[Śaṅkara's comments on them]
   +"Another method for calculating the circumference, which is easier than this, has been told by *himself* as follows (*ito 'pi laghutaraparidhyānayanopāyo 'munaivoktaḥ, tad yathā*)." (p. 379, line 13)
   +[Three and a half stanzas cited].
3. *Yuktidīpikā* 2.206–211. For the details, see Rajagopal & Rangachari, 1978, 92–93. For the date of the work, see Sarma's Introduction (p. lxi) to Nīlakaṇṭha 1977.
4. We have used a computer, PC-1480U, with the BASIC language.
5. *Kriyākramakarī*, p. 386, line 12:
   tatra tāvad uktarūpasaṃskāraḥ sūkṣmo na veti prathamaṃ nirūpaṇīyam/ tadarthaṃ yayā kayā cid viṣamasaṃkhyayā haraṇe kṛte pṛthak saṃskāraṃ kuryāt/ atha taduttaraviṣamasaṃkhyāharaṇānantaraṃ ca pṛthak saṃskāraṃ kuryāt/ evaṃ kṛte labdhau paridhī yadi tulyau bhavatas tarhi saṃskāraḥ sūkṣma iti nirṇīyatām/ katham/ ubhayatra yadi sāmyaṃ bhavati tarhi saṃskārasya sarvasādhāraṇatvād uttarottaram api haraṇānantaraṃ [saṃskāre] kṛte sati tathaiva bhavet/ ataḥ prāg labdham eva sūkṣmam ity avagantavyam iti/
   This passage corresponds to *Yuktidīpikā* 2.220–221. We cite here the two stanzas in order to give the reader some ideas about the parallelism between the two works. See also n. 17 below.
   vicāryam ādau saṃskāra eṣa sūkṣmo na vety ataḥ/
   tadarthaṃ paridhī neyau dvau pṛthak tau samau yadi//220//
   nirantarādhaūrdhvasthayugmasaṃkhyotthasaṃskṛtī/
   tarhi sarvatra sāmyaṃ sambhavatīty avagamyatām//221//
6. *Kriyākramakarī*, p. 386, line 17.
   uttaraviṣamasaṃkhyāptasya tatsaṃskāralabdhasya cāntare pūrvasaṃskāralabdhenatulye saty evobhayatra paridhisāmyaṃ sambhavati/ ato yayā kayā cid viṣamasaṃkhyayā labdhena phalena taduttarasaṃskārasya pūrvasaṃskāraphalasya ca yogo yathā tulyo bhavet tathā saṃskāraḥ kartavyaḥ/
7. *Kriyākramakarī*, p. 388, line 2.



8. *Kriyākramakarī*, p. 388, line 6.
asya sthaulyasya darśanāt tṛptim alabhamānaḥ saṃskārāntaram anviyeṣa/saṃskārahārāptasya phalasyātrābhīṣṭāt kiñcid ādhikyaṃ jātam/ atas tatparihārāya hāradvaye rūpaṃ kṣiptavān/
9. *Kriyākramakarī*, p. 388, line 15.
ataḥ saṃskārahārayogo dvighnarāśinādhiko bhavati/ viṣamasaṃkhyāṃśas tu caturghnarāśinā rūpeṇa ca/
10. *Kriyākramakarī*, p. 388, line 3. See n. 7 above.
11. *Kriyākramakarī*, p. 388, line 17.
ata idānīṃ rāśisthāne 'pi sthaulyaṃ jātam/ prāg rūpasthāne evam[a]/ ataḥ saṃskārahārayoḥ pūrṇarūpakṣepo na kartavyaḥ/
[a]Read -sthāna eva, as in the mss. mentioned in the apparatus.
12. *Kriyākramakarī*, p. 388, line 19.
svabhaktarūpayute hāre tu tadardhatulyena rūpeṇaivādhikyaṃ bhavati saṃskārahārayor dvighnarāśyāsannatvāt/
13. *Kriyākramakarī*, p. 388, line 20.
atra rūpāntaram ekam eva/ catvāri rūpāntarāṇi labhyāni ca/ viṣamasaṃkhyāṃśasyetaraikyāc caturbhī rūpair ūnatvāt/ ataḥ pūrvakalpite saṃskārahāradvaye svabhaktāni catvāri rūpāṇi kṣepyāṇi/
14. *Kriyākramakarī*, p. 389, line 1.
tadānīṃ viṣamasaṃkhyābhavo 'ṃśaḥ prāyeṇāṣṭabhī rūpair adhiko bhavati/ itarāṃśai-

kyasya tu caturbhir evādhikyaṃ syāt/ ata idānīṃ prāyaḥ saukṣmyaṃ bhavatīti nirūpya svāptacatuṣkakṣepaṃ kṛtavān ācāryaḥ/

15. *Yuktidīpikā* 2.263–267.
atha svabhaktarūpe tu yukte saṃskārahārayoḥ/
prāgvat sacchedatānītiḥ pūrvaṃ[a] sthaulyam ihānayet//263//
prāguktād ojasaṃkhyāyāṃ rūpadvayam ato 'dhikam/
saṃskārahārayor dvighnaujarāśyāsannatā yataḥ//264//
saṃskārayor aṃśayutau rūpādhikyaṃ puroditam/
pūrṇarūpayutau dvighnarāśitulyaṃ yato 'ntaram//265//
dvighnarāśihṛte tasmin pūrṇarūpaṃ tato 'ntaram/
yataḥ prāgojasaṃkhyāyāṃ nyūnaṃ rūpacatuṣṭayam//266//
idānīṃ trikam evātaḥ kṣepyaṃ saṃskārahārayoḥ/
svāptaṃ rūpacatuṣkaṃ syāt tat prāyaḥ sūkṣmatā bhavet//267//
[a]Read – ānītipūrvaṃ

16. *Kriyākramakarī*, 389, line 11.
asyāpi saṃskārasya kiyat sthaulyam ity avagantuṃ prāguktenaiva mārgeṇa svāptacatuṣkakṣepe samānajātitvaṃ nayet/

nyāsaḥ/ tatra pūrvakalpitasaṃskārahārau

2	2^0

2	2

etayoḥ svāptacatuṣkaṃ kṣiptā[a] samānajātitve kṛte hārāṃśayor apavartitayor

nyāsaḥ/ prathamo hāraḥ

5	5^0
2	4^0	4

[b] dvitīyaḥ

5	5
2	4	4

[c] viṣamasaṃkhyā

5
5	0

[d] anyonyahārābhihatau harāṃśāv iti[e] samacchedatve kṛte viṣamasaṃkhyāṃśaḥ

4	0	0	0	5	6
4	0	0	5^0	5	0

[f] prathamahārasya

2	2^0	0	4	0	
4	0	0	0	16	0

[g]

dvitīyahārasya

2	2	0	4^0	0	
4	0	0	0	16	0

[h] hārāptayor aikyaṃ yad yac ca punar viṣamasaṃkhyāptaṃ tayor antarasya

16					
4	0	0	0	16	0

asmiṃś caturbhir apavartite viṣamasaṃkhyāyāḥ samapañcahatiḥ caturghnamūlayutā hāraḥ/
catussaṃkhyo 'ṃśaḥ/

[a]Read kṣiptvā. [b]Read

1	1^0	
2	4^0	4

[c]Read

1	1	
2	4	4

[d]Read

1	
1	0

[e]Cited from *Līlāvatī* 30. [f]Read

	4	0	0	0	16
4	0	0	0	16	0

[g]Read

2	2	0	4^0	0	
4	0	0	0	16	0

[h]Read

	2	2^0	0	4	0
4	0	0	0	16	0

17. *Kriyākramakarī*, p. 390, line 17.
athāto 'pi kiñcit sūkṣmataraḥ saṃskāro likhyate/
ante samasaṃkhyādalavargaḥ saiko guṇaḥ sa eva punaḥ/
yugaguṇito rūpayutaḥ samasaṃkhyādalahato bhaved dhāraḥ/ iti/
ayaṃ viṣamasaṃkhyāharaṇānantaraṃ kartavyaḥ saṃskāraviśeṣa iti/
This passage corresponds to *Yuktidīpikā* 2.295–296.
ebhyaḥ sūkṣmataro 'nyo vilikhyate kaś canāpi saṃskāraḥ/
ante samasaṃkhyādalavargaḥ saiko guṇaḥ sa eva punaḥ//295//
yugaguṇito rūpayutaḥ samasaṃkhyādalahato bhaved dhāraḥ/
triśarādiviṣamasaṃkhyāharaṇāt param etad eva vā kāryam//296//.

Paper 43

K. V. Sarma (2002)

Nīlakaṇṭha Somayājī (1444–1545): The astute astronomer of Kerala and his works

Article in

500 years of Tantrasaṅgraha, Ed. by M. S. Sriram, M. D. Srinivas and K. Ramasubramanian, Indian Institute of Advanced Study, Shimla, 2002.

Nīlakaṇṭha Somayāji (1444-1545 AD),

The astute astronomer of Kerala and his works*

Nīlakaṇṭha Somayāji, the astute astronomer of Kerala and author of several works on Indian astronomy, shines as one of the brightest stars in the firmament of Indian astronomy. A versatile scholar, Nīlakaṇṭha initiated several new and accurate methods of astronomical computations relating to the instantaneous velocities of the planets, the eclipses, measurement of time from the Sun's and Moon's shadows and a host of other matters. To his credit go the depiction of a simpler and more accurate planetary model of the interior planets and the computation of the equation of centre. The novel theories envisaged in his works form a pointer to the great strides that the development of the astronomical and mathematical studies had taken in Kerala during medieval times.

Autobiographical details

Our author Nīlakaṇṭha is generally referred to with the title *Somayāji, Somasut, Somasutvan* or *Comatiri*, the last being the Malayalam derivative of the Sanskrit words. A detailed colophon occurring at the end of his *Bhāṣya* on the *Gaṇitapāda* of the *Āryabhaṭīya*, contains a good deal of information about him.[1]

इति श्रीकुण्डग्रामजेन गार्ग्यगोत्रेण आश्वलायनेन भाट्टेन केरलसद्ग्रामगृहस्थेन श्रीश्वेतारण्यनाथपरमेश्वरकरुणाधिकरणभूतविग्रहेण जातवेदपुत्रेण शंकराग्रजेन जातवेदो मातुलेन दृग्गणितनिर्मापकपरमेश्वरपुत्र-श्रीदामोदरात्तज्योतिषामयनेन रवित आत्तवेदान्तशास्त्रेण सुब्रह्मण्यसहृदयेन नीलकण्ठेन सोमसुता विरचितविविधग्रन्थेन दृष्टबहूपपत्तिना स्थापितपरमार्थेन कालेन शंकरार्य(भ)निर्मिते श्रीमदार्यभटाचार्यविरचितसिद्धान्तव्याख्याने महाभाष्ये....

The above-quoted passage informs that Nīlakaṇṭha belonged to the *Gārgya-gotra*,[2] was a follower of the *Āśvalāyana-sūtra* of the *Ṛgveda* and was a Bhaṭṭa. He was the son of Jātavedas and had a

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* *500 hundred years of Tantrasaṅgraha*, Ed. by M.S. Sriram, M.D. Srinivas and K. Ramasubramanian, Indian Inst. of Advanced Study, Shimla, 2002.

younger brother named Śaṅkara. He had an uncle Jātavedas by name and a close friend Subramaṇya. He was a performer of the *Soma* sacrifice. He had composed several works on astronomy, in which subject he had made deep and extensive investigations, a fact, which is well borne out by his available works.

Some more personal details about Nīlakaṇṭha seem to be forthcoming from a Malayalam work entitled *Laghurāmāyaṇam*. This work describes itself as a work of Rāma, son of Nīlakaṇṭha of the *Gārgyagotra* and resident of Kuṇḍagrāma. *Cf.*, colophon at its end[3];

इति श्रीकुण्डग्रामजेन गार्ग्यकुलतिलकेन श्रीनीलकण्ठात्मजेन आर्याम्बागर्भसम्भवेन मन्वादिस्मृतिमर्मज्ञ-संस्कृत-द्राविड-भाषा-त्रयपारीणस्य दक्षिणामूर्तिनाम्नोऽग्रजेन रामेण विरचितं श्रीरामायणं प्रबन्धम्।

This Nīlakaṇṭha is identified by the editor of the work with our author[4]. If this identification is correct, Nīlakaṇṭha's wife was named Āryā, and he had two sons Rāma and Dakṣiṇāmūrti, the latter of whom was well versed in the *Dharmaśāstras* and learned in the three languages, Sanskrit, Tamil and Malayalam. The great Malayalam poet Tuncattu Ezuttacchan is said to have been a student of Nīlakaṇṭha. Nīlakaṇṭha is also said to have composed, at the request of a friend, a panegyric in Malayalam on the goddess Pārvatī, the presiding deity of the temple of Ūrakam in Kochin, in order to ward off the predicted premature death of that friend's daughter[5].

The authenticity of the above work and the sources of the information are however not quite certain, and corroborative evidences have to be found before accepting the above statements.

Native Village and favourite deity

Nīlakaṇṭha hailed from Trik-kaṇṭi-yur (Sanskritised into Śrī-kuṇḍa-pura or Śrī-kuṇḍa-grāma), near Tirur, S.Rly. Ponnani Taluk, South Malabar, a famous seat of learning in Kerala during the middle ages. The name of his *Illam*, as the house of a Nampūtiri Brahmin is called, was Ke̲ḷa̲llūr (sometimes spelt also as Kerallur and Kalannur), Sanskritised into *Kerala-sad-grāma* corresponding to the Malayalam word *Kerala-nall-ūr*[6]. Nīlakaṇṭha's house is identified as the present

Eṭamana Illam, situated a little to the south of the local temple[7]. It is stated that Nīlakaṇṭha's family became extinct and that the family property was inherited by the nearest relations, viz., the *Eṭamana* family.

Nīlakaṇṭha's favourite deity was Lord Śiva installed in the famous temple at Tṛpparaṅṅoḍ (Sanskrit, *Śrī-parakroḍa*, also *Śrī-śvetāraṇya* near his village, *cf*: श्वेतारण्यनाथ-परमेश्वर-करुणाधिकरणभूतविग्रहेण, in the colophon to the *Āryabhaṭīyabhāṣya*.

Patron, Netranārāyaṇa, and brother Śaṅkara

Nīlakaṇṭha refers to his younger brother Śaṅkara in several places in the *Āryabhaṭīyabhāṣya*. Śaṅkara too seems to have been well versed in astronomy and to have followed his elder brother's studies. Thus, after describing some method on the rule of three (*trairāśika*) in his *Āryabhaṭīyabhāṣya* (*Gaṇita*, 26), Nīlakaṇṭha says how his brother who was teaching at the house of his patron explained to the latter some of those theories[8]:

अत्र केषाञ्चिद् युक्तयः पुनः अस्मदनुजेन शंकराख्येन तत्समीपे अध्यापयता वर्तमानेन तस्मै (नेत्रनारायणाय) प्रतिपादिताः।

Nīlakaṇṭha observes at the close of the *Bhāṣya* on the *Golapāda* that he was entrusting the *Bhāṣya* to Śaṅkara for its proper propagation. Thus, just before the final colophon, Nīlakaṇṭha says[9]:

एवमिदम् अस्माभिर्यथामति व्याख्यातम्। नमः स्वयम्भुवे तस्मै यत्प्रसादादिदं कृतम्। नमो भगवते तस्मै श्रीमदार्यभटाय च। शिष्यं तन्त्वेन विचार्यार्यभटसूत्रभाष्यमिदम्। यदि स न्यायाल्लिप्सेदस्मै दातव्यमेव शंकर ते।।

That Nīlakaṇṭha was intimately connected with and was patronised by Kauṣītakī Āḍhya Netranārāyaṇa, known locally as Azhvāñceri Tamparakkal, the religious head of the Nampūtiri Brahmins of Kerala, is known from several references in his writings. It is also clear that the patron had great esteem for Nīlakaṇṭha's erudition in astronomy, in which subject he too was interested and used to discuss difficult points with Nīlakaṇṭha. Thus, in the discussion on the calculation of the motion of planets (*Āryabhaṭīya.*, *Kāla.*, 22-25), Nīlakaṇṭha says[10]:

कर्णभुक्तिः स्फुटेत्यत्र व्याख्याने पारमेश्वरे।
व्यासार्धाप्तं कोटिवर्गात् कर्क्येणादावृणं धनम्।।
कोट्यां तदूनयुग्व्यासदलं गतिविधौ श्रुतिः।
प्रकारान्तरमाहैवं सूक्ष्मभुक्तिप्रसिद्धये।।
गुरूणां मे पितात्रापि स्थौल्यान्मत्सरिणोदिते।
परमेश्वरतच्छिष्या नैव वेलागतिं विदुः।।
इति कौषीतकी श्रुत्वा नेत्रनारायणः प्रभुः।
मह्यं न्यवेदयत् तस्मै तदैवं प्रत्यपादयम्।।

Again, in the long discussion on the calculation of the apparent position of celestial bodies (*Āryabhaṭīya.*, *Kāla*, 17-21), speaking on a method to derive the *sakṛt-karṇa*, our author says[11]:

अन्यदपि कर्म अस्माभिरुपन्यस्यमानं श्रुत्वा आढ्येन कौषीतकिना अनुष्टुभा निबद्धम् -
स्वोच्चोनमध्यमार्कस्य भुताज्याघ्ना त्रिजीविका।
स्वोच्चहीनस्फुटार्कस्य दोर्ज्याभक्ता श्रुतिर्भवेत्।।

This would indicate the intimacy that existed between Nīlakaṇṭha and his patron and the common interest that bound them together. On the compilation of the *Āryabhaṭīyabhāṣhya*, Nīlakaṇṭha observes in one place[12]:

यन्मयात्र केषाञ्चित् सूत्राणां तदुक्तीः प्रतिपाद्य कौषीतकिना आढ्येन नारायणाख्येन व्याख्यानं कारितम्, अतस्तदेवात्र लिख्यते।

Again, in another context, he remarks[13]:

इतीदं प्रथमे वयस्येव, वर्तमानेन मया द्वितीयवयसि स्थितेन कौषीतकिना आढ्येन कारितम्।..।.. तस्मिन् स्वर्गते पुनः.....व्याख्यानमारब्धम्।

It is clear from the above that the credit of enthusing Nīlakaṇṭha in his investigations, and in fact, to have prompted him to write his *Āryabhaṭīyabhāṣhya*, goes to Netranārāyaṇa,[14] the members of whose family are known all through the annals of Kerala history to have been good scholars and, at the same time, liberal patrons of scholarship.

Nīlakaṇṭha's teachers : Ravi and Dāmodara

Nīlakaṇṭha informs us in his *Āryabhaṭīyabhāṣya* that he studied *Vedānta* under Ravi, *cf. Ravita ata-vedānta-śāstreṇa*[15]. That Ravi was

well versed also in *Jyotiśśāstra* and that Nīlakaṇṭha imbibed some of his knowledge in astronomy also from Ravi is clear from the introductory verse to Nīlakaṇṭha's *Siddhāntadarpaṇa*, where his teachers have been mentioned by double entendre:

श्रीमद्दामोदरं नत्वा भगवन्तं रविं तथा।
यत्प्रसादान्मया लब्धं ज्योतिश्शास्त्रमिदमुच्यते।।

A work on astrology, *Ācāradīpikā*, which is a detailed commentary in verse on *Muhūrtāṣṭaka*, is ascribed to this Ravi[16].

The teacher of Nīlakaṇṭha who actually initiated him into the science of astronomy and instructed him on the various principles underlying mathematical calculations was Dāmodara, son of the *Kerala-Dṛggaṇita* author Parameśvara[17], of the *Bhārgava-gotra* and resident of the village of Alattur (Sanskritised into Aśvattha-grāma) which was situated quite near Nīlakaṇṭha's own village. In his *Āryabhaṭīyabhāṣya*, as also in his other works, Nīlakaṇṭha reverentially refers to his teacher and his studies under him. He speaks of how even as a boy he stayed with his *Guru*, at the latter's residence, prosecuting his studies[18]:

मया गुरुकुले वसता बाल्य एव . . . . . निबद्धं च तत् तदैव अस्मद्गुरुभिः पञ्चभिरुपजातिभिः अर्कस्फुटेनानयनं प्रकुर्याद्. . . । तदपि सर्वत्र विष्कम्भदलं श्रुतौ वा व्यासार्धके स्याद् विपरीतकर्णःइत्यस्मद्गुरुणोक्तम्।[19]

Similar quotations and other references, which Nīlakaṇṭha and later authors make, proclaim Dāmodara not only to be a prominent astronomer of the times but also as the author of erudite works on the subject, manuscripts of which are yet to come to light.

Nīlakaṇṭha followed the footsteps of Parameśvara, founder of the *Dṛggaṇita* system of astronomy in Kerala and one of the foremost astronomers of the land. For him Parameśvara was not only the revered father of his *Guru* but was also his *Paramaguru*, by which term he generally refers to him in his works:

यतो भार्गवपरमेश्वराचार्येण अस्मत्परमगुरुणा चलांशास्त्वं (4546) इति कल्यब्दे परीक्ष्य पञ्चदशांशपूर्तिर्निर्णिता।[20]; अस्मत्परमगुरुणापि सिद्धान्तदीपिकायाम् एतत् प्रतिपादितम्।[21]

Date of Nīlakaṇṭha

Indisputable evidences are available for fixing the date of our author. Śaṅkara, Nīlakaṇṭha's pupil, in his commentary on his teacher's *Tantrasaṅgraha* points out that the first and last verses of that work contain chronograms specifying the dates of the commencement and of the completion of the work. Thus, after giving the literal meaning of the first verse of the work[22]:

हे विष्णो निहितं कृत्स्नं जगत् त्वय्येव कारणे।
ज्योतिषां ज्योतिषे तस्मै नमो नारायणाय ते ।।

Śaṅkara says:

आचार्येण इमं श्लोकं आदितो ब्रुवता प्रथमपादेन प्रबन्धारम्भदिनकल्यहर्गणश्च अक्षरसंख्यया उपदिष्टः। समाप्तिसमयाहर्गणश्च 'लक्ष्मीशनिहितध्यान' इत्यन्ते भविष्यति।

These two *Kali* dates, 16,80,548 and 16,80,553, work out to *Kali* year 4601, *Mīna* 26, and 4602, *Meṣa* 1, both dates[23] occuring in A.D. 1500.

The *Siddhāntadarpaṇa* and Nīlakaṇṭha's own commentary thereon give, respectively, the year and actual date of his birth[24],

कलिसन्ध्यष्टमांशे स्वशतांशाढ्ये गते ततः।
धनुर्मिथुनयोर्मध्ये प्रायशस्त्वयने उभे।।

दिव्याब्दशतमिता खलु काले सन्ध्या स्मर्यते। तस्य अष्टमांशः सार्धदिव्याब्दद्वादशकः। स च सौराब्दानां पञ्चचत्वारिंशत् शतम् (४५००)। तस्य शतांशः पञ्चचत्वारिंशदब्दः (४५)। ततः स्वशतांशाढ्यः 'शिवशिवे'ति (४५४५)। कल्यब्दैस्तावति याते उभे अयने उत्तरदक्षिणाख्ये प्रायशो धनुर्मिथुनमध्ये स्तः। तदा अयनचलनांशाः धनात्मकाः पञ्चदशसंख्या बभूवुः । प्रायिकत्वं च कलाष्टकाधिकत्वात्। यतो भार्गवपरमेश्वराचार्येण अस्मत्परमगुरुणा 'चलांशास्स्वं' (४५३६) इति कल्यब्दे परीक्ष्य पञ्चदशांशपूर्तिर्निर्णीता। अतः सन्ध्याष्टमांशशतांशस्य प्रायिकत्वम्। स्वजन्मकालज्ञापनार्थं चैवमुक्तम्। तदा अहर्गणश्च 'त्यजाम्यज्ञतां तर्कैः।' (१६,६०,१८१) इति।

(p. 17 of the edn).

Here, Nīlakaṇṭha himself says that he was born on the *Kali* day 16,60,1881, which corresponds to June 17, 1444 A.D. according to Gregorian calendar.

That Nīlakaṇṭha lived to a ripe old age, even to become a centenarian, is attested by a contemporary reference made to him in a Malayalam work on astrology, viz., the *Praśnasāra* by Mādhava, a *Nampūtiri Brāhmaṇa* of the Iñcakkāḷavā house in Kerala, who wrote his work in A.D. 1542-43. Here, Mādhava says that he could count upon reputed authorities like Keḷanallūr to recommend his work.

āḷ-āyat-ādaravil ādiyil attimaṭṭam
lōkōttaran punar itinn-iha 'keḷanallūr'।
ābhāsar allarivatuḷḷavar ādarippān
porum prasiddhi perikoḷḷavar uṇṭanekam ॥

The date of composition of this work, *Praśnasāra*, is given as Kollam era 718/*Kali* 4644 (A.D. 1543) by the following verse in the work itself:

eḷunūṭṭorupatteṭṭavatu kollam atāya nāḷ ।
varunna viṣuvad, bhāvatattvam (4644) kalyabdam āyatu ॥

Rightly does Nīlakaṇṭha remark in his *Āryabhaṭīyabhāṣya* [25]:

मयाद्य प्रवयसा. . . . . यथाकथञ्चिदेव व्याख्यानमारब्धम् ।

Moreover, we know of at least two more works composed by him subsequent to his writing the *Āryabhaṭīyabhāṣya*, viz., the commentary on the *Siddhāntadarpaṇa* and the *Jyotirmīmāṃsā*, both of which quote the *Āryabhaṭīyabhāṣya*.

Versatility of Nīlakaṇṭha :

For a *Jyautiṣika*, and one who had specialized on its astronomical aspect, Nīlakaṇṭha seems to be very well read. Every other page of his writings substantiates his knowledge of the several branches of Indian philosophy and culture; Sundararāja, the Tamil astronomer calls him '*Ṣaḍdarśanapārangata*' 'One who had mastered the six systems of philosophy'. Nīlakaṇṭha himself informs us that he studied *Vedānta* under *Ravi*: *cf.*, *Ravita ātta-vedānta-śāstreṇa*. He can refer to a *Mīmāṃsā* authority to establish a mathematical point[26] and with equal felicity apply a grammatical dictum to the same purpose.[27] Piṅgala's *Chandas-sūtra*[28] and the lexicons are quoted as

the occasion demanded. The scriptures and the *Dharmaśāstra*s also come in for citation,[29] and, so also the *Purāṇas*[30] like the *Bhāgavata*[31] and the Viṣṇu.[32]

As for *Jyotiṣa* works, Nīlakaṇṭha exhibits a surprising familiarity with a large number of them, from the *Vedāṅgajyotiṣa* down to the treatises of his own times. He uses all types of *Jyotiṣa* texts, *Gaṇita*, *Saṃhitā* and *Horā*, but as became his subject of specailisation, his quotations are mainly from texts dealing with astronomy proper. Some of the more important texts of all India prevalence that Nīlakaṇṭha quotes are: *Vedāṅgajyotiṣa*, *Āryabhaṭīya*, Varāhamihira's *Pañcasiddhāntikā*, *Bṛhajjātaka* and *Bṛhatsaṃhitā*, *Sūryasiddhānta*, Śrīpati's *Siddhāntaśekhara* and Mañjula's *Laghumānasa*. Of the texts popular mainly in Kerala may be mentioned, the *Parahitagaṇita* or *Grahacāranibandhana* of Haridatta, the *Bhāṣya* on the *Āryabhaṭīya* and *Laghu* and *Mahā-bhāskarīyas* by Bhāskara I, Govindasvāmi's *Bhāṣya* on the latter and Parameśvara's super-commentary thereon. Other works of Parameśvara like his *Āryabhaṭīya-vyākhyā* also come in for citation as also passages from his own teacher Dāmodara. Another Kerala author whom Nīlakaṇṭha quotes often is Mādhava, often styled *Golavid*, who was a reputed astronomer of the times.[33] Manuscritps of several works quoted by Nīlakaṇṭha are yet to be unearthed and a detailed study of the numerous authorities quoted by Nīlakaṇṭha is bound to throw welcome light on the annals of Hindu astronomy.

Nīlakaṇṭha's works

Nīlakaṇṭha has written several works which reflect his deep study of and ripe scholarship in astronomy, embodying the results of his investigations in the subject and interpreting the science lucidly.

i. *Golasāra*[34]

The *Golasāra* is a short work in 56 *āryā* verses, divided into three *parichheda*s, containing, respectively, 11, 15 and 30 verses. The first *pariccheda* sets out the basic astronomical constants, viz., the number of civil days and the revolutions of the planets in an aeon, the positions of the higher apses and the ascending nodes of the planets,

their maximum latitudes, their epicycles to the equations of the apses and of conjunction, the diameters of the orbits of the Sun and the Moon and the *yojana* measures of the epicycles.

Pariccheda II is concerned with the presentation of the celestial globe (*Jyotirgola*) from the point of view of astronomical conceptions and observations and the movement, therein, of the heavenly bodies. The position of the great circles, *Ghaṭikā-maṇḍala* (celestial equator) and the *Apakrama-maṇḍala* (ecliptic), their mutual obliquity, the division of the ecliptic, the (apparent) rotation of the celestial globe, the rising point of the ecliptic (*lagna*), the measure of the orbits of the planets, the measurement of the positions of the planets on the ecliptic and the position of the horizon at the equator and elsewhere are noticed here, in order.

In *Pariccheda* III, verses 1 to 15 deal with the circle and the graphical and computational derivation of the sines. Verses 16 to 30 discuss the inter-relationship of the *Manda*, *Śīghra* and *Kakṣya* circles of the different planets and also how the results arrived at by calculation of the positions of the planets are affected by their *Kṣepa* (deflection) from the ecliptic.

ii-iii. *Siddhāntadarpaṇa* and *Vyākhyā* [35]

The importance of *Siddhāntadarpaṇa* lies in the fact that the author presents herein the astronomical constants as verified through his own observations and investigations. The passages expressing these constants having been adopted and commented upon by the author in his commentary, which latter happens to be one of his last works. It could be taken that the values of the constants as given herein are the final figures accepted by Nīlakaṇṭha.

In the first part of the work called *Upadeśa-bhāga* (theory section), the author enunciates in twenty couplets (2-21), his view on the number of revolutions of the planets, their higher apses (*mandocchas*) and ascending nodes (*pātas*) during a definite period of time, the dimensions of the epicycles of equations of the apses (*manda-paridhi*) and of conjunctions (*śīghra-paridhi*), the measure of the aeons

(*yugas*), the velocities of the planets, the measures of the diameters of the Moon and the Sun, the geographical location of the city of Avantī, the situation of the ecliptic (*apakrama-vṛtta*) and the conception of the epicycles.

In the second part called *Nyāyabhāga* (Practical section) verses 21-31), are set forth the eccentric and orbital circles (*pratimaṇḍala* and *kakṣyā-maṇḍala*), the sines etc. of the angles measured on these circles, the geocentric positions of the planets, declination and its measurement. The author also gives his views on the occurrence of the *vyatīpāta*s, the extent of the lunar crescent and the data from which eclipses are to be calculated.

The commentary

In line with his *Bhāṣya* on the *Āryabhaṭīya*, Nīlakaṇṭha's commentary on the *Siddhāntadarpaṇa* is elaborate and discursive. Alongside explaining the text proper, it introduces related topics by way of background, illustration and rationalization. Often, the commentary dilates into verse, a common practice with Kerala astronomers like Parameśvara. This feature is found also in Nīlakaṇṭha's *Āryabhaṭīyabhāṣya*.

Nīlakaṇṭha's discussions are often highly instructive. An instance in point is his detailed analysis of the mental working of a mathematician who proceeds to derive the relation between the sides and the hypotenuse of a right angled triangle (pp.22-24). In the course of this discussion, he makes a mention also of the two methods of approach for solving mathematical problems, viz., that of logical reasoning and that of demonstration on the board. He adds that one should try the method of demonstration, for logical reasoning is limited, endless and sometimes inconclusive (*alpa-viṣayatvāt, ānantyāt, kvacid apy aviśrānteś ca*, p.23). Equally instructive are the methods of constructing the armillary sphere, the defining of the situation of *rāśis* therein (p.14), and the exposition of the computation of the geocentric positions of the planets (pp. 25-31).

The following observations of Nīlakaṇṭha are noteworthy:

(i) *Ayanāṃśa* (measure in minutes of the precession of the equinoxes) should be derived by observation, if it has to be accurate (p.17).

(ii) The use of *Trairāśika* (Rule of three) is justified even in computations concerning moving bodies, since the factors involved are considered, only for specific moments at which they might be taken as stationary (p.25)

(iii) In astronomical computations, it does not make any difference whether the eastward motion of the Earth or the westward motion of the planets are taken, because the motion is relative (p.5).

(iv) The *Bhagola* (*Rāśigola*, ecliptic sphere) revolves as a whole and not *Vāyugola* (having the *Ghaṭikāmaṇḍala* or Celestial equator as one of its great circles); thus the relative distances between the stars remain constant, from which other results follow (pp.16-17).

iv. v. *Candracchāyāgaṇita* and *Vyākhyā* [36]

Chāyāgaṇita or 'Computations concerning the Shadow' constitutes the Hindu astronomer's method for ascertaining the exact time of occurrence of any event on the basis of the shadow cast by the Sun or the Moon. The expediency of this device during ancient and medieval times would be apparent when one considers the limitations of such chronometrical devices as the hour-glass, water disc, etc. The gnomon, being one of the simplest of astronomical instruments that could be set up at any time and at any place, and readings made independent of circumscribing factors like zero-point etc., is the most handy instrument to record such abrupt occurrences as the birth of a child, death of a person, etc. A knowledge of the exact moment of these happenings, as computed by astronomical methods from the measure of the shadow, facilitates the prediction of the future based

on them in accordance with the dictums of astrology. Obviously the shadow cast by the Sun is measured during day and that by the Moon during night.

The *Candracchāyāgaṇita* of Nīlakaṇṭha Somayāji, belongs to this genre of texts and sets out the processes for the computation both of *Kramacchāyā*, 'Shadow from Time' (verses 1-17), and *Viparītacchāyā*, 'Time from shadow' (verses 18-32), of the Moon. The commentary by Nīlakaṇṭha himself gives a lucid exposition of the textual verses.

Kramacchāyā and *Viparītacchāyā*

The utility of *Chāyāgaṇita* in the daily life of the medieval Hindu, whose prescribed way of life required the performance of numerous religious rites, is not far to seek. Ordinary astronomical computations enabled him to calculate auspicious times (*muhūrta*) for sacred rites and social functions, in terms of time-units like *nāḍikā*s and *vināḍikā*s, say, after Sunrise or Sunset. But, in the absence of accurate chronometers, it was not easy to ascertain when that auspicious moment, as calculated, had arrived. *Chāyāgaṇita* came to his rescue in such a situation. For, it was possible to calculate, in advance, the length that the Sun's or the Moon's shadow would attain at the appointed time. One could, then, set up the gnomon and watch for the shadow to reach the stipulated length and perform the rite at the right moment. This process of computing the shadow for any specific time is called *Kramacchāyā*, 'Direct (process of) Shadow (computation)'.

The converse of *Kramacchāyā*, is known as *Viparītacchāyā*, 'Reverse (process of) Shadow (computation)', according to which time is calculated from the length of the shadow. This was equally, if not more, important in everyday life, in that it enabled the accurate ascertainment of the time of abrupt, non-planned occurrences like moments of birth and death, unexpected arrivals, untoward happenings and the like.

Popularity of *Chāyāgaṇita* in Kerala

The medieval astronomers of Kerala seem to have taken the best advantage of this natural phenomenon and devised highly intricate calculations to get the exact times corresponding to the measures of the shadow, taking into consideration also the factors that affected the shadow, such as the latitude and longitude of the place, time of the year, precession of the equinoxes, etc. Practically every one of the numerous astronomical manuals (*Karaṇa granthas*) produced in Kerala contains sections devoted to *Chāyāgaṇita*, both of the Sun and of the Moon. In the *Pañcabodha* manuals, of which about a dozen different texts have been identified and documented, *Chāyā* is one of the five subjects dealt with, the others being *Vyatipāta, Grahaṇa, Śṛṅgonnati* and *Mauḍhya*. Several texts which are devoted solely to computations based on shadow have also been composed.

vi. *Tantrasaṅgraha*

Tantrasaṅgraha composed on the model of the '*Tantra*' class of astronomical texts in which the zero point for computation is taken as the commencemet of the *Kaliyuga* as in texts like the *Āryabhaṭīya*, is a major work of Nīlakaṇṭha. Perhaps, to be in tune with tradition, Nīlakaṇṭha makes use of the *bhūtasaṅkhyā* notation of all India prevalence in place of the Kerala system of *kaṭapayādi*, which he uses in his other works. In about 432 verses, mostly couched in the *anuṣṭubh* metre, the author depicts the entire gamut of theoretical and practical astronomy in eight sections entitled *prakaraṇas*: I. *Madhyama* (Mean planets), II. *Sphuṭa* (True planters). III *chāyā* (Gnomonic shadow), IV. *Chandragrahaṇa* (Lunar eclipse), V. *Ravigrahaṇa* (Solar eclipse), VI. *Vyatīpāta* (The 'aspect' of the Sun and the Moon when the sum of their longitudes is 180° and their declinations (*krānti*) are equal in magnitude and direction), VII. *Dṛkkaraṇa* (Reduction to observation), and VIII. *Śṛṅgonnati* (Elevation of the Moon's horns).

The initial verse of the work gives, in a chronogram, the date of the commencement of the work and the last verse, in another chronogram, the date of completion of the work, both in 1500 A.D.

with five days intervening between them; while this may seem to be too short for the composition of the work, it need not be so; for a literary giant and idea-filled astronomer this is not a difficult feat.

Through *Tantrasaṅgraha*, Nīlakaṇṭha explores new fields, formulates novel ideas and introduces major revisions as in the case of the inner planets Mercury and Venus in the matter of their planetary model and the better formulation of their equation of centre.

The two commentaries, *Yuktidīpikā* and *Laghuvivṛti* both by Śaṅkara, on the *Tantrasaṅgraha* are highly useful in understanding the text, especially the former. It is, indeed, a real *Yuktidīpikā* in that it offers lucid rationales to the statements in the text.

Manuscritps of *Tantrasaṅgraha*

The *Tantrasaṅgraha* is divided into eight chapters, and contains 432 verses in all. Since the work had been popular and was accepted as an authority, a large number of manuscripts of the work exist, of which about fifty have been documented in the *New Catalogus Catalogorum,* (Madras University, Vol. VIII, 1974, p.97). One of these manuscripts, No.475-E of the Kerala University Oriental Research Institute and Manuscript Library, gives the date of its transcription through the chronogram '*sevyo dugdhābdhitalpaḥ*', being *Kali* day 16,99,847, falling in the year A.D. 1551, and so this Ms. is almost contemporaneous with its author Nīlakaṇṭha Somayāji who passed away in about AD 1545. All the said documented manuscripts conform to the division and extent of the work as mentioned above. In several cases there are post-colophonic statements expressing the completeness of the work in eight chapters. In one such manuscript, dated M.E. 928 (AD 1753), being Ms. No. C.224-C, deposited in the said Library, the total number of verses in the work is also stated to be 432, which conforms to the presently available text.

इति तन्त्रसंग्रहे अष्टमोऽध्यायः। तन्त्रसंग्रहः समाप्तः। अध्यायं एट्टिलुं कूटि श्लोकङ्ङळ् नानूट्टिमुप्पत्तिरण्टु। ९२८- माण्ट् चिङ्ङमासत्तिल् एषुतिञ्ञ पुस्तकम्।।

vii. *Āryabhaṭīyabhāṣya*[38]

Āryabhaṭīyabhāṣya is an elaborate commentary on the cryptic and *sūtra* like text of *Āryabhaṭīya* which comprehends in 121 *āryā*s the fields of Mathematics and Astronomy. A perusal of the commentary will amply prove that it is no false claim that Nīlakaṇṭha makes when he designates his work as a '*Mahābhāṣya*' and explains the method of exposition adopted by him [39].

श्रीमदार्यभटाचार्यविरचितसिद्धान्तव्याख्याने 'महाभाष्ये' उत्तरभागे युक्तिप्रतिपादनपरे त्यक्तान्यथाप्रतिपत्तौ निरस्तदुर्वाख्याप्रपञ्चे समुद्घाटितगूढार्थे सकलजनपदजातमनुजहिते निर्दर्शितगीतिपादार्थे सर्वज्योतिषामयनरहस्यार्थनिदर्शके समुदाहृतमाधवादिगणितज्ञाचार्यकृत-युक्तिसमुदाये निरस्ताखिलविप्रतिपत्तिप्रपञ्च-समुपजनितसर्वज्योतिषामयनविदमलहृदय-सरसिजविकासे निर्मले गम्भीरे अन्यूनातिरिक्ते गणितपादगतार्यात्रयस्त्रिंशद्व्याख्यानं समाप्तम्।

In another context, recalling how he came to write the commentary, Nīlakaṇṭha remarks [40]:

मयाद्य प्रवयसा ज्ञाता युक्तीः प्रतिपादयितुं भास्करादिभिरन्यथाव्याख्यातानां कर्माण्यपि प्रतिपादयितुं यथाकथञ्चिदेव व्याख्यानमारब्धम्।

The lucid manner in which the difficult conceptions about the celestial globe and astronomical calculations are made clear, the wealth of quotations, and the results of personal investigations and comparative studies presented herein amply justify the appellation *Mahābhāskarīya* which Nīlakaṇṭha has given to his work.

Nīlakaṇṭha has commented only on the *Gaṇita*, *Kālakriyā* and *Golapāda*s of the *Āryabhaṭīya*, leaving out the *Gītikāpāda*, which he says is covered by the commentary on the other three sections[41]:

तत्रेयं त्रिपाद्यस्माभिर्व्याचिख्यासिता, यतस्तद् व्याख्येयरूपत्वाद् गीतिकापादस्य। एतद् व्याख्यानेनैवार्थः प्रकाशेत।

viii. *Grahaṇanirṇaya*

This is a work on the computation of lunar and solar eclipses. Manuscripts of this work are yet to be discovered, but later authors and Nīlakaṇṭha himself in his *Āryabhaṭīyabhāṣya* quote from this work[42]:

तदैव ग्रहणमध्यं च। स्फुटसाम्ये तु विक्षेपकोटिमण्डलापक्रममण्डलयोः भुक्तभागसाम्यमेव स्यात्। तदुक्तं मया ग्रहणनिर्णये-

परमक्षेपकोटिघ्नः पातोनार्कभुजागुणः।
स्वेष्टविक्षेपकोट्याप्तस्तत्क्षेपकृतियोगतः।।
पदं यच्चापितं यच्च पातोनार्कभुजाधनुः।
तद्विशेषं हतं षष्ट्या गत्यन्तरहृतं क्षिपेत्।।
पर्वान्ते युक्पदे क्षेपे शोधयेद् विषमे पदे।
एवं कृतोऽपि पर्वान्तः सूर्येन्द्वोर्ग्रहणे स्फुटम्।।

These verses are quoted also by Śaṅkara in his commentary *Laguvivṛti* on Nīlakaṇṭha's *Tantrasaṅgraha*[43] with the introductory remark: तदुक्तमनेनैव ग्रहणनिर्णये।

ix. *Sundararāja-praśnottara*

Sundararāja, son of Anantanārāyaṇa, was an astronomer of the Tamil country contemporaneous with Nīlakaṇṭha and author of a detailed commentary on the *Vākyakaraṇa* or *Vākyapañcādhyāyī* which is a manual on the basis of which almanacs are computed in the Tamil districts.[44] Sundararāja had the greatest respect for Nīlakaṇṭha whom he addressed for clarification of certain points in astronomy. Nīlakaṇṭha's detailed answers to these questions formed a regular work, *Sundararājapraśnottara*. Manuscripts of this work are yet to come to light, but both the authors refer to this work. Sundararāja in his commentary on the last verse of Ch.V of the *Vākyakaraṇa* says:[45]

अत्र तु गतियोगांशकेनैव हरणं युक्तमिति श्रीमत्-केरलसद्ग्रामनिवासि-नीलकण्ठार्येण त्रिस्कन्धविद्यापारदृश्वना षड्दर्शनीपारंगतेन आश्वलायनसूत्रेण गर्गगोत्रेण नवकलरु(?)जातेन गोलचूडामणिना अस्मदनुग्रहार्थे सुन्दरराजप्रश्नोत्तराख्ये ग्रन्थे प्रतिपादितम्। तेन गतियोगेनैव विभज्य स्थितिदलं ज्ञेयम्।

Nīlakaṇṭha too refers to this work in his *Āryabhaṭīyabhāṣya*[46], सुन्दरराजप्रश्नोत्तराख्ये मयोक्तमत्राप्यनुसन्धेयम्।

x. *Grahaparīkṣākrama*

The well-known Kerala astrologer, the late Puliyūr Puruṣottaman Nampūtiri, has edited[47] an old incomplete Malayalam summary of a

Sanskrit work under the title *Grahaparīkṣākrama*. The textual verses were not available to the editor and he presumed that the author was Dṛggaṇita-Parameśvara. These verses are, actually, to be found in Nīlakaṇṭha's *Bhāṣya* on the *Golapāda* of the *Āryabhaṭīya*[48]. It is a long tract with about 200 verses, enunciating the principles and methods for verifying astronomical computation by regular observation. The work, as available, ends thus:

इति संक्षेपतः प्रोक्ता परीक्षा ज्योतिषामिह।
कालमानचतुष्कस्य श्रुतस्य विवृतिस्त्वियम्।।

xi. *Jyotirmīmāṃsā* [49]

The *Jyotirmīmāṃsā* of Nīlakaṇṭha is a unique work, the like of which does not seem to exist in Indian astronomical literature. It asserts that astronomy is primarily a discipline of observation, experimentation and computation and deserves to be revised periodically if the results are to be true. The processes to be employed and comparative studies to be made in different cases have also been duly indicated. For this reason, the work deserves to be set out in greater detail than the other writings of Nīlakaṇṭha.

a. Real implication of 'divine instruction'

At the outset, Nīlakaṇṭha draws attention to two eclipses which actually occurred during his times, but whose presence would not have been detected had they been computed through the astronomical constants given in the *Gītikāpāda* of the *Āryabhaṭīya*. Nīlakaṇṭha explains that it was for this very reason that Āryabhaṭa intended astronomy to be a practical discipline and recommended the verification and revision of his own astronomical constants by observation, towards which he had also suggested the method in *Golapāda* 48 (pp. 1-2). To a possible query that the number of planetary revolutions given by Āryabhaṭa are immutable since they form part of divine instruction, Nīlakaṇṭha points out, pertinently, that by the expression 'divine instruction' is not meant any direct instruction by the gods, but only the chastening of the intellect through divine grace, as a result of which the author could express his thoughts logically (pp. 2-3).

b. *Jyotiṣa*, an observational science

Nīlakaṇṭha then takes pains to demonstrate the place of observation and logical inference in the maintenance and furtherance of astronomical tradition (pp. 6-10). A passage from a *Mīmāṃsā* text, which Nīlakaṇṭha quotes in this context, fully expresses his idea:

> The correlation, of the computed Moon etc. with actual observation at a particular place, the revision of computation on the basis of such correlation, logical inference there from being transmitted as tradition, it being again correlated (with observation and again revised) and transmitted further down to others - this is how tradition is continued without interruption, and hence its (continued) authoritativeness[50]. In the same vein Nīlakaṇṭha argues:
>
> One has to accept that each of the five *Siddhāntas* had been authoritative at one time (or other, though they might not be so, now). Therefore, one has to look for a system which tallies with observation. The said tallying has to be verified by contemporary experimenters at the time of eclipses. When the two vary, i.e., when an earlier system does not agree (with observation), experiments have to be conducted with instruments, and revolution numbers of the planets calculated there from. A new system has thus to be expounded. Nobody will be ridiculed for this in this world nor punished in the next[51].

c. Corrections to planetary parameters

Having established the necessity of revising planetary parameters, periodically, Nīlakaṇṭha presents the corrections propounded to the parameters of *Āryabhaṭīya* at different times by certain astronomers including Haridatta, Govinda, Brahmagupta and Lalla (pp. 10-12).

A point of interest centres round two sets of corrections, one by Haridatta, commencing from *Śaka* 444 (*Kali* 3623) as given in his *Grahacāranibandhana*, and the other by Lalla, commencing from *Śaka* 420 (*Kali* 3599) as given his *Śiṣyadhīvṛddhida*. Both these corrections are presumed to have commenced from the date of composition of

the *Āryabhaṭīya* given in verse 10 of the *Kālakriyāpāda*. Nīlakaṇṭha points out that the discrepancy between the dates as accepted by the two authors has arisen on account of the different interpretations, the former taking *Kali* 3600 mentioned in the verse as the date of birth of Āryabhaṭa and the latter as the date of composition of the *Āryabhaṭīya*. While discussing the correction advocated by Haridatta, Nīlakaṇṭha notes that his interpretation was wrong and that *Kali* 3600 was actually the date of composition of *Āryabhaṭīya* (pp. 13-14).

Nīlakaṇṭha takes the occasion to draw attention also to the apparent error in Lalla's commencing the correction from *Śaka* 420 (*Kali* 3599) instead of from *Śaka* 421 (*Kali* 3600), but justifies it on two grounds: (1) 420 which is a round figure is more amenable to mathematical operations and increases the resultant but by one, whatever be the year for which readings are taken, causing thereby but negligible difference in the result. (2) In the case of the Moon such an addition is actually warranted. Especially for the second reason, Nīlakaṇṭha goes to the extent of saying that Lalla had used the figure 420 intentionally for this purpose (pp.14-17).

A similar justification has been made also by Sūryadeva-yajvan (born AD 1191) in his commentary on the *Āryabhaṭīya*. Nīlakaṇṭha extracts the entire section from Sūryadeva in order to buttress his view (pp.19-20).

d. Reason for different Means for same revolutions

The author poses a question as to how the means of certain planets calculated according to different systems differ even though the numbers of their revolutions per *Kalpa* as given in the said systems are the same. In answer, it is demonstrated that it is so because the number of civil days in *Kalpa* differs in the different systems, resulting in a corresponding difference in the means also (pp.17-19).

e. Vedic authority for inference in astronomy

Earlier in the work (pp. 3-6), Nīlakaṇṭha had stressed the significance of logical inference in the matter of astronomical computation. Vedic approbation in the matter is sought to be provided

by a reference to a passage from the *Taittirīya-āraṇyaka*, 1.2.1, as expounded by the well-known Vedic scholiast Sāyaṇa- Mādhava in his work *Kālanirṇaya*. It is specified in the said passage that *Smṛti, Pratyakṣa, Aitihya* and *Anumāna* (Inference) are means to determine the periods of planetary motion (pp. 21-22).

f. Relative accuracy of different systems

For an estimation of the relative accuracy of the different systems, Nīlakaṇṭha adopts the ingenious method of computing the planets for a common date. For this purpose *Kali* day 16,82,112 is chosen, it being a date occurring in A.D. 1504 and, so, contemporary to his own times. This date has the advantage also of being 8/75,00,000 of a *Kalpa*, which makes calculations easy. The results obtained are then correlated to observation.

In this connection, Nīlakaṇṭha takes into consideration the Sunrise and Midnight systems (*Audayika* and *Ārdharātrika pakṣas*), both of Āryabhaṭa, and those of Brahmagupta and Śrīpati, as set out in *Brahmasphuṭasiddhānta* and *Siddhāntaśekhara*, composed respectively by the latter two authors (pp. 22-31).

g. Correction through eclipses

Well defined eclipses being the most convenient natural phenomena for the verification of astronomical systems and effecting corrections thereto, the utilization thereof is stressed by Nīlakaṇṭha. He instructs as follows, in this connection:

The eclipses illustrated in the *Siddhāntadīpikā* might be computed (in all their details) by the investigator. Otherwise, other eclipses handed down by the tradition of one's own school might be computed. In the light of these and of the eclipses actually observed by oneself, future eclipses should be computed and forecast. Or, eclipses occurring at other parts of the country should be computed using the longitude, latitude, etc., of that part of the country and, with that as basis, the true Sun, Moon and Node, (at the relevant times) ascertained. Then, from the Sun, Moon and Node, ascertained (as

above), past and future eclipses at one's own place shall be computed, using the latitude and longitude of one's own place[52].

Nīlakaṇṭha follows up the above instruction with the details of a number of eclipses observed and recorded by Parameśvara, his grand-teacher, in the *Siddhāntadīpikā*[53] (32-35).

At the close of enumeration of the eclipses, Nīlakaṇṭha indicates the procedure for verification. He says:

The computation of the Mean planets having been enunciated by Parameśvara later than Śrīpati, the former will tally (better) with observation. Therefore, in the case of the eclipses enumerated by Parameśvara, those observed by me (Nīlakaṇṭha) and others that might be mentioned hereafter, the Mean Sun etc., shall be computed as directed by Parameśvara. Their True positions shall (however) be computed according to Śrīpati's method. The eclipses should then be computed duly making use of the methods derived from the rationales enunciated by me (in my *Bhāṣya* on the *Kālakriyā* and *Golapāda*s (of the *Āryabhaṭīya*)[54].

Nīlakaṇṭha then illustrates his instructions by means of a practical example (pp.35-36). The precession of the equinoxes which has also a bearing on the results is also touched upon here (pp.36-37).

h. True Motion, Position, etc., of planets

While continuing the previous computation, the rationale underlying the determination of the true motion of a planet at a particular place at a particular moment is explained in detail. Examples are also worked out for true Sun and Moon to illustrate the rationale of the *dvitīya-sphuṭa* of planets (pp.37-41). Another topic treated in a similar manner relates to the correction of the periphery of the *manda* epicycle by means of the *natajyā* (Rsine zenith distance) (pp.42-44). Other corrections to be applied to get the true planet, such as *ayanacalana* (precession of the equinoxes), *prāṇakālāntara* and *caradala* (half ascensional difference), are also set out in turn with the tables of the Rsines of the latter two and the method of application thereof (pp.44-49).

i. Relation of the sides and hypotenuse

After explaining the rationale involved in determining the height of a lamp-post by means of two gnomons (pp.49-50), Nīlakaṇṭha rationalizes, from fundamentals, the graphical proofs for the relation between the sides and hypotenuse of a right angled triangle – the Pythagoras theorem. The discussion is, however, incomplete, the manuscripts having a long break here (p.50).

j. Reduction of angular distances

Errors in results arise on account of the observer being stationed on the Earth's surface, while the basic measures relate to the centre of the Earth sphere or to the horizontal plane thereof. These are corrected by reducing the basic angular distances to the *dṛggola* ('visible celestial sphere'). Nīlakaṇṭha rationalizes this type of correction with reference to the orb of the Moon (pp.51-52).

Conversely, in the case of the gnomonic shadow, it is the measure on the *dṛggola* that needs to be reduced to that of the *bhagola* ('sphere of the zodiac'). The rationalization of the processes involved forms the last section in the present manuscript, which breaks off half-way during the middle of the said discussion (pp.53-55).

The only available manuscript of *Jyotirmīmāṃsā* is incomplete, its beginning and end being lost, and some gaps occur in the middle. However, the nature of the work and the manner in which its author Nīlakaṇṭha is presenting it could give one some idea of the portions lost.

k. The beginning of the work

In the original palmleaf manuscript,[55] the *Jyotirmīmāṃsā* is inscribed in continuation of another work of Nīlakaṇṭha, being the *Siddhāntadarpaṇa* with auto-commentary. In this manuscript, the final portion of *Siddhāntadarpaṇa* [56] along with the initial portion of *Jyotirmīmāṃsā* is lost, the lost in the latter work being presumably, not more than one folio.

A cue for the missing portion could be had from the tenor of the first section of the work (see below, pp.1-2), where it is established

that appropriate corrections based on observation should be applied periodically to astronomical parameters etc., so that they might give results which would tally with observation. The available manuscript commences with arguments based on eclipses supporting the said thesis, introduced with the words *atha grahaṇam*. It may be presumed reasonably that the author must have, earlier to this, presented similar arguments to the same purpose, based on other visible astronomical phenomena like the setting of the planets, the heliacal rising of the signs, gnomonic shadow etc., with the introductory words, *atha mauḍhyam, atha lagnam, atha chāyā* etc.

Besides the above, the lost portion should have contained one or more introductory verses through which the author would have uttered the usual invocation to his favorite deity and stated his purpose in composing the work.

l. Missing portions in the middle

Apart from the minor gaps, with which the manuscript abounds and which have mostly been gilled tentatively, there occur in the manuscript at least three major omissions which could, obviously, not be filled and, which therefore, mar the continuity of reading, The first of these occurs towards the end (?) of section 14, on 'Astronomical corrections through eclipses', where an illustrative eclipse was being computed (p.37). The second omission occurs towards the beginning of section 19, on the 'Height of a lamp post by means of two gnomons', the close of the previous section also having been lost (p.49). A portion towards the close of section 20 and beginning of section 21 is also lost.

m. Omission at the end

The extant manuscript breaks off abruptly towards the middle of section 22, being a discussion on the reduction of the minutes of arc of the visible celestial sphere (*dṛggola*) to those of the zodical sphere (*bhagola*). There is no possibility of knowing, with exactitude, what more Nīlakaṇṭha had intended to include in the work. The comprehensive title *Jyotirmīmāṃsā* given to the work and of

Nīlakaṇṭha's apparent intention (1) to demonstrate that the discipline of *Jyotiṣa* is based on observation, and (2) to study the comparative accuracy of the different systems, would induce one to presume that the work contained all that is conveyed by the title and covered by the said intentions. Thus, after completeing the topic at hand viz., *Grahasphuṭa* (True positions of planets), Nīlakaṇṭha should have taken up for investigation other subjects like *Lagna* (Rising point of the ecliptic), *Chāyā* and *Viparītacchāyā* (Shadow and Inverse shadow), and *Candraśṛṅgonnati* (Elevation of the Moon's horns).

The school of Nīlakaṇṭha

An important work based on *Tantrasaṅgraha* is the *Yuktibhāṣā*, 'Rationale in the Malayalam language', whose sole aim is to rationalize the theories involved in the constants and computations occurring in the *Tantrasaṅgraha*. Thus, after the benedictory verses, the work commences with the statement:

aviṭe naṭe tantrasaṅgrahatte anusariccu grahagatiyinkal upayogamuḷḷa gaṇitaṅṅale muzhuvanāyi colluvān tuṭaṅṅunneṭattu...

'Here, commencing an elucidation in full of the rationales of planetary computations according to the *Tantrasaṅgraha* ...'

The work finds its first reference in modern writings in an article by C.M. Whish in 1835, where it is referred to for verifying the date of the author of *Tantrasaṅgraha*. Whish had contemplated "a farther (sic) account of the *Yuktibhāṣā*... will be given in a separate paper," which, however, does not appear to have been written or published.

Yuktibhāṣā, known also as *Gaṇitanyāyasaṅgraha*, 'Compendium of astronomical rationale', has been a popular text in Kerala for more than four hundred years, since its composition towards AD. 1530. Several manuscripts of the work are known. However, since the work is couched in Malayalam language, which is spoken only in Kerala it has remained practically, beyond the purview of scholars who did not know the language, in spite of its having been published. And the few articles on this important work relate to only certain individual

topics treated herein. It is therefore necessary that a critical appraisal of the nature and contents of this work as a whole is made, so as to enable scholars take up the same for further study.

The introductory verses of the *Yuktibhāṣā* do not mention the name of its author, nor do its manuscripts indicate his name at their closing colophons. There are, however, evidences which point to the correct name of the author of *Yuktibhāṣā* as Jyeṣṭhadeva and his date to be AD 1500-1610.[57]

The astronomical chronology (*granthavari*) in Kerala continued uninterrupted. The post colophonic statement in the Malayalam language found in an old palmleaf manuscript of a Malayalam commentary on *Sūryasiddhāntha* preserved in the Oriental Institute, Baroda, Ms. No, 9886, proves the fact.[58]

Conclusion

The account of Nīlakaṇṭha Somayāji and his work as depicted above is a pointer not only to the work in the field of astronomy done by him, but also to the development of this discipline in this period in India.

FOOTNOTES:

1. *Āryabhaṭīya*, *Trivandrum Sanskrit Series* (*TSS.*), No.101 (Trivandrum, 1930), pp. 1-180.
2. The term *Gārgya* is often affixed to his name in references.
3. Ed. P.R. Menon, *Tuncattu Granthāvali*, No.3 Tuncattu Karyalayam, Chittoor, 2nd edn., 1931.
4. Vide P.R. Menon in his article '*Tuncattu Ezhuttacchan*' in the Malayalam monthly, Tuncattu Ezuttacchan, 3 (1952-53) 127-35.
5. Ibid. This stotra is published in a collection of stotras in Malayalam script entitled *Stavaratnamālā*, Pt. I.

6. It may be noted that in the expression Gārgya Kerala prefixed to the author's name, the word Kerala referes to the name of his house and not to the state, as is sometimes taken.

7. *Cf.*, Vaṭakkumkūr Rājarāja Varma, *History of Sans. Lit. in Kerala*, Vol. I, Trivandrum, 1938, p.384.

8. *Āryabhaṭīya*, *TSS.* No.101 (Trivandrum, 1930), p.156.

9. *Āryabhaṭīya*, *TSS.* No. 185 (Trivandrum, 1957), p. 156.

10. *Āryabhaṭīya*, *TSS. No.* 101 Rpt. 152. (Trivandrum, 1977), p. 85.

11. *Ibid.*, p. 62.

12. *Āryabhaṭīya*, *TSS.* No. 101 (Trivandrum, 1977), p. 152.

13. *Ibid.*, p.208.

14. Even with regard to Nīlakaṇṭha's *Tantrasaṅgraha*, its introductory verse,

    हे विष्णो निहितं कृत्स्नं जगत् त्वय्येव कारणे।
    ज्योतिषां ज्योतिषे तस्मै नमो नारायणाय ते ।। has a veiled reference to his patron (Netra) Nārāyaṇa at whose instance that work too seems to have been written.

15. *Āryabhaṭīya*, *TSS.* No. 101 Rpt. (Trivandrum, 1977), p. 241.

16. Ulloor, *Kerala Sāhitya Caritram*, Vol.II, Trivandrum (1954), p.114. For a manuscript of this work see Kerala Uni. Mss. Lib., No. 3336-B.

17. *Cf.* the detailed colophon quoted above.

18. *Āryabhaṭīya*, *TSS.* No. 110 Rpt. (Trivandrum, 1977), p. 63.

19. *Siddhāntadarpaṇavyākhyā*, on verse 27, p. 30.

20. *Ibid.*, under verse 18.

21. *Āryabhaṭīyabhāṣya*, *Golapāda*, verse 3.

22. *Tantrasaṅgraha*, verse 1, refer to fn. 37.

23. These correspond in the Gregorian calendar to March 22, 1500 and March 27, 1500 A.D. respectively.

24. *Siddhāntadarpaṇa*, verse 18, refer to fn. 35.

25. *Āryabhaṭīya, TSS. No. 101 & 110 Rpt. (Trivandrum, 1977), p.* 208.

26. *Cf., Āryabhaṭīyabhāṣya, TSS.* No. 101, pp. 53, 158, where Parthasarathi Misra's *Vyāptinirṇaya* and *Advaitavivaraṇa* and *Ajitā* (commentary on *Ślokavārttika*) and its commentary *Vijayā* come in for quotation. On *Golapāda*, 50, the *Bṛhhaṭṭīkā* of Kumārila Bhaṭṭa is cited.

27. *Cf.*, quotations from the *Vākyapadīya*, *Āryabhaṭīyabhāṣya*, *TSS.*. No. 110 Rpt. (Trivandrum, 1977), p. 31.

28. See *Āyabhaṭīyabhāṣya*, *TSS.* No.101 Rpt. (Trivandrum, 1977), p. 241.

29. See commentary on *Siddhāntadarpaṇa*, verses 1,3; the *Grahaṇa* work, pp. 48, 49: and *Āryabhaṭīyabhāṣya*, *Golapāda*, verse 48, where the *Taittirīya Āraṇyaka*, *Ṛgveda*, *Parāśarasmṛti*, *Kālanirṇaya* of Sāyaṇa, Manusmṛti etc. are quoted.

30. See *Siddhāntadarpaṇa-vyākhyā*, p.1 refer to fn.35.

31. *Cf.*, *Āryabhaṭīyabhāṣya*, *TSS.*. No.110 Rpt. (Trivandrum, 1977), pp. 16.26.

32. *Cf.*, *ibid.*, p.8.

33. On this Mādhava (c.1340-1425), who was a teacher of Parameśvara, see the present writer's introduction to Mādhava's *Veṇvāroha* (Tripunnithura, Cochin, 1957), and *Sphuṭacandrāpti* (Hoshiarpur, 1973).

34. Critically ed. with Introduction by K.V. Sarma, Vishveshvaranand Institute, Punjab University, Hoshiarpur, 1970.

35. Critically ed. with the author's own commentary and Translation and two Appendices and detailed Introduction by K.V. Sarma, Vishveshvaranand Institute, Punjab University, Hoshiarpur, 1976.

36. Cr. Ed. with Auto-commentary, Introduction and two Appendices by K.V. Sarma, Vishveshvaranand Institute, Hoshiarpur, 1976.
37. Cr. Edited with two commentaries, *Yuktidīpikā* and *Laghuvivṛti*, and detailed Introduction and Indices by K.V. Sarma, Vishveshvaranand Institute, Punjab University, Hoshiarpur, 1977.
38. Ed. in 3 volumes, *TSS.*. Nos. 101, 110, 185 (Trivandrum, 1930, 1931, 1957).
39. See *Āryabhaṭīyabhāṣya*, *TSS.* No. Rpt. (Trivandrum, 1977), p. 101.
40. *Ibid.*, p. 156.
41. *Ibid.*, p.1.
42. See *Āryabhaṭīyabhāṣya*, *TSS.* No..185 (Trivandrum, 1957), p.102.
43. In Chapter IV, verse 27, *TSS.*: No. 188, p.107.
44. Cr. Ed. with Introduction and Appendices by T.S. Kuppanna Sastri and K.V. Sarma, K.S. Research Inst., Madras, 1964.
45. *Ibid.*, p. 119.
46. See *Āryabhaṭīyabhāṣya*, *TSS.* No. 185 (Trivandrum, 1957), p.149.
47. Published by the Astrological Research Institute, Bombay-25, 1950.
48. See *Āryabhaṭīyabhāṣya*, *TSS.* No. 185 (Trivandrum, 1957) pp. 132-49.
49. Ed. with Introduction and Appendices, by K.V. Sarma. Vishveshvaranand Institute, Punjab University, Hoshiarpur, 1977.
50. गणितोन्नीतस्य चन्द्रादेः देशविशेषान्वयस्य प्रत्यक्षेण संवादः, ततो निश्चितान्वयस्य परस्य गणितलिंगोपदेशः, ततस्तस्यापोपदेशावगतान्वयस्य अनुमानं, संवादः, परस्मै चोपदेशः इति सम्प्रदायाविच्छेदात् प्रामाण्यम्।
51. पञ्चसिद्धान्तस्तावत् क्वचित्काले प्रमाणमेव इत्यवगन्तव्यम्। अपि च यः सिद्धान्तो दर्शनाविसंवादी भवति सोऽन्वेषणीयः। दर्शनसंवादश्च तदानीन्तनैः परीक्षकैः ग्रहणादौ विज्ञातव्यः। ये पुनरन्यथा, प्राक्तनसिद्धान्तस्य भेदे सति यन्त्रैः परीक्ष्य ग्रहाणां भगणादिसंख्यां ज्ञात्वा अभिनवसिद्धान्तः प्रणेय इत्यर्थात्, तत् त इहलोकेऽहसनीयाः, परलोकेऽदण्डनीयाश्च इति।

52. अथ सिद्धान्तदीपिकायाम् उदाहृतानि ग्रहणानि परीक्षमाणेन गण्यानि। अन्यानि वा ऐतिह्यसिद्धानि स्ववंश्यैरुपदिष्टानि गण्यन्ताम्। एतैः प्रत्यक्षावगतैश्च अनागतग्रहणं निर्णीय वक्तव्यम्। जनपदान्तरजात् तद्देशान्तराक्षादिभिः गणयित्वा रवीन्दुतुंगपातान् विज्ञाय अतीतमनागतं स्वदेशजं ग्रहणं स्वदेशान्तराक्षाभ्यां विज्ञातैस्तैरेवं ग्रहोच्चपातैर्गणयित्वा निर्णेयाः।

53. See *Siddhāntadīpikā*, the supercommentary of Parameśvara on Govindasvāmi's *Bhāṣya* on the *Mahābhāsakrīya* of Bhāskara, under verse 5.77 (Ed. by T.S.K Sastri, Madras, 1957, pp. 329-32).

54. सिद्धान्तशेखराद्युक्तमध्यमेभ्यः परमेश्वरोक्तानां नूतनत्वाद् अस्य दृष्टिसाम्यं स्यात्। तस्मात् सिद्धन्तदीपिकोदाहृतानि अस्माभिः (नीलकण्ठेन) दृष्टानि च तत्तदवसरे वक्ष्यमाणानि परमेश्वरोक्तप्रकारेण १. अर्कादिमध्यमान्यानीय श्रीपत्युक्तप्रकारेण स्फुटीकृत्य कालक्रियागोलपादोक्ताभिः अस्माभिर्व्याख्याताभिः युक्तिभिः सिद्धैः क्रियाविशेषैश्च गण्यन्ताम्।

55. Ms. no. p. 975 of the Trivandrum Palace Collection, now deposited in the Kerala Univ. Or. Res. Inst. and Ms. Library.

56. This lost portion contains the last five verses, 28 to 32, of the work and the commentary thereon, vide, edn. of *Siddhāntadarpaṇa* with auto-commentary, by K.V. Sarma, Hoshiarpur, 1976, p.31.

57. For detailed reference on the topic see the article *Gaṇitayuktibhāṣā of Jyeṣṭhadeva.*

58. See Teacher - Pupil tradition in the article *Āryabhaṭa and Āryabhaṭan Astronomy: Antecendents, Status and Development.*

Paper 44

William Dunham (2005)

Newton

Excerpts from:
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Notes

This article looks at a few of Newton's early achievements: his generalized binomial expansion for turning certain expressions into infinite series, his technique for finding inverses of such series, and his quadrature rule for determining areas under curves. We conclude with a spectacular consequence of these: the series expansion for the sine of an angle.

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William Wade Dunham (born 1947) is an American writer who was originally trained in topology but became interested in the history of mathematics and specializes in Leonhard Euler. He has received several awards for writing and teaching on this subject. (Wikipedia)

Source': Princeton University

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CHAPTER 1

Newton

Isaac Newton

Isaac Newton (1642–1727) stands as a seminal figure not just in mathematics but in all of Western intellectual history. He was born into a world where science had yet to establish a clear supremacy over medieval superstition. By the time of his death, the Age of Reason was in full bloom. This remarkable transition was due in no small part to his own contributions.

For mathematicians, Isaac Newton is revered as the creator of calculus, or, to use his name for it, of "fluxions." Its origin dates to the mid-1660s when he was a young scholar at Trinity College, Cambridge. There he had absorbed the work of such predecessors as René Descartes (1596–1650), John Wallis (1616–1703), and Trinity's own Isaac Barrow (1630–1677), but he soon found himself moving into uncharted territory. During the next few years, a period his biographer Richard Westfall characterized as one of "incandescent activity," Newton changed forever the mathematical landscape [1]. By 1669, Barrow himself was describing his colleague as

"a fellow of our College and very young . . . but of an extraordinary genius and proficiency" [2].

In this chapter, we look at a few of Newton's early achievements: his generalized binomial expansion for turning certain expressions into infinite series, his technique for finding inverses of such series, and his quadrature rule for determining areas under curves. We conclude with a spectacular consequence of these: the series expansion for the sine of an angle. Newton's account of the binomial expansion appears in his *epistola prior*, a letter he sent to Leibniz in the summer of 1676 long after he had done the original work. The other discussions come from Newton's 1669 treatise *De analysi per aequationes numero terminorum infinitas*, usually called simply the *De analysi*.

Although this chapter is restricted to Newton's early work, we note that "early" Newton tends to surpass the mature work of just about anyone else.

Generalized Binomial Expansion

By 1665, Isaac Newton had found a simple way to expand—his word was "reduce"—binomial expressions into series. For him, such reductions would be a means of recasting binomials in alternate form as well as an entryway into the method of fluxions. This theorem was the starting point for much of Newton's mathematical innovation.

As described in the *epistola prior*, the issue at hand was to reduce the binomial $(P + PQ)^{m/n}$ and to do so whether m/n "is integral or (so to speak) fractional, whether positive or negative" [3]. This in itself was a bold idea for a time when exponents were sufficiently unfamiliar that they had first to be explained, as Newton did by stressing that "instead of $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[3]{a^5}$, etc. I write $a^{1/2}$, $a^{1/3}$, $a^{5/3}$, and instead of $1/a$, $1/aa$, $1/a^3$, I write a^{-1}, a^{-2}, a^{-3}" [4]. Apparently readers of the day needed a gentle reminder.

Newton discovered a pattern for expanding not only elementary binomials like $(1 + x)^5$ but more sophisticated ones like $\dfrac{1}{\sqrt[3]{(1+x)^5}} = (1 + x)^{-5/3}$. The reduction, as Newton explained to Leibniz, obeyed the rule

$$(P + PQ)^{m/n} = P^{m/n} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ + \text{etc.}, \tag{1}$$

where each of A, B, C, . . . represents the previous term, as will be illustrated below. This is his famous binomial expansion, although perhaps in an unfamiliar guise.

Newton provided the example of $\sqrt{c^2 + x^2} = [c^2 + c^2(x^2/c^2)]^{1/2}$. Here, $P = c^2$, $Q = \dfrac{x^2}{c^2}$, $m = 1$, and $n = 2$. Thus,

$$\sqrt{c^2 + x^2} = (c^2)^{1/2} + \frac{1}{2}A\frac{x^2}{c^2} - \frac{1}{4}B\frac{x^2}{c^2} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \cdots.$$

To identify A, B, C, and the rest, we recall that each is the immediately preceding term. Thus, $A = (c^2)^{1/2} = c$, giving us

$$\sqrt{c^2 + x^2} = c + \frac{x^2}{2c} - \frac{1}{4}B\frac{x^2}{c^2} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \cdots.$$

Likewise B is the previous term—i.e., $B = \dfrac{x^2}{2c}$ —so at this stage we have

$$\sqrt{c^2 + x^2} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \cdots.$$

The analogous substitutions yield $C = -\dfrac{x^4}{8c^3}$ and then $D = \dfrac{x^6}{16c^5}$. Working from left to right in this fashion, Newton arrived at

$$\sqrt{c^2 + x^2} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \cdots.$$

Obviously, the technique has a recursive flavor: one finds the coefficient of x^8 from the coefficient of x^6, which in turn requires the coefficient of x^4, and so on. Although the modern reader is probably accustomed to a "direct" statement of the binomial theorem, Newton's recursion has an undeniable appeal, for it streamlines the arithmetic when calculating a numerical coefficient from its predecessor.

For the record, it is a simple matter to replace A, B, C, . . . by their equivalent expressions in terms of P and Q, then factor the common

$P^{m/n}$ from both sides of (1), and so arrive at the result found in today's texts:

$$(1+Q)^{m/n} = 1 + \frac{m}{n}Q + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2\times 1}Q^2 + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{3\times 2\times 1}Q^3 + \cdots. \tag{2}$$

Newton likened such reductions to the conversion of square roots into infinite decimals, and he was not shy in touting the benefits of the operation. "It is a convenience attending infinite series," he wrote in 1671,

> that all kinds of complicated terms . . . may be reduced to the class of simple quantities, i.e., to an infinite series of fractions whose numerators and denominators are simple terms, which will thus be freed from those difficulties that in their original form seem'd almost insuperable. [5]

To be sure, freeing mathematics from insuperable difficulties is a worthy undertaking.

One additional example may be helpful. Consider the expansion of $\frac{1}{\sqrt{1-x^2}}$, which Newton put to good use in a result we shall discuss later in the chapter. We first write this as $(1-x^2)^{-1/2}$, identify $m=-1$, $n=2$, and $Q=-x^2$, and apply (2):

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{(-1/2)(-3/2)}{2\times 1}(-x^2)^2 \\ &\quad + \frac{(-1/2)(-3/2)(-5/2)}{3\times 2\times 1}(-x^2)^3 \\ &\quad + \frac{(-1/2)(-3/2)(-5/2)(-7/2)}{4\times 3\times 2\times 1}(-x^2)^4 + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots. \end{aligned} \tag{3}$$

Newton would "check" an expansion like (3) by *squaring* the series and examining the answer. If we do the same, restricting our attention to terms of degree no higher than x^8, we get

$$\begin{aligned}&\left[1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\frac{35}{128}x^8+\cdots\right]\\&\quad\times\left[1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\frac{35}{128}x^8+\cdots\right]\\&=1+x^2+x^4+x^6+x^8+\cdots,\end{aligned}$$

where all of the coefficients miraculously turn out to be 1 (try it!). The resulting product, of course, is an infinite geometric series with common ratio x^2 which, by the well-known formula, sums to $\frac{1}{1-x^2}$. But if the *square* of the series in (3) is $\frac{1}{1-x^2}$, we conclude that that series itself must be $\frac{1}{\sqrt{1-x^2}}$. *Voila*!

Newton regarded such calculations as compelling evidence for his general result. He asserted that the "common analysis performed by means of equations of a finite number of terms" may be extended to such infinite expressions "albeit we mortals whose reasoning powers are confined within narrow limits, can neither express nor so conceive all the terms of these equations, as to know exactly from thence the quantities we want" [6].

Inverting Series

Having described a method for reducing certain binomials to infinite series of the form $z = A + Bx + Cx^2 + Dx^3 + \cdots$, Newton next sought a way of finding the series for x in terms of z. In modern terminology, he was seeking the inverse relationship. The resulting technique involves a bit of heavy algebraic lifting, but it warrants our attention for it too will appear later on. As Newton did, we describe the inversion procedure by means of a specific example.

Beginning with the series $z = x - x^2 + x^3 - x^4 + \cdots$, we rewrite it as

$$(x - x^2 + x^3 - x^4 + \cdots) - z = 0 \tag{4}$$

and discard all powers of x greater than or equal to the quadratic. This, of course, leaves $x - z = 0$, and so the inverted series begins as $x = z$.

Newton was aware that discarding all those higher degree terms rendered the solution inexact. The *exact* answer would have the form $x = z + p$, where p is a series yet to be determined. Substituting $z + p$ for x in (4) gives

$$[(z + p) - (z + p)^2 + (z + p)^3 - (z + p)^4 + \cdots] - z = 0,$$

which we then expand and rearrange to get

$$\begin{aligned}&[-z^2 + z^3 - z^4 + z^5 - \cdots] + [1 - 2z + 3z^2 - 4z^3 + 5z^4 - \cdots]p\\ &\quad + [-1 + 3z - 6z^2 + 10z^3 - \cdots]p^2 + [1 - 4z + 10z^2 - \cdots]p^3\\ &\quad + [-1 + 5z - \cdots]p^4 + \cdots = 0.\end{aligned} \tag{5}$$

Next, jettison the quadratic, cubic, and higher degree terms in p and solve to get

$$p = \frac{z^2 - z^3 + z^4 - z^5 + \cdots}{1 - 2z + 3z^2 - 4z^3 + \cdots}.$$

Newton now did a second round of weeding, as he tossed out all but the lowest power of z in numerator and denominator. Hence p is approximately $\dfrac{z^2}{1}$, so the inverted series at this stage looks like $x = z + p = z + z^2$.

But p is not *exactly* z^2. Rather, we say $p = z^2 + q$, where q is a series to be determined. To do so, we substitute into (5) to get

$$\begin{aligned}&[-z^2 + z^3 - z^4 + z^5 - \cdots] + [1 - 2z + 3z^2 - 4z^3 + 5z^4 - \cdots](z^2 + q)\\ &\quad + [-1 + 3z - 6z^2 + 10z^3 - \cdots](z^2 + q)^2 + [1 - 4z + 10z^2 - \cdots]\\ &\quad (z^2 + q)^3 + [-1 + 5z - \cdots](z^2 + q)^4 + \cdots = 0.\end{aligned}$$

We expand and collect terms by powers of q:

$$\begin{aligned}&[-z^3 + z^4 - z^6 + \cdots] + [1 - 2z + z^2 + 2z^3 - \cdots]q\\ &\quad + [-1 + 3z - 3z^2 - 2z^3 + \cdots]q^2 + \cdots.\end{aligned} \tag{6}$$

As before, discard terms involving powers of q above the first, solve to get $q = \dfrac{z^3 - z^4 + z^6 - \cdots}{1 - 2z + z^2 + 2z^3 + \cdots}$, and then drop all but the lowest degree terms top and bottom to arrive at $q = \dfrac{z^3}{1}$. At this point, the series looks like $x = z + z^2 + q = z + z^2 + z^3$.

The process would be continued by substituting $q = z^3 + r$ into (6). Newton, who had a remarkable tolerance for algebraic monotony, seemed able to continue such calculations *ad infinitum* (almost). But eventually even he was ready to step back, examine the output, and seek a pattern. Newton put it this way: "Let it be observed here, by the bye, that when 5 or 6 terms . . . are known, they may be continued at pleasure for most part, by observing the analogy of the progression" [7].

For our example, such an examination suggests that $x = z + z^2 + z^3 + z^4 + z^5 + \cdots$ is the inverse of the series $z = x - x^2 + x^3 - x^4 + \cdots$ with which we began.

In what sense can this be trusted? After all, Newton discarded most of his terms most of the time, so what confidence remains that the answer is correct?

Again, we take comfort in the following "check." The original series $z = x - x^2 + x^3 - x^4 + \cdots$ is geometric with common ratio $-x$, and so in closed form $z = \dfrac{x}{1+x}$. Consequently, $x = \dfrac{z}{1-z}$, which we recognize to be the sum of the geometric series $z + z^2 + z^3 + z^4 + z^5 + \cdots$. This is precisely the result to which Newton's procedure had led us. Everything seems to be in working order.

The techniques encountered thus far—the generalized binomial expansion and the inversion of series—would be powerful tools in Newton's hands. There remains one last prerequisite, however, before we can truly appreciate the master at work.

Quadrature Rules from the *De Analysi*

In his *De analysi* of 1669, Newton promised to describe the method "which I had devised some considerable time ago, for measuring the quantity of curves, by means of series, infinite in the number of terms" [8]. This was not Newton's first account of his fluxional discoveries, for he had drafted an October 1666 tract along these same lines. The *De analysi* was a revision that displayed the polish of a maturing thinker. Modern scholars find it strange that the secretive Newton withheld this manuscript from all but a few lucky colleagues, and it did not appear in print until 1711, long after many of its results had been published by others. Nonetheless, the early date and illustrious authorship justify its description as "perhaps the most celebrated of all Newton's mathematical writings" [9].

The treatise began with a statement of the three rules for "the quadrature of simple curves." In the seventeenth century, *quadrature* meant determination of area, so these are just integration rules.

> Rule 1. The quadrature of simple curves: If $y = ax^{m/n}$ is the curve AD, where a is a constant and m and n are positive integers, then the area of region ABD is $\frac{an}{m+n}x^{(m+n)/n}$ (see figure 1.1).

A modern version of this would identify A as the origin, B as $(x, 0)$, and the curve as $y = at^{m/n}$. Newton's statement then becomes $\int_0^x at^{m/n}dt = \frac{ax^{(m/n)+1}}{(m/n)+1} = \frac{an}{m+n}x^{(m+n)/n}$, which is just a special case of the power rule from integral calculus.

Only at the end of the *De analysi* did Newton observe, almost as an afterthought, that "an attentive reader" would want to see a proof for Rule 1 [10]. Attentive as always, we present his argument below.

Again, let the curve be AD with $AB = x$ and $BD = y$, as shown in figure 1.2. Newton assumed that the *area ABD* beneath the curve was given by an expression z written in terms of x. The goal was to find a corresponding

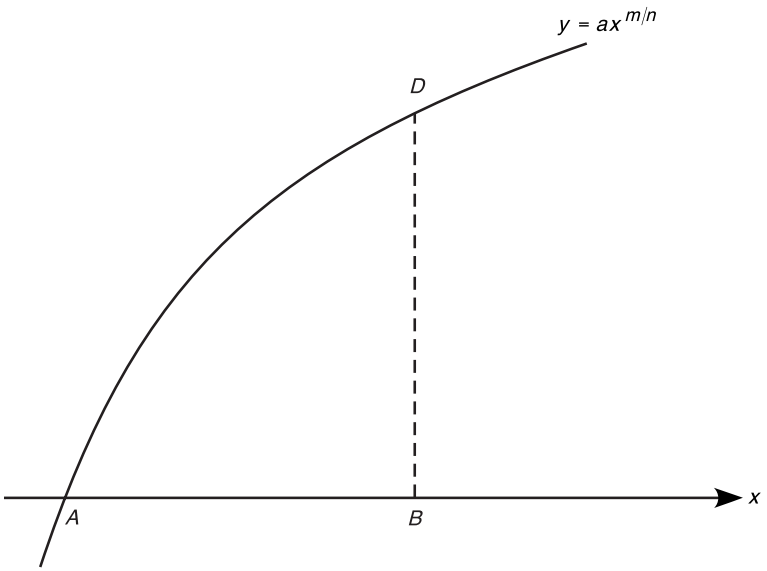


Figure 1.1

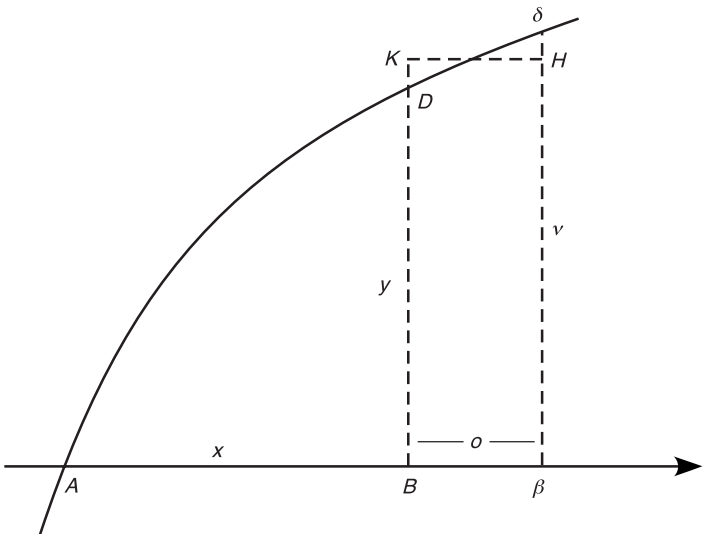


Figure 1.2

formula for y in terms of x. From a modern vantage point, he was beginning with $z = \int_0^x y(t)dt$ and seeking $y = y(x)$. His derivation blended geometry, algebra, and fluxions before ending with a few dramatic flourishes.

At the outset, Newton let β be a point on the horizontal axis a tiny distance o from B. Thus, segment $A\beta$ has length $x + o$. He let z be the area ABD, although to emphasize the functional relationship we shall take the liberty of writing $z = z(x)$. Hence, $z(x + o)$ is the area $A\beta\delta$ under the curve. Next he introduced rectangle $B\beta HK$ of height $v = BK = \beta H$, the area of which he stipulated to be *exactly* that of region $B\beta\delta D$ beneath the curve. In other words, the area of $B\beta\delta D$ was to be ov.

At this point, Newton specified that $z(x) = \dfrac{an}{m+n} x^{(m+n)/n}$ and proceeded to find the instantaneous rate of change of z. To do so, he examined the change in z divided by the change in x as the latter becomes small. For notational ease, he temporarily let $c = an/(m + n)$ and $p = m + n$ so that $z(x) = cx^{p/n}$ and

$$[z(x)]^n = c^n x^p. \tag{7}$$

Now, $z(x + o)$ is the area $A\beta\delta$, which can be decomposed into the area of ABD and that of $B\beta\delta D$. The latter, as noted, is the same as rectangular

area ov and so Newton concluded that $z(x + o) = z(x) + ov$. Substituting into (7), he got

$$[z(x) + ov]^n = [z(x + o)]^n = c^n(x + o)^p,$$

and the binomials on the left and right were expanded to

$$[z(x)]^n + n[z(x)]^{n-1}ov + \frac{n(n-1)}{2}[z(x)]^{n-2}o^2v^2 + \cdots$$
$$= c^n x^p + c^n p x^{p-1}o + c^n \frac{p(p-1)}{2}x^{p-2}o^2 + \cdots.$$

Applying (7) to cancel the leftmost terms on each side and then dividing through by o, Newton arrived at

$$n[z(x)]^{n-1}v + \frac{n(n-1)}{2}[z(x)]^{n-2}ov^2 + \cdots$$
$$= c^n p x^{p-1} + c^n \frac{p(p-1)}{2}x^{p-2}o + \cdots. \tag{8}$$

At that point, he wrote, "If we suppose $B\beta$ to be diminished infinitely and to vanish, or o to be nothing, v and y in that case will be equal, and the terms which are multiplied by o will vanish" [11]. He was asserting that, as o becomes zero, so do all terms in (8) that contain o. At the same time, v becomes equal to y, which is to say that the height BK of the rectangle in Figure 1.2 will equal the ordinate BD of the original curve. In this way, (8) transforms into

$$n[z(x)]^{n-1}y = c^n p x^{p-1}. \tag{9}$$

A modern reader is likely to respond, "Not so fast, Isaac!" When Newton divided by o, that quantity most certainly was *not* zero. A moment later, it *was* zero. There, in a nutshell, lay the rub. This zero/nonzero dichotomy would trouble analysts for the next century and then some. We shall have much more to say about this later in the book.

But Newton proceeded. In (9) he substituted for $z(x)$, c, and p and solved for

$$y = \frac{c^n p x^{p-1}}{n[z(x)]^{n-1}} = \frac{\left[\frac{an}{(m+n)}\right]^n (m+n)x^{m+n-1}}{n\left[\frac{an}{(m+n)}x^{(m+n)/n}\right]^{n-1}} = ax^{m/n}.$$

Thus, starting from his assumption that the area *ABD* is given by $z(x) = \frac{an}{m+n} x^{(m+n)/n}$. Newton had deduced that curve *AD* must satisfy the equation $y = ax^{m/n}$. He had, in essence, differentiated the integral. Then, without further justification, he stated, "Wherefore conversely, if $ax^{m/n} = y$, it shall be $\frac{an}{m+n} x^{(m+n)/n} = z$." His proof of rule 1 was finished [12].

This was a peculiar twist of logic. Having derived the equation of y from that of its area accumulator z, Newton asserted that the relationship went the other way and that the area under $y = ax^{m/n}$ is indeed $\frac{an}{m+n} x^{(m+n)/n}$. Such an argument tends to leave us with mixed feelings, for it features some gaping logical chasms. Derek Whiteside, editor of Newton's mathematical papers, aptly characterized this quadrature proof as "a brief, scarcely comprehensible appearance of fluxions" [13]. On the other hand, it is important to remember the source. Newton was writing at the very beginning of the long calculus journey. Within the context of his time, the proof was groundbreaking, and his conclusion was correct. Something rings true in Richard Westfall's observation that, "however briefly, *De analysi* did indicate the full extent and power of the fluxional method" [14].

Whatever the modern verdict, Newton was satisfied. His other two rules, for which the *De analysi* contained no proofs, were as follows:

> Rule 2. The quadrature of curves compounded of simple ones: If the value of y be made up of several such terms, the area likewise shall be made up of the areas which result from every one of the terms. [15]
>
> Rule 3. The quadrature of all other curves: But if the value of y, or any of its terms be more compounded than the foregoing, it must be reduced into more simple terms . . . and afterwards by the preceding rules you will discover the [area] of the curve sought. [16]

Newton's second rule affirmed that the integral of the sum of finitely many terms is the sum of the integrals. This he illustrated with an example or two. The third rule asserted that, when confronted with a more complicated expression, one was first to "reduce" it into an infinite series, integrate each term of the series by means of the first rule, and then sum the results.

This last was an appealing idea. More to the point, it was the final prerequisite Newton would need to derive a mathematical blockbuster: the infinite series for the sine of an angle. This great theorem from the *De analysi* will serve as the chapter's climax.

NEWTON'S DERIVATION OF THE SINE SERIES

Consider in figure 1.3 the quadrant of a circle centered at the origin and with radius 1, where as before $AB = x$ and $BD = y$. Newton's initial objective was to find an expression for the length of arc αD [17].

From D, draw DT tangent to the circle, and let BK be "the moment of the base AB." In a notation that would become standard after Newton, we let $BK = dx$. This created the "indefinitely small" right triangle DGH, whose hypotenuse DH Newton regarded as the moment of the arc αD. We write $DH = dz$, where $z = z(x)$ stands for the length of arc αD. Because all of this is occurring within the unit circle, the radian measure of $\angle \alpha AD$ is z as well.

Under this scenario, the infinitely small triangle DGH is similar to triangle DBT so that $\frac{GH}{DH} = \frac{BT}{DT}$. Moreover, radius AD is perpendicular to tangent line DT, and so altitude BD splits right triangle ADT into similar pieces: triangles DBT and ABD. It follows that $\frac{BT}{DT} = \frac{BD}{AD}$, and from these two proportions we conclude that $\frac{GH}{DH} = \frac{BD}{AD}$. With the differential notation above, this amounts to $\frac{dx}{dz} = \frac{y}{1}$, and hence $dz = \frac{dx}{y}$.

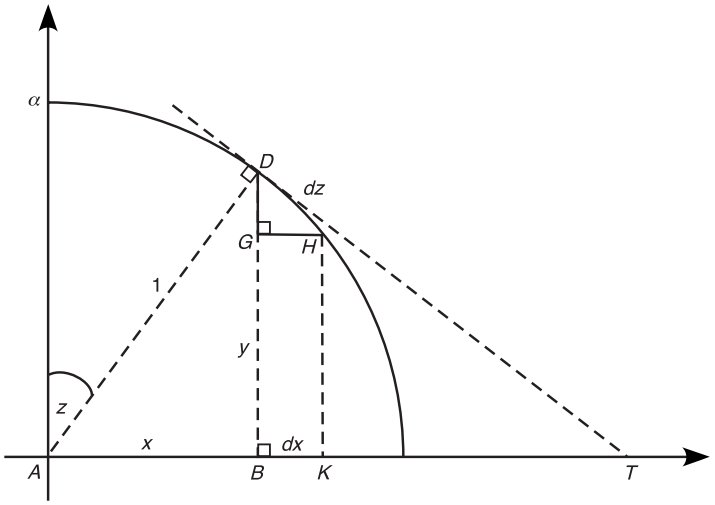


Figure 1.3

Newton's next step was to exploit the circular relationship $y = \sqrt{1-x^2}$ to conclude that $dz = \dfrac{dx}{y} = \dfrac{dx}{\sqrt{1-x^2}}$. Expanding $\dfrac{1}{\sqrt{1-x^2}}$ as in (3) led to

$$dz = \left[1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots\right]dx,$$

and so

$$z = z(x) = \int_0^x dz = \int_0^x \left[1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \frac{5}{16}t^6 + \frac{35}{128}t^8 + \cdots\right]dt.$$

Finding the quadratures of these individual powers and summing the results by Rule 3, Newton concluded that the arclength of αD was

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots. \tag{10}$$

Referring again to Figure 1.3, we see that z is not only the radian measure of $\angle \alpha AD$, but the measure of $\angle ADB$ as well. From triangle ABD, we know that $\sin z = x$ and so

$$\arcsin x = z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots.$$

Thus, beginning with the *algebraic* expression $\dfrac{1}{\sqrt{1-x^2}}$, Newton had used his generalized binomial expansion and basic integration to derive the series for arcsine, an intrinsically more complicated relationship.

But Newton had one other trick up his sleeve. Instead of a series for arclength (z) in terms of its coordinate (x), he sought to reverse the process. He wrote, "If, from the Arch αD given, the Sine AB was required, I extract the root of the equation found above" [18]. That is, Newton would apply his inversion procedure to convert the series for $z = \arcsin x$ into one for $x = \sin z$.

Following the technique described earlier, we begin with $x = z$ as the first term. To push the expansion to the next step, substitute $x = z + p$ into (10) and solve to get

$$p = \frac{-\frac{1}{6}z^3 - \frac{3}{40}z^5 - \frac{5}{112}z^7 - \cdots}{1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \cdots},$$

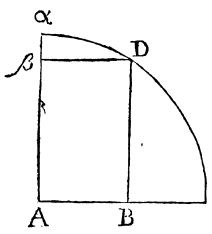
from which we retain only $p = -\frac{1}{6}z^3$. This extends the series to $x = z - \frac{1}{6}z^3$. Next introduce $p = -\frac{1}{6}z^3 + q$ and continue the inversion process, solving for

$$q = \frac{\frac{1}{120}z^5 + \frac{1}{56}z^7 - \frac{1}{72}z^8 + \cdots}{1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \cdots},$$

or simply $q = \frac{1}{120}z^5$. At this stage $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5$, and, as Newton might say, we "continue at pleasure" until discerning the pattern and writing down one of the most important series in analysis:

$$\sin z = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 - \cdots$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.$$

To find the Baſe from the Length of the Curve given.



45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being ſuppoſed that AB $= x$, αD $= z$, and Aα $= 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ &c.

46. And moreover if the Coſine Aβ were required from that Arch given, make Aβ $(= \sqrt{1 - xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c.

Newton's series for sine and cosine (1669)

For good measure, Newton included the series for $\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$ In the words of Derek Whiteside, "These series for the sine and cosine . . . here appear for the first time in a European manuscript" [19].

To us, this development seems incredibly roundabout. We now regard the sine series as a trivial consequence of Taylor's formula and differential calculus. It is so natural a procedure that we expect it was always so. But Newton, as we have seen, approached this very differently. He applied rules of integration, not of differentiation; he generated the sine series from the (to our minds) incidental series for the arcsine; and he needed his complicated inversion scheme to make it all work.

This episode reminds us that mathematics did not necessarily evolve in the manner of today's textbooks. Rather, it developed by fits and starts and odd surprises. Actually that is half the fun, for history is most intriguing when it is at once significant, beautiful, and *unexpected*.

On the subject of the unexpected, we add a word about Whiteside's qualification in the passage above. It seems that Newton was not the first to discover a series for the sine. In 1545, the Indian mathematician Nilakantha (1445–1545) described this series and credited it to his even more remote predecessor Madhava, who lived around 1400. An account of these discoveries, and of the great Indian tradition in mathematics, can be found in [20] and [21]. It is certain, however, that these results were unknown in Europe when Newton was active.

We end with two observations. First, Newton's *De analysi* is a true classic of mathematics, belonging on the bookshelf of anyone interested in how calculus came to be. It provides a glimpse of one of history's most fertile thinkers at an early stage of his intellectual development.

Second, as should be evident by now, a revolution had begun. The young Newton, with a skill and insight beyond his years, had combined infinite series and fluxional methods to push the frontiers of mathematics in new directions. It was his contemporary, James Gregory (1638–1675), who observed that the elementary methods of the past bore the same relationship to these new techniques "as dawn compares to the bright light of noon" [22]. Gregory's charming description was apt, as we see time and again in the chapters to come. And first to travel down this exciting path was Isaac Newton, truly "a man of extraordinary genius and proficiency."

Paper 45

Kim Plofker (2005)

Relations between Approximations to the Sine in Kerala Mathematics

Chapter in *Contributions to the History of Indian Mathematics* edited by Gérard G. Emch, R. Sridharan and M. D. Srinivas, Hindustan Book Agency, 2005, pp. 135–152.

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Source: University of Kentucky

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Relations between Approximations to the Sine in Kerala Mathematics

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Abstract

The mathematicians of the school of Mādhava in late medieval Kerala (South India) described various series expressions for trigonometric quantities. This paper examines the reasoning underlying two such formulas and the possible connections between them.

1 Introduction

The school of Mādhava in fifteenth- and sixteenth-century Kerala is famous for its brilliant mathematical discoveries in a number of areas, including astronomical modeling, trigonometric series, infinitesimals, iterative functions, and solution of equations. Some of their best-known work involves expressions (attributed to Mādhava himself) for the Sine and Cosine that are equivalent to what we now call Taylor series expansions. The modern name can be misleading, however, since formulas that in modern mathematics represent essentially the same concept can differ widely in their original Sanskrit context. Exploring the apparent motivations of these rules, as suggested by their accompanying "yuktis" or rationales, provides a better understanding of their relationships and their underlying concepts of trigonometric series.

Most of the results discussed here date back to the time of Mādhava himself or his student Parameśvara, but the yuktis explaining them in detail did not appear, as far as the existing texts indicate, until about a century later. It is not known how they originated: they may have been written along with the original results in texts now lost, or passed down orally from teacher to pupil, or devised by later members of the school to make sense of the enigmatic formulas, or some combination of all of these.

2 The Sine and Cosine series in the sources

We begin our examination of the Kerala power series by quoting their formulations as they appear in Sanskrit sources, translating them as literally as possible into modern mathematical notation, and showing their relation to the modern forms of these series.

2.1 The "Mādhava-Newton" power series

The following statements about the Sine (here capitalized to indicate that it is scaled to a non-unity trigonometric radius R, instead of being scaled to 1 like the modern trigonometric functions) and Versine (the "versed sine", equal to the Radius minus the Cosine) are part of a yukti laid out by one Śaṅkara in the first half of the sixteenth century. They form part of his commentary *Yuktidīpikā* ("Lamp of Rationales") on an astronomical text called *Tantrasaṅgraha* ("Epitome of Astronomical Treatises") written in 1500 by his teacher Nīlakaṇṭha, who studied under the son of Mādhava's student Parameśvara. (See [12], 184–190, in which similar statements are attributed to a manuscript of the *Tantrasaṅgraha* itself, [7], 169–173, and [9].)

The yukti is derived from a Mālayālam exposition by Jyeṣṭhadeva, another of Śaṅkara's teachers, entitled *Yuktibhāṣa* or "Vernacular [Exposition] of Rationales". This part of Śaṅkara's commentary applies to the beginning of the second chapter, where the *Tantrasaṅgraha*, like other Sanskrit astronomical treatises, introduces trigonometry. Śaṅkara is explaining ([13], 118) a way to find accurate Sine and Versine values for a given arc θ (measured in arcminutes):

> [*Yuktidīpikā* 2, 440–441:] Having multiplied the arc and the results of each [multiplication] by the square of the arc, divide by the squares of the even [numbers] together with [their] roots, multiplied by the square of the Radius, in order. Having put down the arc and the results one below another, subtract going upwards. At the end is the Sine... ([3], 57)
>
> [*Yuktidīpikā* 2, 442–444:] Having multiplied unity and the results of each [multiplication] by the square of the arc, divide by the squares of the even [numbers] minus their roots, multiplied by the square of the Radius, in order. But divide the first [instead] by twice the Radius. Having put down the results one below another, subtract going upwards. At the end is the Versine...

In the computation for the Sine discussed in verses 440–441, the initial "result" is $\theta \cdot \theta^2$, the second $\theta \cdot \theta^2 \cdot \theta^2$, the third $\theta \cdot \theta^2 \cdot \theta^2 \cdot \theta^2$, and so forth. For $n = 1, 2, 3, \ldots$, the nth result is divided by R^2 times a term depending on the corresponding even number $2n$: namely, $(2n)^2 + 2n$. Thus the sequence of results will be

$$\begin{aligned} \frac{\theta^3}{R^2(2^2+2)} &= \frac{\theta^3}{R^2 \cdot 6}, \\ \frac{\theta^5}{R^4(2^2+2)(4^2+4)} &= \frac{\theta^5}{R^4 \cdot 120}, \\ \frac{\theta^7}{R^6(2^2+2)(4^2+4)(6^2+6)} &= \frac{\theta^7}{R^6 \cdot 5040}, \end{aligned} \tag{1}$$

and so forth. Since, for the nth such result, the denominator term

$$\prod_{i=1}^{n}((2i)^2 + 2i) = \prod_{i=1}^{n}(2i(2i+1)) = (2n+1)!, \tag{2}$$

it follows that Śaṅkara's sequence of recursive subtractions of terms from the next higher term will give

$$\operatorname{Sin}\theta = \theta - \left(\frac{\theta^3}{R^2 \cdot 3!} - \left(\frac{\theta^5}{R^4 \cdot 5!} - \left(\frac{\theta^7}{R^6 \cdot 7!} - \ldots\right)\right)\right). \tag{3}$$

The Versine computation in the subsequent verses similarly gives

$$\begin{aligned}\text{Vers}\,\theta &= \frac{\theta^2}{R\cdot 2} - \left(\frac{\theta^4}{R^3\cdot 2(4^2-4)} - \left(\frac{\theta^6}{R^5\cdot 2(4^2-4)(6^2-6)} - \dots\right)\right)\\ &= \frac{\theta^2}{R\cdot 2!} - \left(\frac{\theta^4}{R^3\cdot 4!} - \left(\frac{\theta^6}{R^5\cdot 6!} - \dots\right)\right). \qquad (4)\end{aligned}$$

Since Kerala trigonometry uses the standard value for R of about $3438 \approx 360\cdot 60/2\pi$, the division of $\theta^{(\prime)}$ by R produces what we would call θ in radians. This means that the above expressions are equivalent to the following modern series normalized to $R = 1$:

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\\ 1-\cos(x) &= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots \qquad (5)\end{aligned}$$

Other verses quoted by Śaṅkara in this yukti (*Yuktidīpikā* 2, 437–439: [13], 117–118), and elsewhere ascribed to Mādhava, contain numerical values for the coefficients (up to the fifth-order term) of a slightly different form of the same series for Sine and Versine. Hence it is clear that these power series were known (and probably originally developed) by Mādhava himself.

2.2 Mādhava and the "Taylor-series-like approximation"

Near the beginning of its second chapter, the *Tantrasaṅgraha* states a rule for determining the Sine of some arbitrary arc θ by means of Sines and Cosines from a Sine-table. More detailed analyses of this rule, and of an elaboration of it recorded by Mādhava's student Parameśvara, are given in [4], [5], and [10]. Briefly, choosing from the Sine-table the tabulated arc α closest to θ, and knowing the tabulated Sine and Cosine of α, we manipulate them as follows:

> [*Tantrasaṅgraha* 2, 10cd–12ab:] Having set down the two composite Sines [i.e., the tabulated Sine and Cosine of the tabulated arc] closest to the [arbitrary] arc whose Sine and Cosine are sought, one should compute the arc [of] deficiency

> or excess [depending on whether the desired Sine or Cosine is less or greater than the tabulated one].
>
> And set down as a divisor 13751 divided by twice the arcminutes of that [difference arc], for the purpose of the mutual correction of those [quantities]. Having first divided [either] one [of the tabulated Sine or Cosine by that divisor], add or subtract [the result] with respect to the other, according as the [difference] arc is excessive or deficient.
>
> Now in the same way, apply that [corrected quantity] times two to the other: this is the correction. ([13], 112)

The difference $(\theta - \alpha)$ is found in arcminutes, and used to compute a "divisor" D:

$$D = \frac{13751}{2(\theta - \alpha)} = \frac{2 \cdot 3437;45}{(\theta - \alpha)} \approx \frac{2R}{(\theta - \alpha)}. \tag{6}$$

The Sine and Cosine of α are "mutually corrected" after being divided by D, to give a first approximation to the Sine and Cosine of θ:

$$\operatorname{Sin}\theta \approx \operatorname{Sin}\alpha + \frac{\operatorname{Cos}\alpha}{D}, \qquad \operatorname{Cos}\theta \approx \operatorname{Cos}\alpha - \frac{\operatorname{Sin}\alpha}{D}. \tag{7}$$

The sign of the correction term depends on whether the function of θ is greater or less than that of α. These approximate results are then doubled, divided by D, and applied in another "mutual correction":

$$\begin{aligned} \operatorname{Sin}\theta &\approx \operatorname{Sin}\alpha + \left(\operatorname{Cos}\alpha - \frac{\operatorname{Sin}\alpha}{D}\right)\frac{2}{D}, \\ \operatorname{Cos}\theta &\approx \operatorname{Cos}\alpha - \left(\operatorname{Sin}\alpha + \frac{\operatorname{Cos}\alpha}{D}\right)\frac{2}{D}. \end{aligned} \tag{8}$$

Recalling that $D \approx 2R/(\theta - \alpha)$, and again normalizing to $R = 1$ as for modern trigonometric functions, we get equivalent expressions with x and a in the place of θ and α:

$$\begin{aligned} \sin(x) &\approx \sin(a) + \cos(a) \cdot (x - a) - \frac{\sin(a)}{2} \cdot (x - a)^2, \\ \cos(x) &\approx \cos(a) - \sin(a) \cdot (x - a) - \frac{\cos(a)}{2} \cdot (x - a)^2. \end{aligned} \tag{9}$$

Nīlakaṇṭha describes this method too as "spoken by Mādhava" ([13], 120).

2.3 The common equivalent in the modern Taylor series

Both sets of Mādhava's Sine and Cosine rules, as represented in equations (5) and (9), can be derived from the general Taylor polynomial

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \quad (10)$$

where $f(x)$ is the sine or cosine function whose value at x is sought. Equation (9) gives the first three terms of the Taylor series, while equation (5) is equivalent to the corresponding Maclaurin series where $a = 0$.

The Taylor and Maclaurin expansions *per se*, of course, are theoretically and historically very closely intertwined. They emerged in the seventeenth and eighteenth centuries, in forms very similar to their modern representation, as part of the early calculus toolkit for dealing with arbitrary functions in the form of polynomials. The Maclaurin series was from the beginning considered as a special case of the Taylor series.

3 The series as explained in their yuktis

It is clear that the context in which Mādhava developed his "mutual corrections" of the Sine and Cosine or Sine and Versine was very different from the generalized techniques of successive differentiation employed by Taylor, Maclaurin, and their predecessors. What was the relation, if any, between Mādhava's two methods as perceived by the members of his school?

3.1 Correcting Sines by Versines

Śaṅkara's exposition culminating in the "Mādhava-Newton" series is detailed and intricate, comprising nearly a hundred Sanskrit verses ([13], 109–118). We can only touch upon its highlights here: more thorough summaries (though mostly without translations) of Jyeṣṭhadeva's rationale, from which Śaṅkara derived this explanation, may be found in [11], [7], [9], [12], and [14].

The foundation of the yukti is its division of a given arc into n unit arcs $\Delta\theta$. Each unit arc is bisected by a radius. For each successive arc

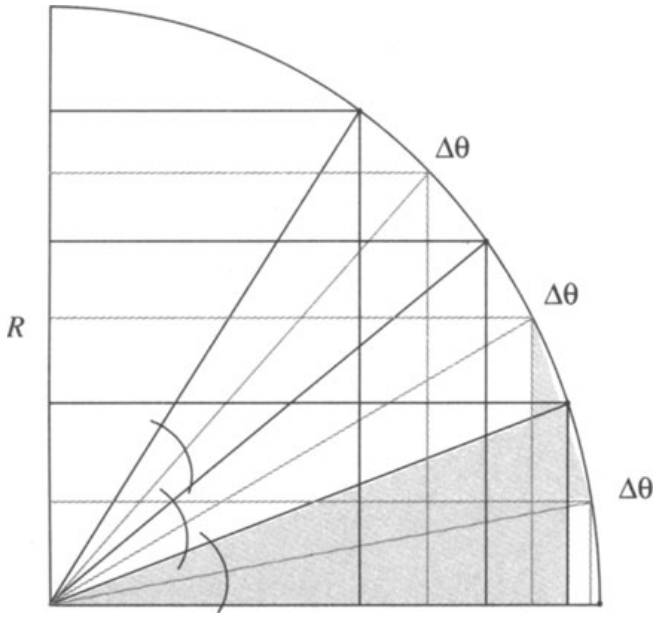

Figure 1: The quadrant with radius R, divided into unit arcs $\Delta\theta$

$\theta_i = i\Delta\theta$, the Sine and Cosine line segments Sin_i and Cos_i are drawn, as are the "medial" Sines and Cosines at the bisection points, $\text{Sin}_{i.5}$ and $\text{Cos}_{i.5}$.

As Figure 1 illustrates, the similar right triangles thus produced imply that the differences $\Delta\,\text{Sin}$ and $\Delta\,\text{Cos}$ between pairs of successive Sines and Cosines can be found from the medial Cosines and Sines. Each triangle with hypotenuse R and vertical leg $\text{Sin}\,\theta_i$ is similar to the smaller triangle with hypotenuse $\text{Crd}\,\Delta\theta$ and horizontal leg $\Delta\,\text{Cos}_{i.5}$, as in the pair of shaded triangles in the figure. As Śaṅkara puts it,

> [*Yuktidīpikā* 2, 349–352:] ... The change in the difference between Sines produced from the junctions of the corresponding arcs should be [proportional] to the change in the Cosine produced from the center of the arc. [Likewise,] the difference in Sine produced from the center of the arc is known as the change in Cosine-difference at the junctions of the corresponding arc. Thus the changes in difference are mutual[ly dependent].
>
> Then divide the Sine produced to the junction of the first arc,

> multiplied by the Chord, by the Radius: [it is] the difference of the Cosines in the centers of those arcs.

In our notation,

$$\Delta \operatorname{Cos}_{i+1} = \operatorname{Sin}_{i.5} \cdot \frac{\operatorname{Crd}}{R}, \quad \Delta \operatorname{Sin}_{i+1} = \operatorname{Cos}_{i.5} \cdot \frac{\operatorname{Crd}}{R};$$
$$\Delta \operatorname{Cos}_{i.5} = \operatorname{Sin}_{i} \cdot \frac{\operatorname{Crd}}{R}, \quad \Delta \operatorname{Sin}_{i.5} = \operatorname{Cos}_{i} \cdot \frac{\operatorname{Crd}}{R}. \tag{11}$$

For any consecutive pair of, e.g., Sine-differences, their "second difference" or "difference of the differences" $\Delta\Delta \operatorname{Sin}$ also depends on these quantities:

> [*Yuktidīpikā* 2, 352–354:] ... Having multiplied that [Cosine-difference, $\Delta \operatorname{Cos}_{1.5}$] by the Chord, divide again by the Radius. [That] should be the difference of the differences of the Sines produced to the junctions of the first and second [arcs]. Therefore, divide the first Sine, multiplied by the square of the Chord, by the square of the Radius. [The] quotient is the difference of the differences of the first and second [Sines].
>
> [*Yuktidīpikā* 2, 358–359:] ... In the same way, the division by the square of the Radius of every Sine multiplied by the square of the Chord should be similarly the difference of [its own difference and] the difference after that.

That is to say, in terms of the relations from equation 11,

$$\Delta\Delta \operatorname{Sin}_i = \Delta \operatorname{Sin}_i - \Delta \operatorname{Sin}_{i+1} = (\operatorname{Cos}_{i.5-1} - \operatorname{Cos}_{i.5}) \cdot \frac{\operatorname{Crd}}{R} = \Delta \operatorname{Cos}_{i.5} \cdot \frac{\operatorname{Crd}}{R}$$
$$= \operatorname{Sin}_i \cdot \frac{\operatorname{Crd}^2}{R^2}. \tag{12}$$

Consequently, any desired Sine-difference $\Delta \operatorname{Sin}_{i+1}$ (for $i > 0$) can be expressed in terms of the previous Sine-difference and the second-difference between them. But the previous Sine-difference itself can be expressed in terms of another second-difference, and so on, until we are left with a sum of second-differences. And each second-difference, we recall from equation (12), is dependent upon the corresponding Sine, as Śaṅkara goes on to state:

[*Yuktidīpikā* 2, 361–363:] However many differences of [Sine-]differences there are, when the sum of that many, beginning with the first, is subtracted from the first difference, the desired difference is obtained. When it is desired to obtain the total of the differences of [Sine-]differences, [the quotient] from the sum of Sines multiplied by the square of the Chord, divided by the square of the Radius, should be the sum of the differences of the differences.

$$\begin{aligned}
\Delta \operatorname{Sin}_{i+1} &= \Delta \operatorname{Sin}_i - \Delta\Delta \operatorname{Sin}_i \\
&= \Delta \operatorname{Sin}_{i-1} - \Delta\Delta \operatorname{Sin}_{i-1} - \Delta\Delta \operatorname{Sin}_i \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
&= \Delta \operatorname{Sin}_1 - \Delta\Delta \operatorname{Sin}_1 - \ldots - \Delta\Delta \operatorname{Sin}_i \\
&= \Delta \operatorname{Sin}_1 - \sum_{k=1}^{i} \Delta\Delta \operatorname{Sin}_k \\
&= \Delta \operatorname{Sin}_1 - \sum_{k=1}^{i} \operatorname{Sin}_k \cdot \frac{\operatorname{Crd}^2}{R^2}. \qquad (13)
\end{aligned}$$

From equation (13) we can see that if we were to add up i consecutive Sine-differences, the result would involve a sum of sums of Sines (or, alternatively, of Cosine-differences, which are equal to Versine-differences). But the sum of i consecutive Sine-differences is also just the ith Sine itself:

[*Yuktidīpikā* 2, 367–371:] Therefore, many sums of Sines one below another, ending with the sum of the first and second, are to be made here successively. [The quotient] from the sum of all those [sums], multiplied by the square of the Chord, divided by the square of the Radius—which is the result produced from the sums of the Sines—is equal to the quotient [from division] by the Radius of the sum of [the successive sums of] the Versine-differences, times the Chord.

However many differences are considered in the case of [any] desired arc, subtract that [sum of sums] from the first difference times that number of differences. That should be

> the Sine of the desired arc in the form of a sum of Sines of [arc-]portions.

$$\begin{aligned}\Delta \mathrm{Sin}_1 + \Delta \mathrm{Sin}_2 + \ldots + \Delta \mathrm{Sin}_i &= i\Delta \mathrm{Sin}_1 - \sum_{k=1}^{i-1}\sum_{j=1}^{k} \mathrm{Sin}_j \cdot \frac{\mathrm{Crd}^2}{R^2} \\ &= i\Delta \mathrm{Sin}_1 - \sum_{k=1}^{i-1}\sum_{j=1}^{k} \Delta \mathrm{Vers}_{j.5} \cdot \frac{\mathrm{Crd}}{R} \\ &= \mathrm{Sin}_i \,. \end{aligned} \tag{14}$$

Śaṅkara now simplifies the above expression by taking $\Delta \mathrm{Sin}_1$ to be approximately equal to the unit arc $\Delta\theta$:

> [*Yuktidīpikā* 2, 375–376:] Because of the uniform smallness of the arc-portions, the first Sine-difference is assumed [equal to the unit arc]. That [$\Delta \mathrm{Sin}_1$] multiplied by the number of differences is the amount of the desired arc. Therefore, subtract from the desired arc the result produced as stated. The desired Sine remains there; it is [made] accurate by the rule to be stated [below].

So we can rewrite equation (14) as

$$\mathrm{Sin}_i \approx \theta_i - \sum_{k=1}^{i-1}\sum_{j=1}^{k} \mathrm{Sin}_j \cdot \frac{\mathrm{Crd}^2}{R^2} = \theta_i - \sum_{k=1}^{i-1}\sum_{j=1}^{k} \Delta \mathrm{Vers}_{j.5} \cdot \frac{\mathrm{Crd}}{R}. \tag{15}$$

The process of "making the desired Sine accurate" commences with several more simplifying assumptions:

> [*Yuktidīpikā* 2, 379–382:] ...Whatever is the sum-of-sums of the arcs—owing to ignorance of [the values of] the Sines—is to be considered the sum [of the sums] of the Sines. But in that case, the last Sine should be the desired arc... Because of the smallness [of the unit], those arc-portions are considered [to be composed] with unity [i.e., as integers]. So the integers in the desired arc are equal to that [arc]. Therefore

> the sum of the Sines is assumed from the sum of the numbers having one as their first term and common difference. That, multiplied by the Chord [between] the arc-junctures, is divided by the Radius. The quotient should be the sum of the differences of the Cosines drawn to the centers of those arcs.
>
> [*Yuktidīpikā* 2, 384–385:] The other sum of Cosine-differences, [those] produced to the arc-junctures, [is] the Versine.[But] the two are approximately equal, considering the minuteness of the arc-division. On account of considering the arc-units [equal] to unity because of [their] minuteness, the Chord too is equal to that [unity], which makes no difference in the multiplier.

Śaṅkara points out, quite rightly, that we cannot use a sum of $i-1$ sums of unknown Sines to compute an equally unknown ith Sine. He proposes instead taking the sum of i sums of the successive arcs, which (if we take $\Delta\theta \approx \mathrm{Crd}\,\Delta\theta \approx 1$) is just a sum-of-sums of successive integers, whose values are of course known. We will also assume that a sum of medial Cosine- (or Versine-) differences, which really equals a medial Versine, is roughly equal to the adjacent non-medial Versine. This gives us the following modified expression for Sin_i:

$$\mathrm{Sin}_i \approx i - \sum_{k=1}^{i}\sum_{j=1}^{k} j\cdot\frac{1}{R^2} \approx i - \sum_{k=1}^{i}\sum_{j=1}^{k} \Delta\,\mathrm{Vers}_j\cdot\frac{1}{R} \approx i - \sum_{k=1}^{i}\mathrm{Vers}_k\cdot\frac{1}{R}. \quad (16)$$

A brief detour then provides a general expression for sums of successive integers and their successive sums:

> [*Yuktidīpikā* 2, 386–387:] Whatever is the product of however many numbers beginning with the first-term and increasing [successively] by one, that [product] is divided by the product of that many numbers beginning with one and increasing by one. The results one after another are the sums-of-sums of those [numbers].

That is, for a sequence of "however many" (say, q) consecutive integers

beginning with any "first-term" p, Śaṅkara notes that

$$\frac{p\cdot(p+1)\cdots(p+q-1)}{1\cdot 2\cdots q}=\frac{(p+q-1)!}{(p-1)!q!}=\sum_{j_{q-1}=1}^{p}\cdots\sum_{j_2=1}^{j_3}\sum_{j_1=1}^{j_2}j_1. \quad (17)$$

Yet another simplifying assumption follows:

> [*Yuktidīpikā* 2, 389–393:] Half that square of the arc should be the sum of the [consecutive] arcs. Because half the product of the first-term and the first-term plus one is a sum: therefore, from the cube and squared-square [etc.] of the desired arc, divided by the product of numbers beginning with one and increasing by one, there are many resulting sums one after another.
>
> So in this case, the first sum of the arc-portions... should be considered the sum of the Sines. But that sum of Sines, multiplied by the Chord and divided by the Radius, is the desired Versine.

We know from the familiar rule for the sum of an arithmetic series that half the product $p(p+1)$ equals the sum of the first p integers. Śaṅkara now assumes that, since the desired arc $i\approx i+1$, the sum of the first i integers is approximately $i^2/2$, and generalizes that assumption (relying on an earlier demonstration in his commentary) to conclude that

$$\sum_{j_2=1}^{i}\sum_{j_1=1}^{j_2}j_1=\frac{i(i+1)(i+2)}{1\cdot 2\cdot 3}\approx\frac{i^3}{6}\approx\sum_{k=1}^{i}\frac{k^2}{2},$$
$$\sum_{j_3=1}^{i}\sum_{j_2=1}^{j_3}\sum_{j_1=1}^{j_2}j_1=\frac{i(i+1)(i+2)(i+3)}{1\cdot 2\cdot 3\cdot 4}\approx\frac{i^4}{24}\approx\sum_{k=1}^{i}\frac{k^3}{6}, \quad (18)$$

etc. From this approximation and from equation (16), we can assume that

$$\sum_{k=1}^{i}k\cdot\frac{1}{R}\approx\mathrm{Vers}_i\approx\frac{i^2}{2R}. \quad (19)$$

We can also make a substitution for the double sum of integers in equation (16):

> [*Yuktidīpikā* 2, 401–403:] ...But the difference of the desired Sine and [its] arc is from the sum of the sum of the Sines. The square of the desired arc should be divided by the Radius; thence is [found] the Versine. But thence whatever [results] from the cube of the same desired arc [divided] by the square of the Radius, from that the quotient with six is approximately inferred [to be] the difference of the arc and Sine...

That is,

$$i - \mathrm{Sin}_i \approx \sum_{k=1}^{i} \sum_{j=1}^{k} j \cdot \frac{1}{R^2} \approx \frac{i^3}{6} \cdot \frac{1}{R^2}. \tag{20}$$

Now Śaṅkara reminds us that this double sum of consecutive integers or arcs should really have been a double sum of Sines, so every arc in the sum needs to be corrected by the above expression for the difference between the arc and its Sine:

> [*Yuktidīpikā* 2, 406–408:] To remove the inaccuracy [resulting] from producing [the Sine and Versine expressions] from a sum of arcs [instead of Sines], in just this way one should determine the difference of the [other] Sines and [their] arcs, beginning with the next-to-last. And subtract that [difference each] from its arc: [those] are the Sines of each [arc]. Or else therefore, one should subtract the sum of the differences of the Sines and arcs from the sum of the arcs. Thence should be the sum of the Sines. From that, as before, determine the sum of the Versine-differences...
>
> [*Yuktidīpikā* 2, 417–419:] ...[W]hatever sum-of-sums is inferred from the determination of the Versine, the difference of the Sine and [its] arc is deduced from the sum-of-sums after that [one]. Thence in this case, the quotient from the product of the cube and square of the desired arc, [divided] by whatever is the product of five numbers beginning with one and increasing by one, [is] also divided by the squared-square of the Radius. The difference of the Sine and [its] arc determined in this way should be more accurate.

All the corrected consecutive terms must be re-summed to produce a more correct Versine, using the approximation for successive sums-of-sums in equation (18):

$$\text{Vers}_i \approx \sum_{k=1}^{i} \left(k - \frac{k^3}{6R^2}\right) \cdot \frac{1}{R} = \sum_{k=1}^{i} \frac{k}{R} - \sum_{k=1}^{i} \frac{k^3}{6R^3} \approx \frac{i^2}{2R} - \frac{i^4}{24 \cdot R^3}. \quad (21)$$

And we can use the same sort of term-by-term correction to modify all the Versines composing the Sine in equation (16):

$$\begin{aligned} \text{Sin}_i &\approx i - \sum_{k=1}^{i} \text{Vers}_k \cdot \frac{1}{R} \approx i - \sum_{k=1}^{i} \left(\frac{k^2}{2R} - \frac{k^4}{4!R^3}\right) \cdot \frac{1}{R} \\ &= i - \left(\sum_{k=1}^{i} \frac{k^2}{2R^2} - \sum_{k=1}^{i} \frac{k^4}{4!R^4}\right) \\ &\approx i - \frac{i^3}{3!R^2} + \frac{i^5}{5!R^4}. \end{aligned} \quad (22)$$

These recursive corrections of the Sine by the Versine and vice versa are then applied indefinitely to produce the general rules of equations (3) and (4).

3.2 Correcting Sines by Cosines

We turn now to the *Tantrasaṅgraha*'s "Taylor-series-like approximations" of equation (8) to see what we can glean of the reasoning behind them. Consider first of all that if we omit their final terms, we get expressions equivalent to the following:

$$\text{Sin}\,\theta - \text{Sin}\,\alpha \approx \text{Cos}\,\alpha \cdot \frac{2}{D}, \quad \text{Cos}\,\alpha - \text{Cos}\,\theta \approx \text{Sin}\,\alpha \cdot \frac{2}{D}, \quad (23)$$

or equivalently

$$\frac{\theta - \alpha}{2R} \approx \frac{\text{Sin}\,\theta - \text{Sin}\,\alpha}{2\,\text{Cos}\,\alpha} \approx \frac{\text{Cos}\,\alpha - \text{Cos}\,\theta}{2\,\text{Sin}\,\alpha}. \quad (24)$$

As discussed in ([10], 287–288), Śaṅkara analyzes in a different commentary[1] on the same text (*Laghuvivṛti*, or "Minor Commentary") an

[1]Śaṅkara does not comment on this approximation in the *Yuktidīpikā*. I'm indebted to the reviewer of this paper for the important observation that such a yukti is in fact provided in the *Yuktibhāṣa* of Jyeṣṭhadeva.

approximation very close to the first of these, namely

$$\frac{\theta - \alpha}{2R} \approx \frac{\operatorname{Sin}\theta - \operatorname{Sin}\alpha}{\operatorname{Cos}\alpha + \operatorname{Cos}\theta}, \tag{25}$$

as follows:

> [*Laghuvivṛti* on *Tantrasaṅgraha* 2, 14cd–15ab:] Here, where the divisor should be made from the Cosine of the medial arc, it is said [to be made] with the sum of the Cosines of both full [arcs], by assuming that that [sum] equals twice the medial Cosine. But in reality, the sum of the Cosines of the two full [arcs] is somewhat less than twice the medial Cosine. Because of the deficiency of that divisor, the result of that is somewhat too big. But actually, that is what is desired: for that result is really the Chord, which is a little less than its arc...

In other words, if we considered the slightly smaller $\operatorname{Crd}(\theta - \alpha)$ rather than its arc, it would be more appropriate to write:

$$\frac{\operatorname{Crd}(\theta - \alpha)}{2R} \approx \frac{\operatorname{Sin}\theta - \operatorname{Sin}\alpha}{2\operatorname{Cos}\left(\alpha + \dfrac{\theta - \alpha}{2}\right)}. \tag{26}$$

But this is essentially the same similar-triangle relation expressed in equation (11): it states the difference between two successive Sines in terms of a linear proportion involving the Chord of the arc-difference, the Radius, and the medial Cosine. This exact linear proportion is what validates the slightly adjusted one, using the arc instead of the Chord, in equation (24). The "Mādhava-Newton series" and the "Taylor-series-like approximation" are therefore ultimately based on the same Rule of Three derived from similar right triangles.

Śaṅkara does not explain here the reason for adding the second-order correction term, but he proposes the option of including several more of them (see [10], 292):

> [*Laghuvivṛti* on *Tantrasaṅgraha* 2, 10cd–14ab:] Although here, prior to that, the quotient from half the Cosine with that same divisor [can] be applied to the Sine—and prior

> to that, the quotient-result from a fourth part of the Sine to the [half-]Cosine, and prior to that [the result] from an eighth part of the Cosine to the [fractional] Sine and [similarly, the result] from a sixteenth part of that [Sine] to the [fractional] Cosine—yet because of the smallness of that, it is to be considered negligible.

That is, the approximations in equation (8) might be expanded (if the size of the correction made it worthwhile) to

$$\begin{aligned}
\operatorname{Sin}\theta \approx \operatorname{Sin}\alpha + \Big(\operatorname{Cos}\alpha - \Big(\operatorname{Sin}\alpha + \Big(\frac{\operatorname{Cos}\alpha}{2} - \Big(\frac{\operatorname{Sin}\alpha}{4} + \Big(\frac{\operatorname{Cos}\alpha}{8} \\
- \frac{\operatorname{Sin}\alpha}{16}\cdot\frac{1}{D}\Big)\frac{1}{D}\Big)\frac{1}{D}\Big)\frac{1}{D}\Big)\frac{1}{D}\Big)\frac{2}{D}, \\
\operatorname{Cos}\theta \approx \operatorname{Cos}\alpha - \Big(\operatorname{Sin}\alpha + \Big(\operatorname{Cos}\alpha - \Big(\frac{\operatorname{Sin}\alpha}{2} + \Big(\frac{\operatorname{Cos}\alpha}{4} - \Big(\frac{\operatorname{Sin}\alpha}{8} \\
+ \frac{\operatorname{Cos}\alpha}{16}\cdot\frac{1}{D}\Big)\frac{1}{D}\Big)\frac{1}{D}\Big)\frac{1}{D}\Big)\frac{1}{D}\Big)\frac{2}{D}.
\end{aligned} \tag{27}$$

Recalling that $D = 2R/(\theta - \alpha)$, we may rewrite these as

$$\begin{aligned}
\operatorname{Sin}\theta \approx \operatorname{Sin}\alpha + \operatorname{Cos}\alpha(\theta-\alpha) - \frac{\operatorname{Sin}\alpha(\theta-\alpha)^2}{2R^2} - \frac{\operatorname{Cos}\alpha(\theta-\alpha)^3}{8R^3} \\
+ \frac{\operatorname{Sin}\alpha(\theta-\alpha)^4}{32R^4} + \frac{\operatorname{Cos}\alpha(\theta-\alpha)^5}{128R^5} - \frac{\operatorname{Sin}\alpha(\theta-\alpha)^6}{512R^6}, \\
\operatorname{Cos}\theta \approx \operatorname{Cos}\alpha - \operatorname{Sin}\alpha(\theta-\alpha) - \frac{\operatorname{Cos}\alpha(\theta-\alpha)^2}{2R^2} + \frac{\operatorname{Sin}\alpha(\theta-\alpha)^3}{8R^3} \\
+ \frac{\operatorname{Cos}\alpha(\theta-\alpha)^4}{32R^4} - \frac{\operatorname{Sin}\alpha(\theta-\alpha)^5}{128R^5} - \frac{\operatorname{Cos}\alpha(\theta-\alpha)^6}{512R^6}.
\end{aligned} \tag{28}$$

And if we set $\alpha = 0$ in order to compare these approximations with their counterparts in equations (3) and (4), they reduce to

$$\begin{aligned}
\operatorname{Sin}\theta \approx \theta - \frac{\theta^3}{8R^2} + \frac{\theta^5}{128R^4}, \\
\operatorname{Cos}\theta \approx R - \frac{\theta^2}{2R} + \frac{\theta^4}{32R^3} - \frac{\theta^6}{512R^5}.
\end{aligned} \tag{29}$$

3.3 Conclusion: the same or different series?

These commentaries of Śaṅkara contain almost the last known treatments within Mādhava's school of these two types of approximation to the Sine and Cosine. (The anonymous source in [3] appears to be somewhat later.) As understood in this mature form by Śaṅkara, the "Mādhava-Newton" series and the "Taylor-like" approximation are clearly independent rules. The former is valued largely for its production of Sine-values without the need for Sine-tables; the latter depends on a known tabulated Sine and Cosine to determine those of a nearby arc. Even if reduced to an equivalent form as in equation (29) (and there is no evidence that the "Taylor-like" approximation was ever actually handled this way in the Indian texts), they would be incompatible, since the integer coefficients in the denominators of the former are factorials, while those of the latter are powers of 2.[2]

Yet the two series may be linked conceptually. They are both founded on the Rule of Three arising from the same pair of similar right triangles in the subdivision of the quadrant. And they are both developed, after an initial approximation from this Rule of Three, via recursive mutual corrections of Sine-terms by Cosine-terms and vice versa. The power and fecundity of these basic concepts is illustrated in the variety of the approximations derived from them. In particular, the origin of these brilliant derivations in an elementary linear proportion recalls Bhāskara's comment on mathematical foundations: "Just as this universe is pervaded by Lord Nārāyaṇa (who removes the sufferings of those who worship him and is the sole generator of this universe), with his many forms—worlds and heavens and mountains and rivers and gods and men and demons and so on—in the same way, this whole type of computation is pervaded by the [rule of] three quantities."

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Paper 46

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Algorithms in Indian Mathematics

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Algorithms in Indian Mathematics

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Abstract

Indian Mathematics is predominantly algorithmic. In fact, the very word "Algorithm" is derived from the name of Al Khwarizmi (c. 9^{th} Century) whose works played a crucial role in the transmission of Indian algorithmic procedures to the Islamic and later to the Western world. We shall discuss a few selected algorithms that are representative of the Indian mathematical tradition from the ancient *Śulbasūtrās* to the medieval texts of the Kerala School. In particular, we shall outline some of the constructions described in the *Śulbasūtras*, the algorithm for computing the cube-root given by Āryabhaṭa (c.499) and the *kuṭṭaka* and *cakravāla* algorithms for solving linear and quadratic indeterminate equations as discussed by Āryabhaṭa (c.499), Brahmagupta (c.628), Jayadeva (prior to the 11^{th} century) and Bhāskara (c.1150). We shall also discuss the efficient algorithms for accurate computation of π and the sine function due to Mādhava (c.14^{th} century) as discussed in the texts of the Kerala School of Mathematics and Astronomy.

1 Introduction

The very word *algorithm* is derived from the name of the famous ninth century Islamic mathematician Al-Khwarizmi who was greatly influenced by the Indian procedures in mathematics and wrote the famous book *Kitab al-hisab al-hindi* on the art of Hindu reckoning [1]. Indian Mathematics is predominantly algorithmic. It abounds in algorithms for quick and efficient computations with numbers, and algebraic, geometric and trigonometric quantities. However, it was understood that

rationale and proofs have to be provided for all mathematical operations and they are to be found in the commentaries [2].

In this article, we describe some representative algorithms in Indian mathematics, as they have been developed over the ages. We begin with the *Śulbasūtras*, which are adjuncts to *Vedas* and contain the procedures for constructing various types of Vedic altars, of which we discuss some interesting examples. *Āryabhaṭīya* (c.499) is one of the earliest texts which gives algorithms for finding the square, cube, square root, cube root, etc. of a number. Detailed discussions of the same are found in the commentary on *Āryabhaṭīya* by Bhāskara I (c.629). We present the algorithms for finding the square and the cube root given in *Āryabhaṭīya* and its commentary. We also describe the *Kuṭṭaka* or the pulverizer algorithm for solving linear indeterminate equations.

We next discuss the solution of quadratic indeterminate equations or *Vargaprakṛti* which was first considered by Brahmagupta in his *Brāhmasphuṭa-siddhānta* (c.628). An optimal algorithm for the same called *Cakravāla* has been discussed by Bhāskara II in his *Bījagaṇita* (C.1150). The algorithm can at least be traced back to Jayadeva who lived prior to 11th century. We shall compare the *Cakravāla* algorithm with the one discovered by Wallis and Brouncker in the 17th century and proved later by Lagrange. We then discuss the algorithms for constructing the sine tables in Indian astronomy both in their earlier versions and as given in *Tantrasaṅgraha* (c.1500) and *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somayājī, the famous Kerala astronomer-mathematician. Finally, we describe the accurate calculation of π with a suitable remainder term, due to Mādhava (14th cent.) as described in the Kerala works *Yuktidīpikā* (c.1530), a commentary on *Tantrasaṅgraha* by Śaṅkara Vāriyar and *Yuktibhāṣā* (c.1530) of Jyeṣṭhadeva.

2 Geometrical construction in *Śulbasūtras* [3]

Śulbasūtras are part of *Kalpasūtras*, which are one of the six *Vedāṅgas*. These are essentially manuals for geometrical constructions of Vedic altars. Baudhāyana, Āpastamba, Kātyāyana etc. are some of the *Śulbasūtras*. Some of the simpler procedures discussed here are for dividing a line into equal parts, drawing perpendiculars, constructing

squares and rectangles, etc. Pythagoras theorem or the 'Theorem of the square of the diagonal' plays a crucial rule in most of these constructions. This theorem was known in India, at least by the time of Baudhāyana *Śulbasūtra*, where right triangles with sides (3,4,5), (15,3,39), (7,24,25), etc., are mentioned. The earliest recorded explicit statement of the theorem is also to be found there.

> दीर्घचतुरश्रस्याक्ष्णया रज्जुः पार्श्वमानी तिर्यङ्मानी च यत् पृथग्भूते कुरुतः तमुभयं करोति ।
>
> (*Baudhāyana Śulbasūtra* 1.48)
>
> The diagonal cord of a rectangle makes both (the squares) that the vertical and horizontal sides make separately.

We give some illustrative examples of geometrical algorithms in the *Śulbasūtras* in the following.

2.1 To draw a square equal to the difference of two squares

> चतुरश्राच्चतुरश्रं निर्जिहीर्षन् यावन्निर्जिहीर्षेत् तस्य करण्या वृद्ध्रमुल्लिखेत् । वृद्ध्रस्य पार्श्वमानीं अक्ष्णया इतरत् पार्श्वम् उपसंहरेत् सा यत्र निपतेत्-तदपच्छिन्द्यात् ॥
>
> (*Āpastamba Śulbasūtra* 2.5)
>
> Wishing to deduct a square from a square one should cut off a segment by the side of the square to be removed. One of the lateral sides of the segment is drawn diagonally across to touch the other lateral side. The portion of the side beyond this point should be cut off.

$ABCD$ is the larger square. AE is the side of the smaller square. AD is drawn diagonally across with point A fixed till D touches EF at P. Considering the triangle AEP.

$$\begin{aligned} EP^2 &= AP^2 - AE^2 \\ &= AD^2 - AE^2 \end{aligned}$$

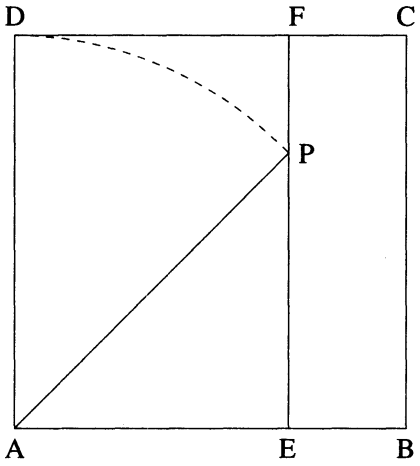

Fig.1

Hence, the square with side EP is the required answer.

2.2 To convert a rectangle into a square

दीर्घचतुरश्रं समचतुरश्रं चिकीर्षन् तिर्यङ्मान्यापच्छिद्य शेषं विभज्योभयत उपदध्यात् । खण्डमागन्तुना संपूरयेत् । तस्य निह्लास उक्तः ।

(*Āpastamba Śulbasūtra* 2.7)

Wishing to turn a rectangle into a square, one should cut off a part equal to the transverse side and the remainder should be divided into two and juxtaposed at the two sides (of the first segment). The bit (at the corner) should be filled by an imported bit. The removal of this has been explained already.

$ABCD$ is the given rectangle (see Fig.2). Consider the square AB_1C_1D with side AD. The remaining rectangle B_1BCC_1 is divided into two equal strips $B_1B_2EC_1$ and B_2BCE. The strip B_2BCE is cut off and applied to the side of the square DC_1. So the original rectangle has been converted into a square with side AD_1 with the small square C_1EC_2F unfilled at one corner. Hence,

$$\begin{aligned} \text{Area of the required square} &= \text{Area of the square } AB_2C_2D_1 \\ &- \text{Area of the square } C_1EC_2F. \end{aligned}$$

Using the previous procedure, this square is easily constructed.

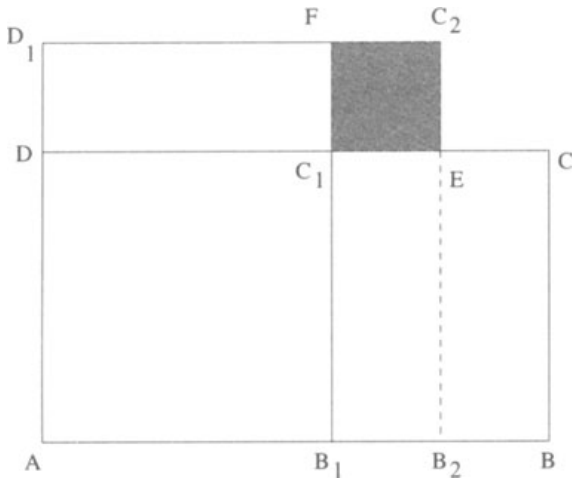

Fig.2

2.3 Combining equal squares

(a) Combining two equal squares

The rule in the *Āpastamba Śulbasūtra* (1.5) is

> **समस्य द्विकरणी ।**
>
> The diagonal of the square is the double-maker.

$ABCD$ is the given square with side a and area a^2. Then the square on the diagonal AC is the square with area $2a^2$. That is, $AC = \sqrt{2}a$. Incidentally, Fig.3 also indicates the proof.

Both *Āpastamba* (1.6) and *Baudhāyana Śulbasūtras* (1.62) give a very good approximation to the value of $\sqrt{2}$:

> **प्रमाणं तृतीयेन वर्धयेत्तच्चतुर्थेन आत्मचतुस्त्रिंशोनेन सविशेषः ।**
>
> The measure should be increased by one-third of itself, which is again increased by its one-forth and diminished by $\frac{1}{34}$ of that (second) increment. This is the *saviśeṣa.*

That is,

$$\sqrt{2} \simeq 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = \frac{577}{408} \simeq 1.414216,$$

which is correct to 5 decimal places.

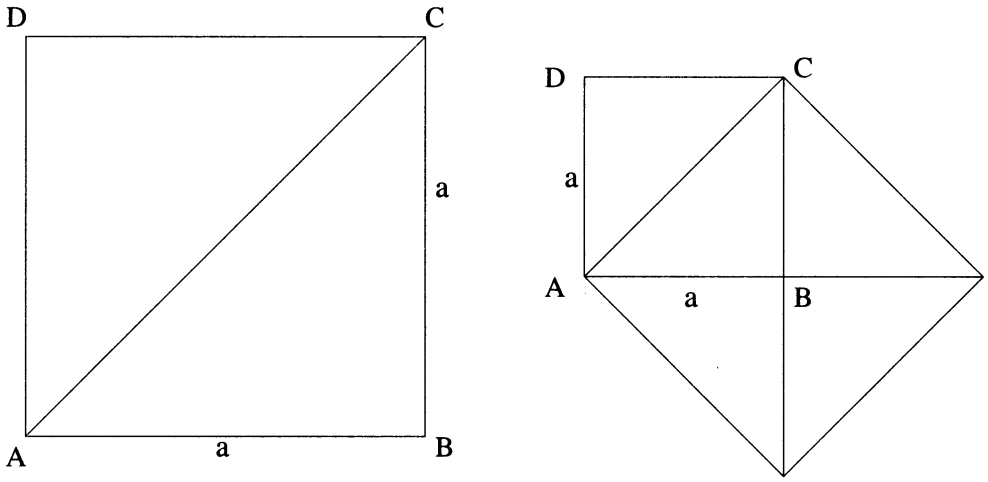

Fig.3

(b) An ingenious method for constructing a square with area n times a given square is given in *Kātyāyana-śulbasūtra* (6.7)

> यावत्प्रमाणानि समचतुरश्राण्येकीकर्तुं चिकीर्षेत् एकोनानि तानि भवन्ति तिर्यक् द्विगुणान्येकत एकाधिकानि । त्र्यस्रिर्भवति तस्येषुस्तत्करोति ॥
>
> As many sides (of equal side) as you wish to combine into one, the transverse line will be(equal to) one less than that; twice a side will be(equal to) one more than that. It will be a triangle. Its arrow(i.e., altitude) will do that.

Let the side of the original square be a. Construct an isosceles triangle ABC (see Fig.4) with base

$$BC = (n-1)a,$$

and sides

$$AB = AC = \frac{(n+1)a}{2}.$$

This is easily achieved by stretching the mid point A of a rope with length $(n+1)a$ away from BC such that the rope is taut. Then the altitude AD is the side of the square with area na^2, for,

$$\begin{aligned} AD^2 &= AB^2 - BD^2 \\ &= \left[\frac{(n+1)a}{2}\right]^2 - \left[\frac{(n-1)a}{2}\right]^2 \\ &= na^2. \end{aligned}$$

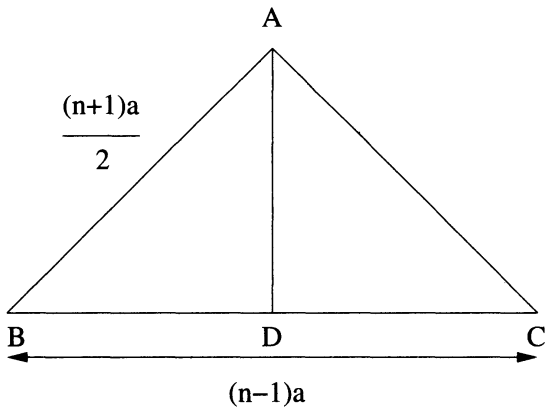

Fig.4

3 Algorithms for square and the cube root

3.1 Square

One of the earliest example of an algorithm in arithmetic is the one for squaring a number. In his *Āryabhaṭīya-bhāṣya* (c.629) Bhāskara I cites the following as an ancient rule for squaring [4]:

> अन्त्यपदस्य वर्गं कृत्वा द्विगुणं तदेवचान्त्यपदम् ।
> शेषपदैराहन्यात् उत्सार्योत्सार्य वर्गविधौ ॥

> To calculate the square of a number, the square of the last digit is to be placed over it. And the rest of the digits, doubled and multiplied by the last are to be placed over them respectively. Then, (omitting the last digit and) moving the rest by one place, the process is repeated.

Consider for example, the square of 125. The procedure is summarized below.

1 5 6 2 5	
25	$5^2 = 25$
4 20	$2^2 = 4,\ 5.2.2 = 20$
1 4 10	$1^2 = 1,\ 2.2.1 = 4,\ 5.2.1 = 10$
1 2 5	

It is noteworthy that this 'ancient rule' for squaring is already optimal. The mechanical method of squaring a n - digit number involves n^2 multiplications, whereas the above method clearly involves only $\frac{n(n+1)}{2}$ multiplications.

3.2 Cube root

An algorithm for square root is given in *Āryabhaṭīya* . Indeed, texts prior to this work also deal with square roots. However, the first clear enunciation of the cube root algorithm is found in *Āryabhaṭīya* , which we proceed to describe [5]

> अघनाद् भजेद् द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण ।
> वर्गस्त्रिपूर्वगुणितः शोध्यः प्रथमाद् घनश्च घनात् ॥
>
> (Having subtracted the greatest possible cube root from the last cube place and then having written down the cube root of the number subtracted in the line of the cube root), divide the second non-cube place (standing on the right of the last cube place) by thrice the square of the cube root (already obtained); (then) subtract from the first non-cube place (standing on the right of the second non-cube place) the square of the quotient multiplied by thrice the previous (cube root); and (then subtract) the cube (of quotient) from the cube place (standing on the right of the first non-cube place) (and write down the quotient on the right of the previous cube root in the line of the cube root, and treat this as the new cube root. Repeat the process if there are still digits on the right).

We consider the cube root of 17,71,561 as an example. Beginning from the units place, the notational places are called cube place (c), first non-cube place (n), second non-cube place(n'), cube place(c), first non-cube place(n), second non-cube place(n'), and so on.

		c	n'	n	c	n'	n	c	121
		1	7	7	1	5	6	1	(line of cube root)
Subtract 1^3		1							
Divide by 3.1^2	3)	0	7	(2					
		0	6						
			1	7					
Subtract $3.1.2^2$			1	2					
				5	1				
Subtract 2^3				0	8				
Divide by 3.12^2	432)			4	3	5	(1		
				4	3	2			
						3	6		
Subtract $3.12.1^2$						3	6		
							0	1	
Subtract 1^3								1	
								0	

The process ends and the cube root is 121. The algorithm is obviously based on the algebraic identity: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

4 *Kuṭṭaka* or 'pulveriser' algorithm for linear indeterminate equations

The subject of first order indeterminate equations was considered very important by the ancient Indian mathematicians and most of them have dealt with the *Kuṭṭaka* or pulveriser method to solve them, beginning with *Āryabhaṭīya* . They are important in astronomy also, for instance, in the calculation of *Ahargaṇa* (the number of days elapsed from a given epoch) from the mean longitudes of planets. This algorithm also plays a crucial role in the solution of the much more difficult second order indeterminate equations.

In one of its versions, the problem is to find an integer N which being divided by two given integers (a, b) will leave two given remainders (r_1, r_2). Thus we have

$$N = ax + r_1 = by + r_2.$$

Kuṭṭaka method for this equation is summarized in two verses in *Āryabhaṭīya* (*Gaṇitapāda*, 32, 33).

अधिकाग्रभागहारं छिन्द्याद्ऊनाग्रभागहारेण ।
शेषपरस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तम् ॥
अध उपरिगुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेषम् ।
अधिकाग्रच्छेदगुणं द्विच्छेदाग्रमधिकाग्रयुतम् ॥

Below we give the translation of these verses by Datta and Singh following the interpretation of Bhāskara I [6]:

> Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. The remainder (and the divisor corresponding to the smaller remainder) being mutually divided, the last residue should be multiplied by such an optional integer that the product being added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients is odd) by the difference of the remainders (will be exactly divisible by the penultimate remainder. Place the quotients of the mutual division successively one below the other in a column, below them the optional multiplier and underneath it the quotient just obtained). Any number below (i.e., the penultimate) is multiplied by the one just above it and then added to the one just below it. Divide the last number (obtained by doing so repeatedly) by the divisor corresponding to the smaller remainder; then multiply the remainder by the divisor corresponding the smaller remainder and add the greater remainder. (The result will be) the number corresponding to the two divisors.

In the following we take $r_1 > r_2$, so that a is the divisor corresponding to the greater remainder, and b the one corresponding to the smaller remainder. Let

$$c = r_1 - r_2.$$

We write down the procedure, when the number of quotients (ignoring the first one q, as is usual with Āryabhaṭa) is even.

$$
\begin{array}{llllll}
b) & a & (q & & & \\
 & \underline{bq} & & & & \\
 & r_1) & b & (q_1 & & \\
 & & \underline{r_1 q_1} & & & \\
 & & r_2) & r_1 & (q_2 & \\
 & & & \underline{r_2 q_2} & & \\
 & & & \cdot & & \\
 & & & \cdot & & \\
 & & & \cdot & & \\
 & & & & r_{2n}) & r_{2n-1} \quad (q_{2n} \\
 & & & & & \underline{r_{2n} q_{2n}} \\
 & & & & & r_{2n+1}
\end{array}
$$

Now a number t (*mati*) is chosen such that $r_{2n+1}t + c$ is divisible by r_{2n} with quotient s. Then these are set down in the form of a *vallī* (column) and the successive columns are generated:

$$
\begin{array}{llll}
q_1 & q_1 & q_1 & \cdots\cdots \quad q_1\beta_{2n-1} + \beta_{2n-2} \\
q_2 & q_2 & q_2 & \cdots\cdots \quad \beta_{2n-1} \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
q_{2n-1} & q_{2n-1} & q_{2n-1}\ \beta_1 + t = \beta_2 & \\
q_{2n} & q_{2n}t + s = \beta_1 & & \\
t & t & & \\
s & & &
\end{array}
$$

Divide $q_1\beta_{2n-1} + \beta_{2n-2}$ by b. The remainder is x and $N = ax + r_1$.

When we divide this by b, the quotient is y and the remainder is r_2. When the number of quotients (omitting q) is an odd integer $2n-1$, the number t is chosen such that $r_{2n}t - c$ is divisible by r_{2n-1}.

Example: To solve $45x + 7 = 29y$.

Here

$$a = 45,\ b = 29,\ r_1 = 7,\ r_2 = 0.$$

$$
\begin{array}{llllll}
29) & 45 & (1 & & & \\
 & \underline{29} & & & & \\
 & 16) & 29 & (1 & & \\
 & & \underline{16} & & & \\
 & & 13) & 16 & (1 & \\
 & & & \underline{13} & & \\
 & & & 3) & 13 & (4 \\
 & & & & \underline{12} & \\
 & & & & 1 &
\end{array}
$$

Here the number of quotients (omitting the first) is odd. t should be chosen such that $1 \times t - 7$ is divisible by 3. Hence t is chosen to be 10. Therefore we have,

$$
\begin{array}{llll}
1 & 1 & 1 & 92 \\
1 & 1 & 51 & 51 \\
4 & 41 & 41 & \\
10 & 10 & & \\
1 & & &
\end{array}
$$

Now, $92 = 29 \times 3 + 5$. Therefore,

$$N = 49 \times 5 + 7 = 29 \times 8.$$

Hence,

$$x = 5,\ y = 8.$$

5 *Varga-prakṛti*: Quadratic indeterminate equations

The quadratic indeterminate equation

$$x^2 - Dy^2 = 1,$$

for a non square integer D is generally referred to as Pell's equation, though the 17th century English mathematician Pell had very little to

do with either posing the problem or solving it. The problem of finding the integer solutions x, y to this equation was posed as a challenge to the European mathematicians by Fermat in 1653, for specific values of $D = 61$, 109, 149, etc. The English mathematicians Brouncker and Wallis solved the equation. Fermat is credited with proving that the equation has infinite number of solutions.

In fact, the quadratic indeterminate equations of the form

$$x^2 - Dy^2 = K,$$

known as *Varga-prakṛti* had been considered nearly thousand years earlier by Brahmagupta in his *Brāhmasphuṭa-siddhānta* (c.628). D, the given non-square integer is called "*prakṛti*", K, a given integer, is called *Kṣepa* and the integer solutions x and y are called *Jyeṣṭha-mūla* and *Kaniṣṭha-mūla* respectively. The fact that the equation has infinite number of solutions is implied by Brahmagupta's *Bhāvanā* Principle, to be described below. The equation $x^2 - Dy^2 = 1$, had been solved for all D by Jayadeva (c.11th century or earlier) as cited by Udayadivākara (c.11th century) in his commentary *Sundarī* on *Laghu-bhāskarīya* of Bhāskara I [7]. Bhāskara II has also discussed this *Cakravāla* algorithm in his *Bījagaṇita* (c.1150) [8].

The motivation for solving such equations was probably to find rational approximation for surds. From $x^2 - Dy^2 = 1$, we find

$$\left|\sqrt{D} - \frac{x}{y}\right| \leq \frac{1}{2xy},$$

so that $\sqrt{D} \approx \frac{x}{y}$, if x and y are large. For instance,

$$577^2 - 2 \times 408^2 = 1,$$

so that

$$\sqrt{2} \approx \frac{577}{408}.$$

It is noteworthy that this is the value of $\sqrt{2}$ given in *Śulbasūtras*. The *Cakravāla* algorithm makes crucial use of the *Bhāvanā* Principle of Brahmagupta which is summarised below.[1]

[1]Further details regarding the *Bhāvanā* principle may be found in the contribution

5.1 Brahmagupta's *Bhāvanā*

If

$$x_1^2 - Dy_1^2 = K_1 \qquad \text{and} \qquad x_2^2 - Dy_2^2 = K_2,$$

then

$$(x_1x_2 \pm Dy_1y_2)^2 - D(x_1y_2 \pm x_2y_1)^2 = K_1K_2.$$

So, if the solution of the *Varga-prakṛti* for the *Kṣepas* K_1 and K_1 are known, the solution for the *kṣepa* K_1K_2 can be found immediately [9].

In particular,

$$x^2 - Dy^2 = 1 \qquad \Rightarrow \qquad (x^2 + Dy^2)^2 - D(2xy)^2 = 1$$

Thus, if one solution of the equation is known, an infinite number of solutions can be found.

Brahmagupta's *Bhāvanā* not only helps in finding any number of solutions from just one solution, it also enables us to solve the $K = 1$ case, provided we know a solution for $K = -1$, or ± 2 or ± 4. We give the solutions (x, y) of $x^2 - Dy^2 = 1$ in terms of the solutions (x_1, y_1) of $x_1^2 - Dy_1^2 = K$, when K assumes any of these five values below:

$$\begin{aligned}
K = -1 : x &= x_1^2 + Dy_1^2, \quad y = 2x_1y_1. \\
K = \pm 2 : x &= \frac{(x_1^2 + Dy_1^2)}{2} \quad y = x_1y_1. \\
K = -4 : x &= (x_1^2 + 2)\left[\frac{1}{2}(x_1^2 + 1)(x_1^2 + 3) - 1\right], \\
y &= \frac{x_1y_1(x_1^2 + 1)(x_1^2 + 3)}{2}. \\
K = +4 : x &= \frac{(x_1^2 - 2)}{2}, \quad y = \frac{x_1y_1}{2}, \quad \text{if } x_1 \text{ is even}, \\
x &= x_1\frac{(x_1^2 - 3)}{2}, \quad y = \frac{y_1(x_1^2 - 1)}{2}, \quad \text{if } x_1 \text{ is odd}.
\end{aligned}$$

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of Amarthya Kumar Dutta in this volume.

5.2 *Cakravāla* algorithm for the *Varga-prakṛti* $x^2 - Dy^2 = 1$

In his *Bījagaṇita*, Bhāskarācārya gives the *Cakravāla* algorithm for solving the equation $x^2 - Dy^2 = 1$ in four verses (71 -75) [10].

ह्रस्वज्येष्ठपदक्षेपान् भाज्यप्रक्षेपभाजकान् ।
कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते ॥
गुणवर्गे प्रकृत्योनेऽथवाल्पं शेषकं यथा ।
तत्तु क्षेपहृतं क्षेपः व्यस्तः प्रकृतितश्च्युते ॥
गुणलब्धिः पदं ह्रस्वं ततो ज्येष्ठमथोऽसकृत् ।
त्यक्त्वा पूर्वपदक्षेपान् चक्रवालमिदं जगुः ॥
चतुर्द्व्येकयुतावेवं अभिन्ने भवतः पदे ।
चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना ॥

Considering the lesser root, greater root and interpolator (*Kṣepa*) as the dividend, addend and divisor respectively of pulversier) the indeterminate multiplier of it should be so taken as will make the residue of the *prakṛti* diminished by the square of that multiplier or the latter minus the *prakṛti* (as the case may be) to be the least. That residue divided by the (original) interpolator is the (new) interpolator; it should be reversed in sign in case of the subtraction from the *prakṛti*, The quotient corresponding to that value of the multiplier is the (new) lesser root, like wise is obtained the greater root. The same process should be followed putting aside (each time) the previous roots and the interpolator. This process is called *Cakravāla* (or the cyclic method). By this method, there will appear two integral roots corresponding to an equation with $\pm 1, \pm 2$ or ± 4 as interpolator. In order to derive integral roots corresponding to an equation with additive unity from those of the equation with the interpolator ± 2 or ± 4, *Bhāvanā* (should be applied).

We describe algorithm in the following. To solve

$$x^2 - Dy^2 = 1,$$

we have to consider equations of the form

$$x_i^2 - Dy_i^2 = K_i,$$

in the intermediate stages, and also an indeterminate multiplier p_i. Here y_i is the lesser root, x_i is the greater root and K_i is the *kṣepa*. We start with,

$$x_0^2 - Dy_0^2 = K_0, \text{ with } x_0 = 1,\ y_0 = 0,\ K_0 = 1,$$

and set $p_0 = 0$. Given x_i, y_i, K_i and p_i, y_{i+1} and p_{i+1} are obtained by solving the *kuṭṭaka*,

$$y_{i+1} = \frac{y_i\, p_{i+1} + x_i}{|K_i|},$$

with the extra condition that $|p_{i+1}^2 - D|$ is chosen to be minimum. Then x_{i+1} and K_{i+1} can be found from:

$$x_{i+1} = \frac{x_i\, p_{i+1} + Dy_i}{|K_i|},$$

$$K_{i+1} = \frac{p_{i+1}^2 - D}{K_i}.$$

They satisfy:

$$x_{i+1}^2 - Dy_{i+1}^2 = K_{i+1}.$$

Note that the above equation arises by doing *Bhāvanā* between

$$x_i^2 - Dy_i^2 = K_i,$$

$$p_{i+1}^2 - D.1^2 = p_{i+1}^2 - D,$$

and dividing the new x, y and K by $|K_i|, |K_i|$ and $|K_i^2|$ respectively.

This process is repeated till we get one of the values $\pm 1, \pm 2, \pm 4$ for the *Kṣepa*. If we obtain the *Kṣepa* to be 1, we have solved the problem. If the *Kṣepa* is -1 or ± 2 or ± 4, *Bhāvanā* can be used to obtain the solution for $x^2 - Dy^2 = 1$, as explained earlier.

Bhāskara II or his commentators have not outlined any proof that *Cakravāla* algorithm always leads to a solution in a finite number of steps. In 1929, A.A.Krishnaswami Ayyangar proved that the *Cakravāla* algorithm always leads to a solution in a finite number of steps [11].

He also showed that the above procedure is equivalent to the one in which the *kuṭṭaka* equation for p_{i+1} is replaced by the condition that K_i divides $p_i + p_{i+1}$, with the other conditions being the same (including minimization of $|p_{i+1}^2 - D|$).

Bhāskara himself considered the example of $D = 61$ in his *Bījagaṇita*. The successive iterated values of p_i, K_i, x_i, y_i are given in Table 1. The smallest solution $x = 1766319049$, $y = 226153980$ is reached after the 14th step, if *Cakravāla* algorithm is applied mechanically. However, just after two steps we find $K = -4$, so that the solution is found immediately using the *Bhāvanā*.

Cakravāla **algorithm for** $x^2 - 61y^2 = 1$

i	p_i	K_i	x_i	y_i
0	0	1	1	0
1	8	3	8	1
2	7	-4	39	5
3	9	-5	164	21
4	6	5	453	5
5	9	4	1523	195
6	7	-3	5639	722
7	8	-1	29718	3805
8	8	-3	469849	60158
9	7	4	2319527	296985
10	9	5	9747957	1248098
11	6	-5	26924344	3447309
12	9	-4	90520989	11590025
13	7	3	335159612	42912791
14	8	1	1766319049	226153980

Table 1

After the first step, we have

$$8^2 - 61 \times 1^2 = 3.$$

In step 2, $8 + p_2$ is divisible by 3 and the minimum value of $|p_2^2 - 61|$ is obtained when $p_2 = 7$. Then $x_2 = 39$, $y_2 = 5$, $K_2 = -4$ and we have

$$39^2 - 61 \times 5^2 = -4.$$

Since $K = -4$, we can use *Bhāvanā* to obtain

$$\begin{aligned} x &= (39^2 + 2)\left[\left(\frac{1}{2}\right)(39^2 + 1)(39^2 + 3) - 1\right] = 1766319049, \\ y &= \left(\frac{1}{2}\right)(39 \times 5)(39^2 + 1)(39^2 + 3) = 226153980. \end{aligned}$$

The Brouncker-Wallis-Euler-Lagrange algorithm for solving the equation $x^2 - Dy^2 = 1$, is based on the continued fraction expansion of $\sqrt{D}$. It can be shown to be identical to *Cakravāla* algorithm, except that the condition

$$\left|D - p_{i+1}^2\right| \text{ is minimum,}$$

is replaced by

$$D - p_{i+1}^2 \text{ is minimum and positive.}$$

The *Cakravāla* algorithm often skips some of the steps encountered in the Brouncker-Wallis-Euler-Lagrange algorithm. Table 2 gives the successive values of p_i, K_i, x_i, y_i in the Brouncker-Wallis-Euler-Lagrange algorithm for $D = 61$ where arrows indicate *Cakravāla* steps. The solution is reached after 22 steps using this algorithm, in contrast to 14 steps in the case of *Cakravāla* algorithm. If the number of steps in the *Cakravāla* algorithm and the Brouncker-Wallis algorithm are denoted by n_C and n_B respectively, then it has been shown [12] empirically that for large D,

$$\frac{n_B}{n_C} \approx 1.44.$$

Thus, the *Chakravāla* algorithm is about 44% more efficient for large D. Quadratic indeterminate equations in which ancient Indians did such pioneering work is an active field of research even today and is at the heart of computational mathematics [13].

Brouncker-Wallis-Euler-Lagrange algorithm for $x^2 - 61y^2 = 1$

	I	p_i	K_j	x_i	y_i
→	0	0	1	1	0
	1	7	-12	7	1
→	2	5	3	8	1
→	3	7	-4	39	5
	4	5	9	125	16
→	5	4	-5	164	21
→	6	6	5	453	58
	7	4	-9	1070	137
→	8	5	4	1523	195
	9	7	-3	5639	722
	10	5	12	24079	3083
→	11	7	-	29718	3805
	12	7	12	440131	56353
→	13	5	-3	469849	60158
→	14	7	4	2319527	296985
	15	5	-9	7428430	951113
→	16	4	5	9747967	1248098
→	17	6	-5	26924344	3447309
	18	4	9	63596645	8142716
→	19	5	-4	90520989	11590025
→	20	7	3	335159612	42912791
	21	5	-12	1431159437	183241189
→	22	7	1	1766319049	226153980

Table 2

6 Construction of the Sine-table

In the above figure $AP = R\sin\theta$. This is the Indian *jyā* , where R the radius or *trijyā* is normally chosen such that one minute of arc on the circle corresponds to one unit of distance. Hence R is the number of minutes in a radian, and its value is very close to 3438, which is the value mentioned in the earlier texts. More exact values are mentioned in later texts, especially those of the medieval Kerala school.

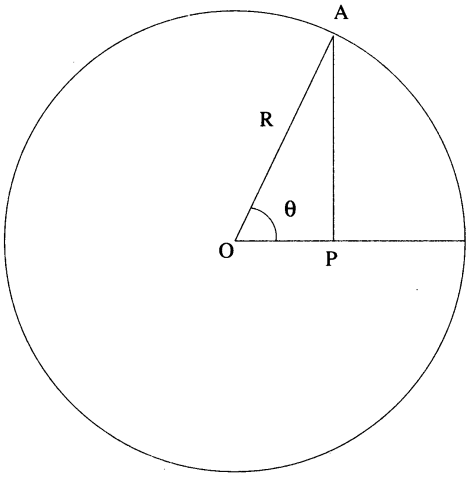

Fig : 5

Normally a quadrant is divided into 24 equal parts, so that each arc bit $\alpha = \frac{90}{24} = 3^0 45' = 225'$. Then the procedure for finding $R\sin i\alpha$, $i = 1, 2, \ldots 24$ is explicitly given. The R sines of the intermediate angles are to be determined by interpolation.

Āryabhaṭīya gives the explicit algorithm for constructing the sine-table (verse 12, *Gaṇitapāda)*:

> प्रथमाद्यापज्यार्धादृरूनं खण्डितं द्वितीयार्धम् ।
> तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि ॥
>
> The first Rsine divided by itself and then diminished by the quotient gives the second Rsine difference. The same first Rsine diminished by the quotients obtained by dividing each of the preceding R-sines by the first Rsine gives the remaining Rsine-differences.

This tells us that

$$R\sin 2\alpha - R\sin\alpha = R\sin\alpha - \frac{R\sin\alpha}{R\sin\alpha} \qquad (\alpha = 225'),$$

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin\alpha - \frac{R\sin\alpha + R\sin 2\alpha + \ldots + R\sin i\alpha}{R\sin\alpha}.$$

The second equation is equivalent to the relation:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - \frac{R\sin i\alpha}{R\sin\alpha}.$$

The values of the 24 *Rsines* are explicitly noted in another verse, and the first *Rsine* is given by

$$R\sin\alpha \approx 225'$$

which obviously uses

$$\sin\alpha \approx \alpha,$$

when α is small. The exact recursion relation for the Rsine differences is:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - R\sin i\alpha \;\; 2(1-\cos\alpha).$$

Now, $2(1-\cos\alpha) = 0.0042822$, which is approximated in *Āryabhaṭīya* to be

$$\frac{1}{R\sin\alpha} = \frac{1}{225} = 0.00444444.$$

The *Āryabhaṭīya* 's sine values are given in the Table 4. The same procedure with the same values are to be found in many other works including *Sūryasiddhānta* [14]. However the Rsine values given by Govindasvāmi in his commentary on *Mahābhāskarīya* of Bhāskara I which are reproduced in Table 4 are more accurate (correct to five decimal places) [15,16].

Far more accurate values for the sines were given by Mādhava of Saṅgamagrāma (1340-1425), the father figure of the Kerala school of astronomy and mathematics. These are cited in the two commentaries on Nīlakaṇṭha Somayāji's *Tantrasaṅgraha* (c.1500)[17,18] namely *Laghuvivṛtti* [17] and *Yuktidīpikā* [18], both composed by Śaṅkara Vāriyar.

These results are based on the series expansion for $\sin\theta$, which we write in the following form:

$$R\sin\theta = R\theta - \frac{(R\theta)^3}{3!R^2} + \frac{(R\theta)^5}{5!R^4} - \frac{(R\theta)^7}{7!R^6} + \frac{(R\theta)^9}{9!R^8} - \frac{(R\theta)^{11}}{11!R^{10}} + \ldots,$$

where $R\theta$ is the arc in minutes [19]. Explicit values of the magnitudes of the terms starting in the reverse from sixth and up to the second in (the RHS of) the above equation, given by Mādhava when the arc $R\theta = 5400' = 90°$, are mentioned in *Yuktidīpikā* in the *kaṭapayādi* notation. The verse giving these values is quoted below[18].

विद्वांस्तुन्नबलः कवीशनिचयः सर्वार्थशीलस्थिरः
निर्विद्धाङ्गनरेन्द्ररुङ्गिगदितेष्वेषु क्रमात् पञ्चसु ।

These values along with the modern values for comparison are listed in Table 3. The last column in this table was computed using $R = 3437.747$ and $R\theta = 5400$.

Term no. in RHS	Sanskrit equivalent in *kaṭapayādi*	Mādhava's value according to *Yuktidīpikā*	Modern value
VI	विद्वान्	$0'0''44'''$	$0'0''44.54'''$
V	तुन्नबलः	$0'33''6'''$	$0'33''5.6'''$
IV	कवीशनिचयः	$16'05''41'''$	$16'05''40.87'''$
III	सर्वार्थशीलस्थिरः	$273'57''47'''$	$273'57''47.11'''$
II	निर्विद्धाङ्गनरेन्द्ररुक्	$2220'39''40'''$	$2220'39''40.10'''$

Table 3

We find that the values given by Mādhava are indeed very accurate. For an arbitrary arc $R\theta$ (in minutes), the procedure is given in the later half of the verse quoted above [18]:

आधस्त्यात् गुणितादभीष्टधनुषः कृत्या विहृत्यान्तिमस्-
यासं शोध्यमुपर्युपर्यथ धनेनैवं धनुष्यन्ततः ॥

It is the following formula for $R\sin\theta$ that is delineated by the above verse:

$$\begin{aligned} R\sin\theta &= R\theta - \beta^3(2220'39''40''') + \beta^5(273'57''47''') - \beta^7(16'5''41''') \\ &\quad +\beta^9(0'33''6''') - \beta^{11}(0'0''44'''), \end{aligned}$$

where $\beta = \frac{R\theta}{5400}$. The 24 Rsines corresponding to $R\theta = 225', 450', 675' \ldots$ were also explicitly stated by Mādhava in the *kaṭapayādi* notation and they have been given in *Laghuvivṛtti* in the set of verses beginning with **श्रेष्ठं नाम वरिष्ठानां हिमाद्रिर्वेदभावनः** and ending with **तत्परादिकलान्तास्तु महाज्या माधवोदिताः**। They coincide with the modern values up to "thirds" (corresponding to an accuracy of sines up to seventh or eighth decimal places).

	R sin θ according to		
θ in min.	*Āryabhaṭīya*	Govindasvāmi	Mādhava(also Modern)
225	225	224 50 23	224 50 22
450	449	448 42 53	448 42 58
675	671	670 40 11	670 40 16
900	890	889 45 08	889 45 15
1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 7
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
2250	2093	2092 45 46	2092 46 03
2475	2267	2266 38 44	2266 39 50
2700	2431	2430 50 54	2430 51 15
2925	2585	2584 37 43	2584 38 06
3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3176 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

Table 4

In Table 4, we compare the values of the *Rsines* in *Āryabhaṭīya* , Govindasvāmin's commentary of *Mahābhāskarīya*, and Mādhava's values as

stated in *Laghuvivṛtti* [16] and the modern values.

It may be noted that the table gives the Rsine values corresponding to arc lengths which are multiples of 225'. In the third and the fourth columns, the values are in minutes, seconds and thirds. Note that the modern values coincide with the Mādhava's values in the last column.

7 Accurate computation of π using an error-minimization algorithm

It is well known that an infinite series for π was first given by the Kerala mathematicians, who invariably ascribe the result to Mādhava of Saṅgamagrāma (1340 - 1425 AD):

$$\frac{Circumference}{Diameter} = \pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \quad).$$

An ingenious geometrical proof of this is to be found in the celebrated Malayalam text *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530) [20,21]. As is well known, this series converges very slowly. An algorithm for accurate and efficient computation of π using the technique of correction terms was also given by Mādhava. This result is cited in *Yuktidīpikā* of Śaṅkara Vāriyar [22].

व्यासे वारिधिनिहते रूपहृते व्यासंसागराभिहते ।
त्रिशरादि-विषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात् ॥
यत्सङ्ख्ययात्र हरणे कृते निवृत्ता हृतिस्तु जामितया ।
तस्या ऊर्ध्वगता या समसङ्ख्या तद्दलं गुणोऽन्ते स्यात् ॥
तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत् ।
ताभ्यामाप्तं स्वमृणे कृते घने क्षेप एव करणीयः ॥
लब्धः परिधिः सूक्ष्मः बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात् ॥

The diameter multiplied by four and divided by unity (is found and stored). Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and

the results are subtracted and added in order (to the earlier stored result).

Take half of the succeeding even number as the multiplier (*guṇa* term) at whichever (odd) number the division process is stopped, because of boredom (by the slow converging process). The square of that (even number) added to unity is the divisor. The ratio has to be multiplied by the product of the diameter and four as (stated) earlier.

The result obtained has to be added if the earlier term (in the series) has been subtracted and subtracted if the earlier term has been added. The resulting circumference is very accurate; in fact more accurate than the one which may be obtained by continuing the division process (with a large number of terms in the series).

Essentially,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \frac{1}{p} \mp \frac{\left(\frac{p+1}{2}\right)}{(p+1)^2 + 1},$$

to a fairly good approximation.

Describing different ways by which better approximations can be obtained, finally Śaṅkara Vāriyar states a more accurate correction term [22]:

एभ्यः सूक्ष्मतरोऽन्यो विलिख्यते कश्चनापि संस्कारः ।
अन्ते समसङ्ख्यादलवर्गः सैको गुणः स एव पुनः ॥
युगगुणितो रूपयुतः समसङ्ख्यादलहतो भवेद् हारः ।
त्रिशरादि विषमसङ्ख्याहरणात् परमेतदेव वा कार्यम् ॥

Now a correction term better than the others will be presented. Here square of half of the even term increased by unity, is the multiplier. The same (square) increased by unity and multiplied by four, and further multiplied by half the even term is the divisor. This correction term should be applied after dividing by the odd numbers three, five, etc.

In other words, a much better approximation for $\frac{\pi}{4}$ is:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \frac{1}{p} \mp \frac{\left(\frac{p+1}{2}\right)^2 + 1}{[(p+1)^2 + 4 + 1]\left(\frac{p+1}{2}\right)}.$$

In *Yuktibhāṣā* , these correction terms are derived using an error-minimization algorithm [21,23]. We give a summary of this procedure as explained in the above references. If the series is truncated at odd number $p-2$ with a correction term $\frac{l}{S_1}$, $\frac{\pi}{4}$ can be written as:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots - \frac{1}{p-2} + \frac{1}{S_1}.$$

On the other hand, if the series is truncated at p with a correction term $\frac{l}{S_2}$, we will have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots - \frac{1}{p-2} + \frac{1}{p} - \frac{1}{S_2}.$$

If the correction terms are exact, then both should yield the same result. That is,

$$\frac{1}{S_1} = \frac{1}{p} - \frac{1}{S_2} \qquad \text{or} \qquad \frac{1}{S_1} + \frac{1}{S_2} = \frac{1}{p}.$$

$S_1 = S_2 = 2p$ actually satisfies the equality. However both the corrections should follow the same rule. If $S_1 = 2p-2$ and $S_2 = 2p+2$, that is S is twice the even number above the last odd number in the series, then the *Sthaulya* or inaccuracy is

$$\Delta = \frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{p} = \frac{1}{2p-2} + \frac{1}{2p+2} - \frac{1}{p} = \frac{4}{4p^3 - 4p},$$

and it can be shown that the error (corresponding to this choice of S) is minimum. For other choices such as $S_1 = 2p-3$ and $S_2 = 2p+1$, the inaccuracy Δ will have a term proportional to p in the numerator, whereas the denominator $\approx 4p^3$. So the inaccuracy Δ will be much larger than its value for $S_1 = 2p-2$, when p is large. By choosing $S_2(p)$ of the form,

$$S_2(p) = \frac{1}{2p + 2 + \frac{A}{2p+2}},$$

it can be shown that the inaccuracy is reduced further and inaccuracy Δ is minimum at $A = 4$. So,

$$S_2(p) = \frac{1}{(2p+2) + \dfrac{4}{2p+2}} = \frac{\left(\dfrac{p+1}{2}\right)}{(p+1)^2+1}.$$

This leads us to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \pm \frac{1}{p} \mp \frac{\left(\dfrac{p+1}{2}\right)}{(p+1)^2+1},$$

which is the first rule given in *Yuktidīpikā* . Proceeding in the same manner, the error is further reduced and will be minimum when we take

$$S_2(p) = \cfrac{1}{(2p+2) + \cfrac{4}{2p+2+\cfrac{16}{2p+2}}},$$

which leads to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \pm \frac{1}{p} \mp \frac{\left(\dfrac{p+1}{2}\right)^2 + 1}{[(p+1)^2 + 4 + 1]\left(\dfrac{p+1}{2}\right)}.$$

This is the more accurate second rule given in *Yuktidīpikā* . When $p = 99$ (that is, considering 50 terms in the series), the resulting value of π is found to be accurate to 11 decimal places. This is same as the value $\pi \approx 3.14159265359222\ldots$ given by the rule, विबुधनेत्रगजाहिहुताशन..., attributed to Mādhava by Nīlakantha in his *Āryabhatīyabhāṣya Gaṇitapāda*, verse 2.

When the error-minimization algorithm is continued along the above lines two steps further,

$$S_2(p) = \cfrac{1}{(2p+2) + \cfrac{4}{2p+2+\cfrac{16}{2p+2+\cfrac{32}{2p+2+\cfrac{64}{2p+2}}}}}$$

When $p = 99$, this yields $\pi = 3.141592653589793241$, which is correct to 17 decimal places. This is the same as the value for π stated by Rājā Śaṅkara Varma in his *Sadratnamālā* (c.1823).

Acknowledgement

I thank M.D.Srinivas, Centre for Policy Studies, Chennai, and K.Ramasubramanian, IIT Bombay, for many useful discussions on this topic.

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Paper 47

P. P. Divakaran (2007)

The First Textbook of Calculus: "Yuktibhāṣā"

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P. P. Divakaran

For a brief biographical sketch of P. P. Divakaran, see page **??**.

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The First Textbook of Calculus: *Yuktibhāṣā*

P. P. Divakaran

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Introduction

In 1832, Charles M. Whish, an Englishman who had worked for the East India Company, presented to the Royal Asiatic Society of Great Britain and Ireland an account[1] of the contents of four palm leaf manuscripts which he had found in the environs of Cochin (or Kochi as it is now known) in coastal central Kerala. One of the four is a work in Malayalam, the language of Kerala, on mathematics and astronomy entitled *Yuktibhāṣā* (*abbreviated to YB* from now on; the others are also mathematical/astronomical texts, by now similarly well-known, in Sanskrit). Thus was brought to the notice of scholarly Europe, perhaps for the first time, the text of which I speak here. Whish's effort seems to have made no impression on historians of mathematics till the 1940s. But the decades since then have seen increasing attention being paid[2] to the quite astonishingly sophisticated mathematics produced by the socalled Kerala

[1] Charles M. Whish (1835, p. 509).

[2] As far as I can tell, Whish's baton was first picked up by Mukunda Marar and Rajagopal (1944, p. 65). Of the numerous subsequent publications, two are worthy of special mention: T. A. Sarasvati Amma's thesis (later published as a book 1979; second revised edition 1999) and George Gheverghese Joseph (1992). The first is largely faithful to the methods actually employed by the Kerala mathematicians. Joseph's book has brought the story to a large readership. There are a fair number of good websites that discuss the work of the Kerala school and there are also some which are not fully reliable.

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school in a relatively brief period of 200 years or so, beginning in the second half of the 14th century CE.

The text of *YB* that I use here is the one that virtually everyone interested in the work relies on directly or indirectly, that given in the annotated Malayalam edition of Rama Varma Tampuran and A. R. Akhilesvara Ayyar, published in 1948.[3] It is based on four separate manuscripts which, according to the editors, are largely in concordance. Tampuran belonged to a local royal family and was a well-known scholar of Malayalam (and hence of Sanskrit) and of the *śāstra* literature, and Ayyar was a schoolteacher with a master's degree in mathematics; in the complementarity of their domains of expertise, a more ideal pair of collaborators is hard to imagine. The main part of the book is 290 printed pages long of which roughly half is commentary. The editing and the commentary are impeccably done, in the best Indian *bhāṣya* tradition. Especially noteworthy are the meticulously drawn geometric figures without which the work of those who followed them would have been made greatly more arduous.

The edition of Tampuran and Ayyar covers only Part I of *YB*. Part II contains applications to astronomy and has so far not been published, in any language. The late K. V. Sarma, whose efforts more than of anyone else brought the main texts of Kerala mathematics and astronomy to the attention of the scholarly world, had completed (in association with M. D. Srinivas, M. S. Sriram and K. Ramasubramanian) an English translation of both parts of *YB* at the time of his death in January 2005, but it is yet to appear. As of now, a faithful presentation of what *YB* contains, in any language other than Malayalam (unfortunately beyond the acquaintance of most interested scholars) does not exist. This should be a matter of regret because what it does contain, primarily, is a very thorough account of the techniques of the infinitesimal calculus that the Kerala school created and developed for the study of the geometry of the circle.

The above claim and the title of this article raise the question: what exactly constitutes the mathematical discipline we call calculus?

The question is no easier to answer today than it has been in the past. Contemporary mathematics almost unanimously holds the view that the central concern of calculus is with the properties of 'functions'. In the early days of calculus in Europe, these were limited to real functions of a real variable, explicitly given by formulae, of a small number of types such as powers of low degree and functions defining the conic sections; e.g., $y = x^2$, $y = (1 - x^2)^{1/2}$. Over the next two centuries and a half, the concept of a function was greatly broadened, culminating in a very abstract and general definition, free of any explicit geometric or numerical connotation, as a 'map' from one 'space' to another, a 'black box' that takes an input from an initial space and gives an output in a final space. In this generality, functions are not amenable to the methods of calculus; for that to be possible, there must be a criterion for deciding when two points are nearby in both the initial and final

[3] Rama Varma (Maru) Tampuran and Akhilesvara Ayyar (1948).

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spaces as well as a rule for adding (and subtracting) two nearby points.[4] No further limitations are necessary.

In its early avatars, in India or in Europe, calculus provided a set of conceptual and technical tools, essentially geometric, with which to approach and find answers to an array of questions regarding simple functions. The remarkable fact is that these tools, suitably sharpened and broadened, are recognisably the very same ones as are in use in today's greatly more general form of the discipline. An instance of this dramatic increase in generality is provided by the invention in the middle of the last century of the theory of distributions, inspired largely by physics. The mathematical objects called distributions are maps only in an indirect sense; they stretch the idea of a function to a point where some people, at least for a time, even denied them a mathematical existence. Nevertheless, distributions are just as readily manageable by the methods of calculus as the simple functions for which they were first fashioned. It seems then fair to characterise the discipline of calculus primarily as this collection of principles and methods formulated in essence by the pioneers, whose ever expanding field of applicability we are far from having fully charted. Most of the problems the creators of calculus tackled existed from antiquity; what was new and revolutionary is their method of solving them.

It is in this perspective that *YB* can be described as a book on calculus rather than on trigonometry or infinite series. No matter how impressive or novel in their time, these well-known results (of which the main ones I shall write down later as the occasion arises) are only the first in a long list of applications of calculus whose end is not in sight. Neither the Kerala mathematicians of the 15th–16th centuries nor the Europeans of the 17th would have recognised a distribution if it blew up (as they are liable to do) in their face, but they would have recognised the processes of differentiation and integration which are used to elucidate their properties. They would also have understood each other's achievements and the many parallels in the way they, vastly separated in space and time, developed the subject. There were significant differences as well; while the early Europeans had a broader view of the uses of calculus from the beginning, the Kerala work as presented in *YB* is, though limited to the circle, more systematic and sophisticated. (Such a declaration of course requires justification and I hope to take up the question elsewhere).

The Background

Till about 700 CE or so, Kerala formed part of the Tamil linguistic landscape. The rich cultural, especially literary, life of those days is well documented and

[4] More technically, spaces on which calculus can be done have to have a topological structure (providing a measure of nearness or locality) and, locally (with respect to the given topology), a linear structure. I thank M. S. Narasimhan for explaining to me the modern sense of what d'Alembert called the metaphysics of calculus.

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we also have some knowledge of the polity. There is however no serious evidence of any astronomical or mathematical activity at that time in Tamil Kerala nor indeed in the Tamil country as a whole. Very plausibly, the taste for astronomy and mathematics travelled to Kerala with a series of waves of migration from further north, generally along the coast, of vedic brahmins. Beginning in the 7th–8th century CE, this influx continued for half a millennium or longer. The early settlers bestowed Aryan legitimacy on local potentates who built temples for them in return, with extensive land grants for the maintenace both of the temples and, in some comfort, of the temple managers. But these brahmins, who came to be known generically as Namputiris, brought with them a great deal more than vedic expertise; they were, at least some of them, a learned people, the repositories and practitioners, creators and transmitters of all the classical sciences. Thus it is that they carried to Kerala a strong heritage in mathematics and astronomy, most notably the legacy of Āryabhaṭa. They also brought with them their language of learning, Sanskrit.

But largely independent of northern influences, the old Tamil language of the region had already begun to take on a distinct local personality. On top of this came the impact of Sanskrit, resulting in a rich and versatile new language, Malayalam. By the time of the composition of *YB*, Malayalam had settled down to its definitive form, hardly different from what it is today. Nevertheless, with the exception of *YB* and a few minor texts, Sanskrit continued as the language of mathematical scholarship and (writing) until its final decline and disappearance, as sudden as was its beginning.[5]

A number of literary references and inscriptions from this period of transition testify to the existence all over Kerala of educational institutions, ranging from the veda schools attached to temples to veritable quasi-autonomous colleges. Over the first half of the second millennium CE, these schools, together with the academies that flourished under royal patronage, were an integral part of the intellectual vibrancy of the time. It is in this setting that almost all of the literally thousands of scientific manuscripts now lying scattered in private and public collections came to be composed.[6]

The earliest known mathematical work which can be unambiguously attributed to Kerala actually predates this golden age. This is a Sanskrit verse

[5] The question of *why YB* was composed in Malayalam is an intriguing one, not addressed here.

[6] This extremely concise sketch of the medieval history of Kerala is an inadequate summary of the pioneering studies of Ilamkulam (or Elamkulam) P. N. Kunjan Pilla. Some aspects of his work have recently been questioned; despite this, his research provides the one coherent overall picture we have of the history of Kerala, especially of the time relevant to us. He wrote in Malayalam, but there is a volume of English translations of a selection of his essays (Ilamkulam 1970). He has the added distinction of having been the first to seriously read Śankaranārāyaṇan's commentary on the *Laghubhāskarīyam*, thereby establishing a first direct link between astronomical activities in Kerala and elsewhere in India.

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commentary by Śankaranārāyaṇan[7] on the first Bhāskara's *Laghubhāskarīyam*. According to the text itself, Śankaranārāyaṇan was the royal astronomer to the Cera emperor Sthāṇu Ravi Varma—the Ceras had lately consolidated their control over most of Kerala—working in the observatory in the capital Mahodayapuram (the famed Muziris or Musiris, emporium to the world; modern Kodungallur). The text gives the date of composition as 869 CE and refers to the author as hailing from Kollapuri in Paidhyarashtra (the region of Pratishthana or Paithan) which has been identified (see footnote [6]) very persuasively with modern Kolhapur in Maharashtra. This is rich and rare fare: in one text, we have a precise date and place of composition and a precise provenance for the author or his immediate ancestors.

As far as we know, the emergence of astronomical scholarship in Kerala with Śankaranārāyaṇan was an isolated event. His intellectual ancestry is unknown and no one seems to have carried on after him; in fact there is no sign of any serious mathematical and astronomical activity in Kerala in the next five hundred years. To an extent, this can be blamed on the war and turmoil that visited Kerala from the Tamil country at the turn of the first millennium. By the end of the 11th century, the prosperous Cera kingdom was gone forever, its capital razed, the royal observatory only a memory.

Mādhavan and his Followers

In the chaotic aftermath of this '100 year war', the Cera kingdom broke up into a number of principalities, of which the most powerful was that ruled by the Zamorin (samutiri) dynasty from Calicut. This period also saw a renewed influx of brahmins and their rise to a position of social and cultural preeminence. The southern part of the Zamorin's domain became a particularly favoured area for Namputiri settlements. It is in this region, on either bank of the river Nila (more commonly known now as the river Bharata), that there emerged, beginning in the second half of the 14th century, a succession of astronomers and mathematicians of truly exceptional quality. They formed a tight-knit group, all linked together by the traditional teacher-disciple bonds. There was at least one father-son pair among them, Parameśvaran and Dāmodaran. Most were Namputiris of one variey or another. According to local tradition, three were native to one village and two taught at the same temple school. Virtually all lived in close geographic proximity and produced their work within a time span of roughly two centuries.

[7] I have transcribed all names as they are conventionally written in Malayalam. Modern names, place names, etc. have been exempted from diacritical markings. The marks on technical terms follow the way they are written in Malayalam script in *YB*. Though the Malayalam transliteration of Sanskrit and Sanskrit-origin words does not always conform to their original forms, the deviations are minor and easily spotted. The Malayalam distinctions between the short and long *e* and *o* as well as the two *l* s have been ignored.

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The most original and creative of this brilliant lot, by the acclamation of those who followed him, was also the first: Mādhavan, said to be of Sangamagramam.[8] Mādhavan has remained a somewhat shadowy figure. We know nothing about his life except that it spanned the second half of the 14th century and the first quarter of the 15th and that the great astronomer Parameśvaran was his disciple. The few manuscript fragments that are attributed to him are primarily astronomical, having little mathematical content of any originality, and it is through the writings of his intellectual heirs that his achievements are known to us. Parameśvaran's student Nīlakaṇṭhan, in particular, quotes mathematical verses attributed to Mādhavan quite liberally in his Sanskrit work *Tantrasaṃgraha*, making it clear that it is meant to be a compendium of his teaching. The overwhelming significance of *YB* lies in the fact that it contains, in chapters 6 and 7, the most thorough exposition we have of the new mathematics that Mādhavan created and *Tantrasaṃgraha* summarised. If only to place *YB* itself in context, it is useful to touch on the highlights of Mādhavan's legacy.

Since at least the time of Āryabhaṭa, the one abiding theme of Indian mathematics was the circle, more particularly the relationship between an arc and the corresponding chord, and a variety of questions linked to it. This was entirely in line with the uses to which mathematical reasoning was put in India, first and foremost for the study of the geometry of the celestial sphere and the motion of heavenly bodies in their epicyclical orbits. Āryabhaṭa himself famously recognised the difficulty of giving an exact number for the ratio of the circumference of a circle to its diameter, qualifying his value 3.1416 for π as approximate, *āsanna*. Fundamentally, what Mādhavan did was to push the quest for precise values for such ratios to its logical and mathematical limit. (That π was understood by this time to be an irrational number is clear from a well-known passage from Nīlakaṇṭhan's commentary on the *Āryabhaṭīya*, see below). In the process, he was led, at a technical level, to express such ratios (trigonometric functions) as appropriate power series, infinite series of smaller and smaller terms with none exactly equal to zero. (We know them as the 'Gregory-Leibniz' series for the arctangent and the 'Newton' series for the sine and the cosine, the quotation marks indicating a compromise between current usage and historical accuracy). Mādhavan also did some wonderful and practical things with these series: techniques for accelerating their convergence; accurate estimates of the remainder as rational functions when a series is truncated; interpolation formulae for values of trigonometric functions at a point in terms of the values at a neighbouring point; and so on. Most of this is described in detail in *YB*.

[8] It is possible to make the case that Mādhavan or his family was a recent arrival from up north. He is said to have been an Emprantiri (brahmins recently arrived from coastal Karnataka, an established staging area on the migrant route). There is no known Sangamagramam ('the village at the confluence') in Kerala, if we set aside some fanciful etymology of place names. Temples to Sangamesvara (Siva) stand at several confluences of rivers or streams in northern Karnataka and southern Maharashtra.

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But more profoundly, Mādhavan arrived at a general method, a philosophy almost, of addressing such questions by passing to the infinitesimally small and then summing the infinite number of the resulting infinitesimal contributions to the relevant geometrical quantity. Stated simply, Mādhavan invented calculus, as it applies to circular arcs. Subject only to the limitation to circular arcs, *YB* conveys clearly that the key conceptual step in this was the recognition that local approximation by linear functions (tangents), in other words differentiation, and their subsequent summing up, integration, are converse processes–in essence, the earliest version of what came to be known in its sharp subsequent form as the fundamental theorem of calculus. What is most striking in all this to the present day reader brought up in the mathematical culture of the 19th and 20th centuries is the easy mastery with which the supposed twin demons of the infinite and the infinitesimal are simultaneously tamed.

About *YB*

K. V. Sarma's date for the composition of *YB*, around 1550–1560 CE, seems now to be generally accepted, as also his attribution of the authorship to Jyeṣṭhadevan.[9] We know a little more about Jyeṣṭhadevan than about Mādhavan: he was a disciple of Nīlakaṇṭhan and Dāmodaran, both disciples of Mādhavan's disciple Parameśvaran and, like Parameśvaran, was connected with the Rama temple (or to a school attached to it) in the village of Alattiyur, just north of the river Nila not far from its mouth. And, most importantly, we have a whole book, extensive and intact, attributed to him.[10] Alattiyur still has memories of his name and of its past mathematical-astronomical glories, but astrology has long since supplanted astronomy as its chief source of pride.

In the context of Indian mathematical writing, *YB* is singular in several respects: (i) it is not in Sanskrit verse but in Malayalam prose (*bhāṣā*), (ii) it provides detailed lines of reasoning (*yukti*) and, as a consequence, (iii) it is inordinately long. It is also very 'theoretical'; there are few worked examples or numerical illustrations (*udāharaṇams*) unlike say in the work of the second Bhāskara and, despite the strongly geometric methods of proof, no diagrams at all. (Tampuran and Ayyar have included a profusion of them in their commentary and these have been freely and gratefully duplicated in the subsequent literature). It is relevant to ask why it is three-fold singular in this particular way. Given our ignorance of the circumstances of its writing, the answers can only be tentative at best. But, reading the text, it is difficult

[9] Sarma and Hariharan (1991, p. 185). The date suggested in this article is about 1530 CE but Sarma later tended to favour a slightly later dating. Once it is accepted that *YB* is an account of Mādhavan's mathematics, transmitted *via* Nīlakaṇṭhan and others, a sharp date is of little consequence except to historiographers.

[10] K. V. Sarma's view (2002) that Jyeṣṭhadevan also authored *Dṛkkaraṇam*, an astronomical chronicle datable to 1607, apparently on the basis of hearsay gathered by Whish, is difficult to reconcile with the accepted date of *YB* and the dramatically different linguistic styles of the two texts.

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to escape the conviction that, unlike the traditionally favoured *sūtra* format—mnemonic verses serving as an *aide-mémoire* with the reasoning being explained in face-to-face sessions—*YB* was meant to be autonomous, written down from the guru's mouth, to be read, struggled with and, hopefully, mastered in course of time, away from the guru and the classroom. The traditional compact verse format in mathematical writing had at least one function other than that of facilitating (primarily oral) communication and text-preservation; concise expressions are commonly made to stand in for numbers, formulae and even whole collections of ideas and methods, 'packages' to be unzipped in the mind. *YB* does have such packages scattered throughout, of its own and in the form of quoted verses from other sources, but on the whole stands at the opposite extreme. The entire text is in a natural, too natural perhaps, Malayalam, explicit and even verbose, with the Sanskrit technical terms embedded in this matrix. The style is colloquial and often repetitive, especially in its use of connecting words common in conversation but not serving any functional (let alone mathematical) purpose. Altogether, its writing was a startlingly original endeavour. One cannot escape the feeling of reading the faithful notes of a masterly but informal set of lectures—face-to-face presentations—but without the easy present day option of an escape to equations and diagrams.[11]

YB claims no originality of content; the first sentence is: "In order to explain[12] all the mathematics useful for the motion of heavenly bodies (*graha*), according to *Tantrasaṃgraha*,[13] I begin by describing the general (or common) mathematical operations such as addition and so on". Three short chapters deal routinely with these basic operations. An even shorter (considering the heavy use made later of the properties of similar triangles) chapter 4 concerns the 'rule of three' (*trairāśikam*). A substantial chapter 5 (together with a long appendix, explicitly stated to be taken over from *Tantrasaṃgraha*, with several numerical examples) is devoted to the method of 'pulverisation' (*kuṭṭākāram*) for the Diophantine solution of an equation of the first degree in two unknowns. This will find its use in Part II of *YB*. As already noted, the meat of the work is in the last two chapters, 6 (the relationship between the circumference and the diameter) and 7 (the theory of chords). Including the commentary, they run to over 200 printed pages.

The inevitable question that arises is how effective the use of an almost totally conversational Malayalam is in conveying the subtle and complex

[11] An independent confirmation of Sarma's view (see footnote [10]) will strengthen the case for someone other than Jyeṣṭhadevan, a disciple for example, doing the actual writing down of the material. Edward Stadum, who was present at the Workshop, later used the phrase 'literary calculus' to describe *YB*. In my mind I can see Jyeṣṭhadevan teaching under a coconut tree, with a sign above saying 'calculus spoken'.

[12] The Malayalam verb used is *colluka* which can mean state, relate, narrate, speak, describe, explain, etc. and is employed in all these senses in *YB*.

[13] There are other roughly contemporaneous texts, though none in Malayalam, based on Nīlakaṇṭhan's *Tantrasaṃgraha*, for example Śankara Vāriyar's *Yuktidīpika*. In many ways, not least in the breadth of his interests, Nīlakaṇṭhan appears to have been the real inheritor of Mādhavan's mantle.

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reasoning that this new knowledge is based on and, more generally, what obstacles the reliance on such a natural style of thought and communication places in the way of development of what is after all a highly abstract undertaking. To enable readers to judge for themselves how well these challenges are met, I now present in translation two passages from *YB*, both relating to calculus, before expressing my own views. The first describes the computation of the surface area of a sphere (from chapter 7) and the second is the section in which the integrals of the positive integral powers of a variable are worked out (in chapter 6).

The translation is as literal as I can sensibly make it. The only liberties I have taken are to provide basic punctuation and paragraph breaks, to drop the too-frequent conversational connectives, to supply personal pronouns where English requires them and to add a few brief explanations (not present in the original) or comments [within square brackets]. In particular, I provide no diagrammatic guide to the reasoning (the basic figure which I use to define notation for my own explanation of the passage on integration of powers should not count)—those so inclined will find it instructive, at least as a start, to supply their own figures before looking up Tampuran and Ayyar (1948) or Sarasvati Amma (1979). The first passage requires only a minor transformation to make it 'modern'; Sarasvati Amma's version is largely faithful. The second passage holds a few points of mathematical interest which it will be worthwhile to return to later. It is also the part of Kerala calculus that today's scholars, unlike Tampuran and Ayyar, sometimes tend to handle with less than complete fidelity, lapsing occasionally into a reliance on later western developments like the binomial theorem for negative exponents, theorems on limits and their interchangeability with infinite sums etc., in place of what is actually in *YB*. For this reason, in the introductory remarks on this passage and in the few explanations within it, I have made an effort to guard against my use of current notation distorting in any way the material I am trying to convey.

The Surface Area of a Sphere

The section headed *golapṛṣṭhakṣetraphalānayanam* gives the proof of the formula: surface area of a sphere = diameter × circumference [of a great circle]. It occurs at the very end of chapter 7 (with only the calculation of the volume of the sphere to follow) though, with the exception of the use of one 'package', the work required is much less demanding than in the proofs of the various trigonometric series earlier in the book. It is probably the case that Mādhavan invented the infinitesimal method first for the determination of π via the 'Gregory-Leibniz' series and then found it to be a powerful general tool for settling several other interesting problems[14] (also see below).

[14] Bhāskara II arrived at the correct formulae for the surface area and volume of the sphere, but by seminumerical methods which the inventor of the infinitesimal method must surely have scoffed at.

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Here is the translation.

Now I narrate that, combining two principles just explained, [namely that] from *piṇḍajyāyogam* can be produced *khaṇḍāntarayogam* and [that] knowing the diameter at one place, we can apply the rule of three (do a *trairāśikam*) as we please, the area of the surface of a sphere will arise.

A uniformly rounded object is called a sphere (*golam*). Through the middle of such a sphere, imagine two circles, one along east-west and the other along south-north. Then imagine circles, one shifted slightly to the south and the other slightly to the north of the east-west circle [the equator]. Their distances from the east-west circle should be the same for all parts (*avayavam*) [the word *samāntaram* for parallel is not used in this section though it is, for straight lines, elsewhere]. Consequently, these two will be slightly smaller than the first (or original) one. Then, starting from these, imagine slightly smaller and smaller circles, all of them at equal distance one from another [i.e., between successive latitudes], so as to end at the south and north edges [the poles]. Their separation along the south-north circle must be seen to be equal. This being so, imagine that the circle-shaped gap between two circles [successive latitudes] is cut at one place, removed and straightened (or spread). Then, of the circles on the two sides of the gap, the larger one will be the base (*bhūmi*) and the smaller one the opposite side (*mukham*) of a trapezium (*samal-aṃbacaturaśram*) whose lateral sides (*pārśvabhuja*) will be the separation (*antarālam*) along the south-north circle of [two successive latitude] circles. Now cut out the part outside the altitude (*laṃbam*) [from an upper vertex to the base], turn it upside down and transfer it to the other side [the opposite edge]. This is a rectangle whose length is half the sum of the base and the oposite edge and whose width is the altitude. In this way, think of all the gaps (*antarālam*) [elsewhere a non-Sanskrit Malayalam word is also used] as rectangles (*āyatacaturaśram*). Their widths are all equal. Lengths have various (or varying) measures (*pramāṇam*). The result of multiplying the length and the width is the area (*kṣetraphalam*). The widths of all being equal, add the lengths of all and multiply by the width. Thus will arise the area of the surface of the sphere. [Nothing so far about the passage to the limit].

Next, the method (*upāyam*) to know how many gaps there are and what their lengths and width are. The radii of the circles that we have imagined above [the latitudes] are half-chords (*arddhajyā*) of a circle whose radius is the radius of the sphere. Hence, multiplying these half-chords by the circumference of the sphere and dividing by the radius of the sphere [i.e., multiplying by 2π] will result in circles [their circumferences to be precise] having the half-chords as radii. These will be the lengths of the rectangles if the chords are taken at the midpoints of the gaps. Multiplying the sum of all half-chords (*arddhajyāyogam*) [by 2π, left

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unsaid] will result in the total length of all the figures (*kṣetrāyāmam*) [i.e., the sum of the lengths of all the rectangles]. The gap between two arbitrary [the Malayalam phrase is *yāva cilava* = *yāvat tāvat*] circles [an arbitrarily chosen pair of consecutive latitudes is meant] at the south-north circle mentioned earlier is an arc segment (*cāpakhaṇḍam*) of the circumference of the sphere [i.e., of the S-N circle].

Next, the method of getting the sum of the chords (*jyāyogam*). Multiply the square of the radius by *khandāntarayogam* [I make no attempt at an exact translation of this 'package'. The expression stands for the sum of the second differentials of the half-chords with respect to the arc (or, equivalently, the angle) and has already been treated earlier in the chapter in connection with the series expansions of sine and cosine. The result about to be used below is also derived there.] and divide by the square of the full chord of the arc segment (*cāpakhaṇḍas-amastajyāvarggam*). The result is the sum of the half-chords. Multiply this by the width. The width is the chord of the arc segment. The *khaṇḍāntarayogam* is the first chord segment. Because of smallness [of the arc segment] these [the first chord segment and the width] are [both] almost equal to the full chord. These are the multipliers and the square of the full chord is the divisor. But multiplication and division are unnecessary [the square of the full chord cancels from the numerator and denominator]. What remains is the radius. This has to be multiplied by the circumference and divided by the radius. Only the radius will survive. Since we have to get the result for both halves of the sphere, double the radius. Therefore, multiplying the diameter of the sphere by the circumference of the sphere will produce the area of the surface of the sphere.

We have here a straightforward narration in natural spoken Malayalam, understandable literally by any literate person but for the technical terms. By and large, the technical nomenclature is that used in mainstream mathematical writing in Sanskrit over many centuries and is consequently unambiguous. But these linguistic resources are now asked to be at the service of a sophisticated novel mode of reasoning demanded by Mādhavan's infinitesimal mathematics, a service they were not designed to provide in full. Added to this burden is the yoke of a foreign tongue, Malayalam, which was not sung, at least mathematically speaking, at the cradle of the author. The strain shows. Even in translation it is evident that, while the first part of the proof (the geometric construction) is simply presented and easy to follow, the transition to the 'infinitesimal mode' is marked by an increasing degree of opacity. What the translation cannot show is that the Malayalam itself is, apart from being very informally used, quite unrefined in relation to other literary material of the period.

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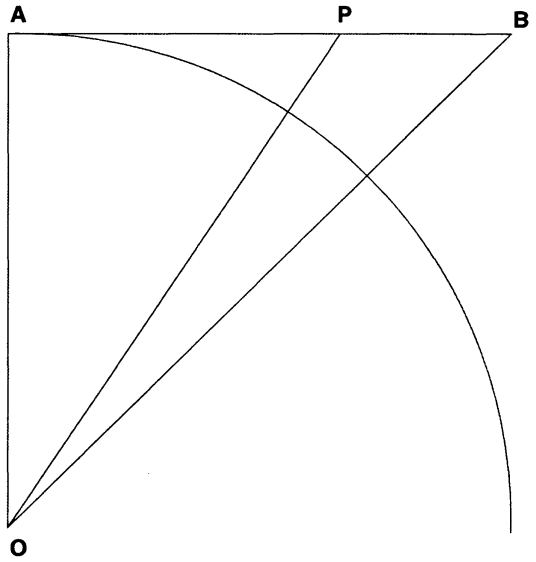
Integrals of Powers

Chapter 6 of *YB* is devoted to establishing the validity of the 'Gregory-Leibniz' series, first for the angle $\pi/4$ and then for general angles, and to various improvements on it. Apart from the mathematical content itself, it is noteworthy as the first illustration of the power of the new infinitesimal geometry when combined with the equally new technique of integration or 'summation in the limit', *saṃkalitam*.

The geometry considers a circle of radius r and a square of side $2r$ in which the circle is inscribed, focussing on one octant of the circle as in the diagram below:



The side AB is divided into n equal segments of length d (P is a typical point of this division), with the limit $d \to 0, n \to \infty$ (with $nd = r$ fixed) to be taken subsequently. After a series of steps including the usual clever choices of similar triangles (*trairāśikam*), some simple algebraic identities, a process of iteration and a judicious neglect of terms of order $1/n^2$ (i.e., second order infinitesimals in the limit), the geometry is shown to lead to the result

$$\text{arclength of the octant} = \frac{\pi}{4}r = r \lim \sum_{j=1}^{n} \left(\frac{1}{n} - \frac{j^2}{n^3} + \frac{j^4}{n^5} - \cdots \right),$$

lim meaning the limit described above. Each term within the bracket (including the sum over j and in the limit but ignoring the minus signs) is referred to as a *saṃkalitam* which I shall translate as an integral, in anticipation.[15] In addition to the integrals of even powers contributing to the arc

[15] Anterior to the Kerala work, *saṃkalita* is the standard Indian term for the sum of any (finite) series. According to Datta and Singh (1993), this usage goes back at least to the (ambiguously dated) Bakhshali manuscript.

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length, *YB* considers those of the odd powers since the induction procedure requires their evaluation. Thus, the *k*th *saṃkalitam* is defined to be

$$I_k := \lim \sum_{j=1}^{n} \frac{rj^k}{n^{k+1}}, \quad k = 0, 1, 2, \ldots$$

The introductory paragraph of the section headed 'Integrals' is:

> Here I describe the method of producing the integrals. First the simple integral (*kevalasaṃkalitam*) is described. Then the integral of two equals multiplied together. Then, even though it is not useful here, I describe also integrals of equals multiplied among themselves three, five, etc. times, since they occur in the midst of those which are useful [namely, the even powers].

The term *kevalasaṃkalitam* does not reappear in the section. In this passage it evidently means the simple integral of the first power, which is referred to in the rest of the section as *mūlasaṃkalitam*. The distinction is with the sequence of repeated *k*-fold integrals of *x* (*ādyadvitīyādisaṃkalitam*); i.e., in modern notation,

$$\int dx \int dx \ldots \int dx = \frac{x^{k+1}}{(k+1)!}$$

which are taken up further on to generate the factorial denominators in the 'Newton' series. With this potential confusion out of the way, here is the subsection entitled *mūlasaṃkalitam* on the computation of the integral

$$I_1 = \lim \left(\frac{r}{n^2} + \frac{2r}{n^2} + \cdots + \frac{(n-1)r}{n^2} + \frac{r}{n} \right) = \lim \frac{d}{r} \left(d + 2d + \cdots + (n-1)d + r \right),$$

the second form being the expression with which *YB* actually works.

> In *mūlasaṃkalitam*, the last side (*bhuja*) [a *bhuja* is the side *AP* of the right triangle *OAP*, *OP* being the corresponding *karṇṇam*; so the last *bhuja* means *AB*] is equal to the radius of the circle, the one below (or before) that is one segment (*khaṇḍam*) less and the one before that two segments less. Suppose all the sides are equal to the radius. In that case, if the radius is multiplied by the number of sides [i.e., *n*], that will be the result of the *saṃkalitam*. But here only one side is equal to the radius. Starting from this, the sides of the other smaller and smaller diagonals (*karṇṇam*) are, in order, one unit (*saṃkhya*) at a time less. Whatever is the number of units the radius is supposed to have, imagine that the number of segments of the side [here it means the full side *AB*] is the same. That makes it easy to remember. The last but one side will be one unit less. The next shorter one will be two less than the number of units of the radius. The missing part (*aṃśam*), starting with one, will increase

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one [unit] by one [unit] progressively, the last deficit (*ūnāmśam*) almost equal to the radius, just one unit less. Now if the deficits are all added, this number (*saṃkhya*) will eventually equal the sum [the word used for this *finite* sum is *saṃkalitam*, as is traditional, see footnote [15]] of terms starting with one, increasing by one and ending with the radius, less one radius [i.e., it is the original sum minus the radius]. Therefore, multiply the number of units in the radius by one added to the number of sides [these two numbers are the same by assumption]. Its half is the *bhujāsaṃkalitam*. *Bhujāsaṃkalitam* means the sum of the sides of all the diagonals.

The smaller the segment, the more accurate (*sūkṣmam*) the result. Therefore imagine that each segment (*bhujākhaṇḍam*) [the text has *bhujāsaṃkhya*, which is probably a result of sloppiness somewhere in the transcription] is divided (cut) into atoms (*aṇu*) and then carry out the *saṃkalitam*. For this, if the division is by *parārddham* [any very large number will do; here, specifically, it is 10^{17}], add one to the product of the *bhuja* and *parārddham*, multiply by the radius and halve it. Also, divide by *parārddham*. This is approximately half the square of the radius. To make it a whole number, divide by *parārddham*. Thus, to the extent that the segment is short, only a small part has to be added to the *bhuja* to produce the *saṃkalitam*. [I understand this sentence to mean that the correction to the integral due to the finite length of the segment can be neglected in the limit]. Therefore, adding nothing to the *bhuja*, multiplying it by the radius and halving it will result in the *saṃkalitam* of the extremely finely segmented *bhuja*. This is how half the square of the radius is the accurate *bhujākhaṇḍasaṃkalitam*.

The terminological imprecisions in the passage (for instance the different senses in which the words *bhuja, saṃkhya* and even *saṃkalitam* itself are employed) are evident as is the increasing lack of clarity (the *parārddham* business for example), once again, of the writing when it comes to describing the reasoning about the passage to the limit.

After this account of the computation of the integral of x, the next subsection deals with x^2 in some detail. The passage from x to x^2 involves conceptual and technical novelties which then carry over smoothly to the general inductive step, from x^k to x^{k+1}. Here is how the general method of computing the integral of an arbitrary positive power (*saṃkalitānayanasāmānyannyāyam*) is summarised:

....To make integrals of higher and higher powers, multiply the particular integral by the radius and remove from it the result divided by the number which is one greater [$I_{k+1} = xI_k - xI_k/(k+1)$ in our notation]. Thus divide the square of the radius by two, the cube by three, the fourth power by four, the fifth power by five. Thus divide the consecutive powers starting with the first (*ekaikottarasamaghātam*) by [the same] consecutive numbers; the results will be the integrals of powers in

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> increasing order. As the simple integral comes from the square, the integral of the square from the cube, the integral of the cube from the fourth power and so on, the power of an unknown (*rāśi*) when divided by the same number as the power will give the integral of one less power of the unknown. This is the method of producing integrals of all powers.

There are several mathematically interesting reasons for reproducing this brief paragraph, to which I will return shortly. But for the present, let us overlook the familiar prolixity (perhaps it was a class of dull students, the *mandabuddhi* so often addressed by Indian mathematicians when they wrote their expositions) and note the clarity with which the theorem itself is stated.

The Language

It bears repetition that, as in the excerpted passages, so in the entire text, no symbols are employed to represent the mathematical objects being manipulated, no formal notation for relations among them and operations on them, no diagrammatic guide to the geometric constructions invoked. The only devices which can be considered as perhaps artificial are, first, the use of common natural (generally Sanskrit-origin) words to denote precisely understood geometric entities: examples include *cāpam, jyā, śaram, bhūmi, bhuja, mukham*, etc; several of these and other similar terms, when translated, have exactly the same significance in European geometry. (The use of natural words as units of an artificial language appropriate for a given discipline is of course a widespread and contemporary practice even in extremely abstract contexts, see the current literature in mathematics and physics). Then there are the phrases whose sense is generally clear from the way they are formed as prescribed in the usual rules of compounding in Sanskrit (and Malayalam), e.g., *samalambacaturaśram, vyāsārddhavarggam*, etc.—they have no extra or hidden meaning apart from the literal. Finally, beyond these, there occur certain expressions whose literal meanings are inadequate to convey in full what they represent or may even be misleading. The most widespread instance of this in *YB* is *trairāśikam*, 'the rule of three', encompassing all properties of a set of four numbers in proportion, and hence all properties of a pair of similar triangles. In the passage on the area of the sphere, the term *piṇḍajyāyogam* is also such a 'package', standing for a certain refined procedure which effectively computes an integral: it means the sum of semichords of arcs of equal length into which a quadrant of a circle is divided, *in the limit of vanishing arc length*. The expression *khaṇḍāntarayogam* stands for an even more subtle procedure—*bhujākhaṇḍam* corresponds to the differential and [*bhujā*]*khaṇḍāntaram* to the second differential of the sine of an angle. (Thus the introductory sentence of the passage on the surface area of the sphere is equivalent to the statement that the second differential of sine is proportional to itself). Probably the best known instance in Kerala mathematics of such esoteric usage is the phrase *jīveparasparam* attributed (in *Tantrasaṃgraha*) to

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Mādhavan and representing the formulae for the sines of the sum and difference of two angles (together with the proof). Unwrapping such packages is essential for the understanding of the *yukti*, but they circumscribe the maximum deviation from the natural use of language that can be found in *YB*.

One other point about the language, or rather the general style of discourse, merits mention. The second section of chapter 7 of *YB* has the heading 'technical terms and definitions'. A demanding reader looking here for precise characterisations of the concepts and terminology necessary for the formulation and proof of the great theorem on the 'Newton' series to follow is likely to be disappointed. The section is neither systematic nor complete, but only a partial account of some of the geometric constructions and the associated nomenclature needed for the proof. In all of *YB*, the terminology itself does not have an invariable significance. To give one instance, the crucially important word *saṃkalitam* is used to mean both a finite sum as in earlier writing (see footnote [15]) and, in chapter 6, integrals of several different types. In the same chapter 6, the term *yogam* never means an integral, but only a finite sum whereas in chapter 7 we have, for example, *arddhajyāyogam* denoting the finite sum as well as the limiting integral. On the other hand, *saṃkalitam* is never used for the integrals occurring in surface and volume computations and, elsewhere in chapter 7, *khaṇḍāntarayogam* and *khaṇḍāntarasaṃkalitam* are distinguished.

Such instances can be multiplied. The question is: Is this lack of precision related in some intrinsic way to a reluctance or an inability to transcend the limitations of natural means of communication? Following from this: To what extent has the use of a dominantly natural language impeded the processes of conceptualising and developing the mathematics and then communicating it? The answer to the first question requires a thorough acquaintance with texts contemporary with and anterior to *YB* (many of them written in the not very natural language of terse mathematical Sanskrit) and is not for me to attempt. As for how well the mathematics was conveyed, if Tampuran and Ayyar could read the work four centuries after it was written and still make perfectly good sense of it—though they confess to having sometimes to guess at the author's train of thought—Jyeṣṭhadevan clearly has not made too bad a job of it.

In concluding these scattered remarks on the language of *YB*, I should note that its semi-natural style is in line with that of mathematical writing in Sanskrit over a long period.[16] We should also remember that the use of a natural language and comprehensibility do not always go together. While there are quite a few texts which were clearly meant to be understood by an informed reader without the need of a detailed commentary—Śankara Vāriyar's works are good examples from Kerala—there are also those, generally terse to the point of being cryptic, whose main purpose appears to have been the transmission of the core content of a body of knowledge and whose proper understanding requires thorough preparation or instruction. The prime example of such an esoteric text is of course *Āryabhaṭīya*.

[16] See Frits Staal's (1995) wide-ranging study of scientific Sanskrit as utilised in several disciplines, including mathematics.

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The Mathematics

What, now, are the other mathematical ideas and techniques, aside from the originality of the infinitesimal method and novel results it was used to prove, that are conveyed to us by the imperfect vehicle that is the natural Malayalam of *YB*? The order in which the various series occur in *YB*—first the numerical series for $\pi/4$, then the 'Gregory' series for general angle and then, in a veritable *tour de force*, the two 'Newton' series for sine and cosine—would seem not to be a matter of chance. It is a logical way to proceed if we accept that the motivation behind the search for a series of ever smaller terms whose sum approaches, in the limit, the value $\pi/4$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

has probably to be sought in the conviction, going back to Āryabhaṭa, that π is an irrational number and therefore cannot be written as the sum of a finite number of fractions. Nīlakaṇṭhan, Mādhavan's true mathematical heir and Jyeṣṭhadevan's teacher, has this comment (in his *Āryabhaṭīyabhāṣyam*) explaining why Āryabhaṭa said the value $\pi = \frac{62832}{20000}$ was only *āsanna* and, along the way, defining an irrational number. In loose translation: "Why is an approximate value given here rather than the true (*vāstava*) one? Because it cannot be expressed. A measure which measures the diameter without a remainder cannot measure the circumference without a remainder and *vice versa*. We can only ensure the smallness of the remainder, not its absence". The lack of 'remainderlessness' is naturally accommodated by an infinite series expansion (though of course every infinite series of fractions does not sum up to an irrational number, as Nīlakaṇṭhan well knew, *cf.* his remarks on convergent geometric series in the same text), in a manner not very different from the way in which infinite continued fractions arise in the use of the Euclidean algorithm.

As described in *YB*, the generalisation from an infinite series representation for the number $\pi/4$ to a power series expansion for an arbitrary angle $\theta(\theta \leq \pi/4)$ in terms of $\tan\theta$:

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$$

involves, technically, only a small step, an elementary application of *trairāśikam*. It has no computational difficulty or significance and helps only marginally in determining a more accurate value for π—replacing $\pi/4$ by $\pi/6$ ($\tan\pi/6 = 1/\sqrt{3}$), for example, improves the convergence slightly. But conceptually it is a giant leap, leaving behind the prop of the irrationality question and towards a notion of what we call today a function. It can be credibly argued that it is this step and the series expansions of $\sin\theta$ and $\cos\theta$ in powers of θ that together mark the advent of calculus, not just as a technique of

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calculation as in the expansion of $\pi/4$, but as the beginning of a new discipline of analysis. The basic operations of calculus make their appearance as needed in the course of these developments: differentials of the first and second order (which are all that is required for the sine and cosine functions), definite and indefinite integrals, the rule for integration by parts (in computing recursively the integral of a general power), the notion of iterated integrals (which make sense only if an indefinite integral is understood as defining a new function, as we would say today), the solution of an elementary differential equation, namely $d^2y/dx^2 + y = 0$ for the sine and cosine functions, etc. Though the motivating impulses may not have been the same, it is uncanny to see infinite series ('equations with an infinite number of terms' in Newton's language) playing such a decisive part alongside the basic concepts of differentials and integrals ('fluxions' and 'fluents') also in the early evolution of European calculus and analysis.

The assurance with which *YB* handles these power series should not really be surprising once we recognise their close affinity with the recursive methods used systematically in diverse areas of Indian thought, including earlier mathematical work. The most direct parallel is perhaps with the positional notation for, say, positive integers. In his talk at this Workshop on the positional 'language' for the writing of numbers, John Kadvany[17] highlighted the fact that the positional representation of any positive integer is no more than an abbreviation for a polynomial whose value it is, with coefficients from a finite set of non-negative integers (0 to 9 in the decimal base case) when the variable is fixed at a positive integer (=10 in the decimal case)—as Kadvany stresses, what makes this representation possible and powerful is the fact that integers can be added and multiplied to get other integers; in technical language, they form a ring (once we include also negative integers).[18] So do polynomials, in the present case in one variable, and this common abstract structure is the key to the positional notation. (When the variable is fixed at an integer, say 10, the uniqueness of the positional representation is assured only if the coefficients are restricted to be less than 10 and the usual carry over rules are imposed; an exact correspondence between integers and their positional representatives requires this to be factored in). Arbitrarily large integers are so representable by polynomials of arbitrarily large degree; there is no upper bound on the integers that can be positionally represented. Indeed, *YB* invokes this property of positionally defined numbers in the very first section (entitled "The nature of numbers", *saṃkhyāsvarūpam*) of chapter 1 with the remarkable sentence: "If we endow numbers with multiplication and positional variation, there is no end to the names of numbers; hence we cannot know [all] the numbers and their order".

[17] John Kadvany (2007, this issue).

[18] A ring is a set whose elements can be added and subtracted as well as multiplied. Division need not be defined and the multiplication may be noncommutative, i.e., depend on the order of the factors. The rings mentioned in the text are all commutative.

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The transition from the positional representation of unboundedly large integers to the trigonometric power series now appears natural: substitute for the value of the base used (10 for decimal) the appropriate variable ($\tan\theta$ for 'Gregory', θ for 'Newton') and allow the coefficients to take values in certain determinate fractions, both positive and negative. The set of formal (i.e., ignoring possible nonconvergence) power series with coefficients which are fractions do form a ring just as polynomials do.

Related to this is the point that the coefficients of a power series obviously cannot all be enumerated. They can be specified by, and effectively only by, recursive rules in some variant or another. In *YB* this is done in a number of slightly different but equivalent ways. In the statement of the 'Gregory' series expansion, the first three coefficients are numerically given and then one is instructed to take the general coefficients as the reciprocals of consecutive odd integers with the signs alternating. For the 'Newton' series:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots.$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots,$$

the recursive specification of the coefficients is more explicit. The coefficient s_{2k+1} of θ^{2k+1} in the sine series is to be computed as

$$s_{2k+1} = \frac{s_{2k-1}}{(2k)^2 + 2k}.$$

Together with the value $s_1 = 1$, also specified, repeated use of this recursion formula leads to

$$s_{2k+1} = \frac{s_{2k-1}}{2k(2k+1)} = \frac{s_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots = \frac{1}{1.2\ldots(2k+1)}.$$

A similar rule is given for the cosine series. It should be noted that the use of recursion is not limited to just the statement of these formulae in an economical form. It is an organic part of the development; the formulae for the coefficients introduce themselves recursively, so to speak, in the very process of their computation. There is, moreover, a unity in the way the coefficients in all these series arise. In the 'Gregory' series they are definite integrals (from 0 to 1 or 0 to $\tan\theta$) of even powers and in the two 'Newton' series they are iterated integrals of even and odd powers respectively, with positive and negative signs alternating. It is accurate to say that the constructive proof of their correctness is itself recursive (though I have to add that the proof is one of the least transparent passages in all of *YB*).

Perhaps the best illustration from *YB* of the use of recursion to establish the validity of an infinite sequence of propositions is the short passage on the

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calculation of the general *saṃkalitam* given in translation earlier together with the two sections of *YB* which precede it. The untranslated (for reasons of space) sections give a recursive proof for the correctness of the values of the integrals of powers. First I_2 is computed in terms of I_1, then I_3 in terms of I_2, both in detail. This is followed by the statement that I_4 can be computed from I_3 and, generally, I_{k+1} from I_k in the same way. It is perfectly obvious that the recursive method cannot work if all values of k, even and odd, are not considered together, even though the series for $\pi/4$ requires I_k to be determined only for even k. Once the first step in the recursion, I_2 from I_1, is done, generalisation is straightforward, involving as the common basic step a recognisable finite form of integration by parts. Indeed, a hint that the recursive nature of the computation was fully understood is available in the choice of the term *mūlasaṃkalitam* for I_1, the root *saṃkalitam* from which all others are generated.

How close does all this come to the method of mathematical induction as it is understood today? A modern formulation of the principle of induction, essentially deductive, will go something like: The truth of an infinite sequence of propositions $P_k, k = 1, 2, \ldots$ is established if (i) P_1 is true and if (ii) P_k is true implies P_{k+1} is true for all k. Lacking the necessary symbolic aptitude, Mādhavan and his followers could not possibly have expressed their computation-oriented reasoning in such abstract and elegant language. But for one not versed in logical niceties, it is difficult to see that their use of recursion, not only to generate an infinite sequence of true mathematical statements but also to prove them, is in any fundamental way different, once allowance is made for the general preference in India for constructive as opposed to deductive proofs.

From a modern perspective, there are of course other issues raised by *YB*'s handling of infinite series, chief among them the question of convergence. Not surprisingly, *YB* is not concerned in any serious way with this question. It is striking however that all the series occurring in *YB* are convergent for the considered range of values of the variable. In particular, the geometry of the derivation of the 'Gregory' series works as it is described only for angles in the first octant which, together with the fourth, fifth and eighth octants, is also the range in which the series converges.

But overshadowing everything else is the pervasive presence of a new way of addressing long-standing problems arising in the geometry of the circle and the forging of the tools required for their successful solution—in other words, the conceptual basis and the techniques of calculus.

The Generosity of Language, the Power of Abstraction

Reading *YB* from today's vantage point, it is possible to imagine several directions along which the development of calculus as depicted in it could have been done with greater generality, without straining the abundant conceptual and computational resources it draws upon. To try to identify these roads not taken and to speculate on why they were not taken is, in spite of the

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risks, a temptation not easily resisted. I shall limit myself to one particular instance of a missed turn which, in my view, clearly brings out the possible role of the language in which the mathematics was 'done', as distinct from how it was communicated, in inhibiting the requisite degree of abstraction and generalisation.

But first a remark, not directly related to language, on an issue *YB* does not concern itself with, that of the irrationality of π. At first sight this is a surprising omission, given the central position occupied by the geometry of the circle in *YB*'s calculus, especially if, as I have suggested above, the infinite series expansions grew out of the search for a method of 'controlling' an irrational number like π. Also as we have seen, the person who asserted this irrationality was none other than Nīlakaṇṭhan, Jyeṣṭhadevan's teacher and the author of *Tantrasaṃgraha*, the source book for *YB*. One reason for the silence may be technical, that a *yukti* for the assertion was beyond the computational methods at the command of the Kerala school—after all, it was only in 1761 that in Europe Johann Heinrich Lambert proved Nīlakaṇṭhan's conjecture (without of course knowing it as such). But a more likely explanation may lie in the philosophical underpinning of Indian science as a whole throughout its long history. Indian mathematicians, unlike the Greeks, seem never to have come to grips with proofs of the irrationality of any number, not of the far easier case of $\sqrt{2}$ nor of $\sqrt{10}$, often used as an approximation for π at the time of Āryabhaṭa. Almost certainly, this failure has to do with the necessity of having to use *reductio ad absurdum* methods in any such putative proof. For the Greeks, proof by contradiction, involving as it does the principle of the excluded middle, was legitimised by the authority of Aristotle. But in India the view that *P* and not*P* are complementary and mutually exclusive was regarded with distrust at least by some and from at least the time of the Buddha.[19] Subsequently, reasoning by contradiction (*tarka*, a word which has other meanings as well) was much debated but generally not accorded full admissibility.[20] Like all scholars, mathematicians would have had a good

[19] According to the Pali canon, scepticism or agnosticism about the validity and/or usefulness of the rule of the excluded middle was expressed by the Buddha himself. The list of questions on which the Buddha is said to have declined to take a position includes some concerning the nature of the physical world, e.g.: Does the universe have a finite extent or not or neither or both? The great congress (3rd century BCE) which saw the parting of the Mahasanghikas from the orthodoxy following a ferocious debate, and whose proceedings are recorded in the *Kathāvattu*, not only made references to such 'multi-valued' propositions, but also discussed rules of debate and disputations. It appears reasonable to suppose that a degree of scepticism about the disjointness between a property and its negation marked the early Buddhist approach to physical phenomena, regardless of what the Buddha actually might or might not have said on the subject.

[20] In Indian philosophy, the term *tarka* signifies "some sort of *reductio ad absurdum* where an appeal to some absurdity or absurd consequence is made in order to lend an indirect support to a positive thesis [by showing] in fact that the opposite thesis leads to absurdities" (Bimal Krishna Matilal (1986, p. 79)). The widespread reluctance to accept *tarka* as "a means leading to a positive piece of knowledge" is also discussed there.

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grounding in other sciences (this certainly was the case in Kerala at the relevant time[21]), and it is almost inconceivable that they did not know of the reservations regarding recourse to *tarka* as a means of establishing a logical truth—*YB* contains not a single instance of it; elsewhere in India, it is not till a hundred years later that we come across its first hesitant and insecure use in mathematics.[22] The fact remains that this disdain denied Indian mathematics the use of a powerful proof device. Contrast this with Newton's free use of it in the *Principia* (Book 1) whose reliance on infinitesimal geometry (but with diagrams supplied), especially of similar triangles, is otherwise so very reminiscent of the Kerala techniques.

The missed turn I wish to focus on concerns the generalisation of Mādhavan's series expansion of $\sin\theta$ in powers of θ with numerical coefficients ('around $\theta = 0$') to an expansion around some nonzero value of θ, namely an expansion of $\sin(\phi + \theta)$ in powers of θ with coefficients depending on ϕ. This would amount to a generalisation of the Maclaurin series for the sine function (and the cosine function) to the corresponding Taylor series. Since such expansions of general (sufficiently 'good') functions f:

$$f(x+y) = f(x) + y\frac{df}{dx} + \frac{y^2}{2!}\frac{d^2f}{dx^2} + \cdots$$

are relatively early landmarks in European calculus, the question whether Mādhavan's interpolation formula can be thought of as giving the first few terms of the Taylor series of the sine function has recently provoked some debate.[23] It may or may not be a satisfactory interpolation, but it certainly cannot be thought of as the beginning of the Taylor expansion—there is a numerical mismatch in the coefficient of the cubic term.

If I were time-transported to the 15th–16th century as a worthy member of the Kerala community of mathematicians I could have pointed out (in *YB*'s irresistible prose style) that, combining (i) *jīveparasparam* (the formula for $\sin(\phi + \theta)$), (ii) *jyānayanam* (the method of determining an arbitrary chord to arbitrary accuracy, i.e. Mādhavan's sine and cosine series) and (iii) *bhujākhaṇḍam* and *bhujākhaṇḍāntaram* (the first and second differentials of sine and cosine), the (correct) Taylor series for the sine function will result. In detail,

[21] For an examination of the epistemological roots of Kerala mathematics, from the writings of its most thoughtful and articulate representative (Nīlakaṇṭhan), see Roddam Narasimha (2007, to appear). It would seem that the study of the impact of philosophical systems on mathematical thinking in the Indian context is still in its infancy. I thank Frits Staal for an elucidation of the status of the 'principle of non-contradiction' in Indian thought and of the nuances one must be sensitive to in interpreting Indian philosophical and logical writings on this issue.

[22] Srinivas (2005).

[23] A good starting point for following the debate regarding Mādhavan's 'wrong' interpolation formula is Kim Plofker (2005). The article is also a valuable step-by-step guide to the computation of the interpolation formula as given in Śankara Vāriyar's *Yuktidīpika*.

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$$\begin{aligned}\sin(\phi+\theta) &= \sin\phi\cos\theta + \cos\phi\sin\theta \\ &= \sin\phi(1-\frac{\theta^2}{2!}+\cdots) + \cos\phi(\theta-\frac{\theta^3}{3!}+\cdots) \\ &= \sin\phi + \theta\cos\phi - \frac{\theta^2}{2!}\sin\phi - \frac{\theta^3}{3!}\cos\phi + \cdots\end{aligned}$$

This is computationally trivial and, by truncation, produces an interpolation formula to any order in θ. Furthermore, *all* derivatives of the sine function are determined by *khaṇḍāntaram* (as also I would have pointed out):

$$\frac{d^{2n+1}\sin\phi}{d\phi^{2n+1}} = (-1)^n\cos\phi, \quad \frac{d^{2n}\sin\phi}{d\phi^{2n}} = (-1)^n\sin\phi$$

and similarly of the cosine function. So Mādhavan's expansion, when shifted from $\theta = 0$ to some (any) other point on the circle, together with a realisation that, abstractly, a circle (a 'uniformly round' figure) does not distinguish one radius from another, is exactly the Taylor series.

At issue here is not whether calculus was invented in India. That question is already answered in the affirmative in the material available in *YB*: though in an apparently different language, it is unmistakably the calculus of Newton and Leibniz, but applied *only to functions corresponding to arcs of the circle*, $y = (1-x^2)^{\frac{1}{2}}$ or, parametrically, $x = \cos\theta, y = \sin\theta$. What Kerala mathematics did not have was an appreciation of the great generality of the concepts and methods it had in hand and deployed so effectively; what was missed is the power of abstraction.

To fully appreciate the extent of this power, we only need to call to mind how mathematics (and, to a lesser extent, the other exact sciences) has evolved over the past few centuries, driven as it is by an increasing emphasis on the structural properties of the sets of objects being studied, to the disadvantage of predominantly calculational techniques. When mathematical objects belonging to a set are characterised by the operations that can be carried out on them and the rules the operations have to obey (the relations among them) we have, effectively, defined those objects in terms of the structure of the set. An object has those properties and only those that come from its belonging to a structurally defined set, bringing with it the freedom to transcend the particular circumstances in which it may first have presented itself. Thus we may wish to study the set of transformations that take a 3-dimensional cube into itself and may soon realise that the resultant of any two such symmetry transformations is also a symmetry transformation, their product, uniquely determined by the first two taken in order; that every symmetry can be undone by another, its inverse, also uniquely determined; etc., in short that it belongs to a class of sets called groups. Already at this stage we have gone through two levels of abstraction. At the first level, we have written down the multiplication table—the specification of the resultant of each ordered pair of symmetry transformations—for the group of

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symmetries of the cube. With this table in hand, we know the group fully: there is no proposition concerning such transformations, no matter how complicated or subtle, whose truth cannot be decided by the table alone, without reference to the cube. It follows that all groups of transformations of any 'physical' entity whatever, indeed all groups not even necessarily of transformations, are abstractly the same group if they have the same multiplication table. At the second, deeper, level, we would recognise that, though the group of symmetries of a cube and of, say, a regular tetrahedron are not the same, i.e., do not have the same multiplication table, not even the same number of elements, they still share a common abstract structure, the structure of a group. We would then proceed to elaborate the (fundamentally syntactic) properties that *all* groups must have; in short we would make a general theory of groups. Essentially identical considerations apply to all algebraic structures, for instance to the rings that we invoked in connection with the positional notation.

It is obvious that I could have made my point more concisely and elegantly, in fact more powerfully, if I had taken recourse to the language that developed to accommodate the abstract structural point of view, a language primarily algebraic: symbols for the various groups under consideration and for their elements, a notation for the group operations, an economical statement of the relations, etc., the language in which theorems/'truth' will be presented/expressed. Everyone now knows that group structures occur in virtually all sciences,[24] that modern physics in particular has been revolutionised by a systematic exploitation of the formal, syntactic, understanding we have acquired of these structures. As for mathematics, and staying with groups, I only mention the recently completed classification of all finite simple groups (the recursive building blocks of all finite groups) as a general illustration of the generative power inherent in the artificial language of algebra. Without that as a vehicle of thought and communication, it seems inconceivable that we could have reduced mathematical reasoning and creativity to an application of "rules without meaning" so successfully.

Rules without meaning but not without purpose. Decisions about what abstract (syntactic) structures are likely to prove 'interesting' and even judgements about what results are 'deep' or 'beautiful' have still to be made semantically, by reference to the contexts in which questions arise and are resolved. The tension between problem-solving and theory-making, the virtual inseparability of content and language, is arguably the main source of the extraordinarily wide scope and power of mathematics as it is practised now (and is perhaps also the reason for the "unreasonable effectiveness of mathematics in the physical sciences"). Once the creative process of discovery is thus algebraised and put on autopilot as it were, there is no limit to what is discoverable—algebra is truly, infinitely, generous.

Moving back from these generalities to the Kerala of the 15th and 16th centuries, one cannot of course be dogmatic about the extent to which pro-

[24] For an unusual instance from the social sciences, see A. Weil (1949).

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gress towards greater abstraction and hence greater generality was inhibited by the spurning of symbolic methods. But on the evidence of *YB*, it cannot be doubted that obvious general points of view were overlooked or neglected. As the episode of the Taylor series makes clear, an intuitive feeling for the symmetry of the circle (the corresponding group being the group of *all* rotations around the centre, which is not a finite group but still, structurally, a group) did not translate into the precise understanding that this symmetry denies a privileged role to any particular diameter, even if it is called the east-west direction. I like to think that if only they could have been persuaded to designate a diameter as the line AB or $\alpha\beta$ or a Malayalam equivalent thereof, the Kerala mathematicians might have gone on eventually to a more general, less concrete, appreciation of their own achievement.

What is puzzling in all this, at least to a non-expert, is that abstraction and symbolic representations of objects and their relationships were not strangers to Indian thought. Grammar (first and foremost), prosody and the classification of meters, cosmogonic speculation, systematisation of the rules of reasoning and several other areas of intellectual inquiry come to mind as having been greatly enriched by a fundamentally structural point of view. In mathematics, the positional notation for numbers is itself a triumph of structure over clumsy description; as we have seen, it is pure syntax. Bhāskara the second, who lived not more than three centuries before the efflorescence of mathematics in Kerala and whose work was known and cherished there, was a true algebraist; his *Bījagaṇita* proposes the use of the names of colours to symbolically represent variables in equations, as distinct from numerical values for them, and states the rules for manipulating them, exactly as if they were numbers. (In a strange echo, modern physics employs the term chromodynamics for the interactions of quarks of different 'colours'). Bhāskara's algebraic legacy seems to have been better nourished in the Arab lands than among his own mathematical heirs in India. Perhaps, at the time and place we are talking about, mathematics was seen as no more than a handmaiden of astronomy, with no high intellectual standing of its own. But then, what else but intellectual curiosity can have inspired a Nīlakaṇṭhan to cogitate on the irrationality of π or a Mādhavan to compute it to thirteen decimal places?

Close to three hundred years after Mādhavan, Newton and Leibniz laid the foundations of calculus as it is taught today and their 18th century followers gave it the form and language that we still follow, more or less faithfully. That came after the first integrations were performed in Europe, by Cavalieri, Fermat and a host of others (perhaps only coincidentally, the integrals of powers). The context in which this happened was vastly different from the one in which the heroes of our narrative perfected their infinitesimal geometry of

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the circle. Descartes had already algebraised geometry[25] and both Newton and Leibniz were well versed in the Cartesian method and philosophy. The first ideas about a general function of a variable were taking shape and the first correspondences between functions and the simplest curves, the conic sections, beginning to be understood. The Newton-Leibniz calculus reflects this great opening up, especially in the willingness and ability to handle more general functions. In the case of Newton, the Newton of the *Principia* in particular, a degree of generality could not, in principle, have been avoided; the position of a mass-particle as a function of time is *a priori* unknown, being determined, in terms of a given force, only after the equations of motion are solved. (In contrast, with the early exception of Āryabhaṭa with his belief in a spinning earth and ideas on relative motion, Indian theories of the motion of celestial bodies were almost entirely descriptive, paying little heed to possible underlying causes). It is in this fundamental respect, in the recognition that the infinitesimal method is of universal applicability, that the European calculus of the late 17th century—even while admitting that an acceptable foundation for it was not laid till the middle of the 19th—can be seen to have gone far beyond Mādhavan's pathbreaking achievements. If history were different and granted them the time, could Mādhavan's followers have taken their own road one day to this high ground? Going by what *YB* tells us, we must remain sceptical. For one thing, there is the much-discussed difference in the European and Indian approaches to doing mathematics, summarised in the catch words 'deductive' or 'axiomatic' and 'constructive' or 'computational'. Our look at (a tiny sample of) *YB*'s contents provides, I think, enough evidence that a more severe obstacle was the aversion to a spare and refined language and the consequent absence of a structural or general point of view.

But what history brought was the Portuguese incursion into Kerala, signalling another period of strife and disorder. The resistance to the invaders was led by none other than the Zamorin who, in his other traditional role, was overlord and protector of the temples and patron of learning. The delta of the river Nila saw much violence and bloodshed during the very time in which, half a day's walk away, Nīlakaṇṭhan, Jyeṣṭhadevan and others were teaching and writing down their new mathematics in their temple villages. All the great texts date from this turbulent 16th century and with its end mathematics in Kerala also went into terminal decline.

[25] Two and a half centuries before Descartes, Nicole Oresme (or Nicolas d'Oresme) had taken the first step to liberate space from its Euclidean literalness and give it a more metaphorical role by plotting on a plane the graph of a 'function' of a 'variable'. I owe my acquaintance with Oresme's work to David Mumford, who is also responsible for the characterisation above of its significance. The Indian view of geometry is, if anything, more down-to-earth than the Greek—compare (i) the use of the terms *bhūmi* and base for the 'horizontal' line in a polygon in the two traditions, (ii) *YB*'s instruction to 'drop a plumbline' with drawing (or dropping!) a perpendicular, not necessarily to a base, and several other similar instances. Notable coincidence: Oresme and Mādhavan (if we accept the conventional chronolgy) lived at about the same time, the former being the senior by 20 or 30 years.

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Paper 48

Kim Plofker (2009)

The School of Mādhava in Kerala

Excerpt from:
Kim Plofker, *Mathematics in India*, Princeton University Press, 2009, Chapter 7.

Kim Plofker

For a biographical note on Kim Ploker see p. 648.

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Chapter Seven

The School of Mādhava in Kerala

7.1 BACKGROUND

Probably the most famous school in Indian mathematics, and the one that produced many of its most remarkable discoveries, is the *guru-paraṃparā* or "chain of teachers" originating with Mādhava in the late fourteenth century and continuing at least into the beginning of the seventeenth. These scholars lived in the region known as Kerala on the southwestern coast of India, in its central part between modern Kochi (or Cochin) and Kozhikode (or Calicut). What survives of their work includes writings in Sanskrit and in the local Dravidian vernacular called Malayalam. In astronomy they are generally considered to be followers of the Ārya-pakṣa, but they also wrote on texts in other *pakṣas*, as well as on astronomical systems unique to Kerala.[1]

A narrow strip of land between the Western Ghat mountains and the Arabian Sea, Kerala in the mid-second millennium maintained a distinct regional culture without being entirely isolated from the neighboring parts of southern India. Moreover, its pepper production and geographical location had made it a major international hub, with trading connections stretching back for many centuries. Its political rulers were generally members of the group known as Nayars, a set of non-Brāhmaṇa castes specializing in war, defense, and governance. Less politically powerful but socially superior to Nayars were the landowning Brāhmaṇa castes, especially the preeminent Nampūtiris, who officiated in (and controlled the property of) Hindu temples. The non-Brāhmaṇa castes known as Ambalavāsis also served in temples, in subordinate capacities. As a trading center, Kerala supported significant populations of foreigners and non-Hindus. Arab merchants wielded great influence in the coastal port towns, and substantial Muslim communities of expatriates and local converts formed there. There were also numerous Nestorian Christians and a few Jews.[2]

A number of non-Brāhmaṇa Hindu groups, including the Nayars and the Ambalavāsis, followed matrilineal systems of descent and inheritance. The

[1] The terms "Keralese" and "Keralan," commonly used to identify this mathematical school and the culture of its practitioners, are widely deprecated as incorrect; the preferred adjectival derivative of "Kerala" is "Malayali," which is probably unfamiliar and potentially confusing to most readers. In this work I have relied on paraphrase to avoid the issue.

[2] [Pani1960], pp. 4–16.

Nampūtiris, on the other hand, maintained not only the more usual Indian patrilineal tradition but also a sort of extreme primogeniture custom, according to which only the eldest son could be the head of the family, enter into a formal marriage, or father legitimate children. These two types of family structure developed a sort of symbiosis whereby a Nampūtiri younger son could form a *saṃbandham* or recognized quasi-marital relationship with a woman in a matrilineal caste, without detriment to the legitimacy of her children within that caste. Professionally, too, the Nampūtiris were associated with lower caste Ambalavāsis in their joint service to the temples.[3]

These social factors allowing for collaboration and even close kinship between members of technically separate groups may help explain the rather unusual caste flexibility in the Mādhava school. Despite traditional prohibitions on low-status groups sharing the Sanskrit intellectual heritage that was considered the province of Brāhmaṇas, it appears that some Ambalavāsis, such as Śaṅkara Vāriyar and Acyuta Piṣāraṭi (see below), were accepted as not only students and commentators but even teachers of Nampūtiri Brāhmaṇas. Apparently such flexibility persisted in Kerala scholarship, judging from the complaints of an anonymous family member of the Azhvāñceri Tamprākkaḷ (the title of the leader of the Nampūtiri caste), writing probably in the mid-eighteenth century: "In the Malayalam area, those who are not eligible to learn *jyotiṣa* study it. And Brāhmaṇas ask them about [astrological topics]. Since neither of these things is proper, I, with a view to change this state of affairs through making the Brāhmaṇas learn *jyotiṣa*, summarized the most essential matters thereof."[4] As we will see, though, many Nampūtiri Brāhmaṇas were masters of *jyotiṣa* themselves. They acquired their Sanskrit learning through instruction at their family estates or *illams*, and perhaps also in temple schools.[5]

7.2 LINEAGE

Nothing is known about Mādhava's own background except that he was a Brāhmaṇa of the caste known as Emprāntiri, somewhat inferior in status to the Nampūtiris, and that his home or *illam* Ilaññipaḷḷi (Sanskrit Bakulavihāra) was in Saṅgamagrāma (modern Irinjalakuda, lat. 10°20′ N, near Kochi/Cochin). His most famous mathematical achievements are the Mādhava-Leibniz series for $\pi/4$ and the Mādhava-Newton power series for the Sine and Cosine, but this work now survives only in a few verses recorded by later members of his school. There are, however, several existing astronomical texts by Mādhava in which Śaka-era dates corresponding to 1403 and 1418 CE are mentioned, so it is generally presumed that he was born sometime around the middle of the fourteenth century. His successors frequently

[3] [Put1977], p. 26.
[4] [SarK1972], p. 71.
[5] [Put1977], pp. 25–27.

referred to him in Sanskrit as "*Gola-vid*," "one who knows the sphere."[6]

The only known direct pupil of Mādhava was a Nampūtiri named Parameśvara, whose *illam* Vaṭaśśeri (Sanskrit Vaṭaśreṇi, lat. 10°51′ N) lay on the Bharathapuzha (or Niḷā) River.[7] A very long-lived and prolific scholar, Parameśvara wrote at least twenty-five separate works on astronomy, mathematics, and astrology, several of which were commentaries on the major medieval *siddhāntas*. His work in astronomy included a long series of eclipse observations beginning around 1393, and he was apparently still active during the training of his son's student Nīlakaṇṭha (born 1443), whose writings mention certain statements that "Parameśvara told me."[8] Moreover, Nīlakaṇṭha remarks that Parameśvara's 1431 astronomical text *Dṛg-gaṇita* was composed "after studying many *śāstras*, observing [for] a period of fifty-five years, [and] observing eclipses, planetary conjunctions, and so forth by means of instruments."[9] If correct, this would push the start of Parameśvara's observational activities back to 1376, giving him a lifespan of well over ninety years from perhaps the 1360s to the 1450s.[10]

Parameśvara's chief student was his own son Dāmodara, whose works do not survive and whose dates are unknown. Dāmodara's astronomical writings are briefly quoted by one of his pupils, the abovementioned Nīlakaṇṭha. This Nīlakaṇṭha (sometimes additionally identified as Gārgyakerala, from his membership in the Garga *gotra*, and Somayājin or Somasutvan, "performer of the *soma*-sacrifice") was a Nampūtiri of the Keḷallūr *illam* (Sanskrit Keralasadgrāma) in Tṛkkaṇṭiyūr (Sanskrit Kuṇḍapura) near modern Tirur. He pursued his *jyotiṣa* studies with Dāmodara at nearby Vaṭaśśeri and was apparently a protégé of the contemporary Azhvāñceri Tamprākkaḷ, one Netranārāyaṇa, who was interested in astronomical theories. Nīlakaṇṭha's own dozen or so compositions all dealt with astronomy; in the best known of these, the *Tantra-saṅgraha*, he gives a date for his birth that corresponds to about 14 June 1444. Nīlakaṇṭha quotes the *Tantra-saṅgraha* in another and hence later work, a commentary on the *Āryabhaṭīya*, which he says he wrote when "advanced in age"—but evidently not too advanced to write at least two more books thereafter, both of which quote this commentary. In fact, if it is correct to identify Nīlakaṇṭha with one "Keḷallūr" who is mentioned as a contemporary authority in a 1543 Malayalam astrological text, he remained active in the field even as a near-centenarian![11]

[6] [Pin1970a], vol. 4, pp. 414–415, and [SarK1972], pp. 51–52.

[7] The location of Vaṭaśreṇi, Aśvatthagrāma, is frequently identified with the village of Ālattūr [Pin1985], [Pin1970a] 4, p. 187, [SarK1972], p. 52. [Kup1957], p. l, describes Vaṭaśreṇi as located where the Niḷā "joins the sea," from Parameśvara's own phrase, "on the north bank of the Niḷā on the ocean" (or "lake"?). Modern Ālattūr, however, is located not at the mouth of the Niḷā but inland on the Gayatripuzha, a tributary of the Niḷā.

[8] *Āryabhaṭīya-bhāṣya* Gola 48, [Kup1957], p. lii.

[9] *Āryabhaṭīya-bhāṣya* Gola 48, [Kup1957], p. lii, and [Will2005], sect. 4.14.4, p. 261.

[10] Slightly varying estimates of Parameśvara's dates are given in [Pin1970a], vol. 4, pp. 187–192, [SarK1972], pp. 52–54, and [Kup1957], pp. l–liv.

[11] [Pin1970a], vol. 3, pp. 173–177, and vol. 4, 142; [SarK1977b], pp. xxiv–xxxviii; and [SarK1972], pp. 55–57.

Dāmodara had a second student, named Jyeṣṭhadeva, also a Nampūtiri Brāhmaṇa, from the Paraññoṭṭu (Parakroḍa) *illam* in Ālattūr. His major (and possibly only) work, the *Yukti-bhāṣa* or "Vernacular [exposition] of rationales," explains in Malayalam prose detailed proofs of the mathematical procedures stated in the *Tantra-saṅgraha*. Although this is the first known work in the Mādhava school to preserve such rationales in detail, it seems reasonable to think that the reasoning involved in some of them must have been worked out much earlier, especially the ones that deal with rules attributed to Mādhava himself.[12]

Nīlakaṇṭha and perhaps also Jyeṣṭhadeva were the primary teachers of Śaṅkara, an Ambalavāsi of the Vāriyar caste, whose family were employed at the shrine of Śiva at Tṛkkuṭaveli (Sanskrit Śrīhutāśa) near modern Ottappalam on the Niḷā River. Śaṅkara wrote an extensive verse commentary on the *Tantra-saṅgraha*, the *Yukti-dīpikā*, or "Lamp of rationales," which is largely based on the *Yukti-bhāṣa* of Jyeṣṭhadeva, as well as a concise prose one, the *Laghu-vivṛti*, or "Short commentary," said to be his last composition. Other works known or thought to be by Śaṅkara include a lengthy but unfinished mostly prose commentary *Kriyā-kramakarī* ("Performing the steps for calculation") on the *Līlāvatī*, which mentions a date in 1534; an astronomical handbook composed probably about the time of its epoch date in 1554; and an astronomical commentary dated to 1529. Śaṅkara's working career thus seems to have roughly coincided with the middle third of the sixteenth century.[13] Śaṅkara's other acknowledged teachers included Nīlakaṇṭha's patron Netranārāyaṇa and another Nampūtiri Brāhmaṇa, Citrabhānu, known as the author of an astronomical work dated to 1530 and a small treatise with solutions and proofs for algebraic equations.[14]

Other, more obscure scholars in this pedagogical lineage include Nīlakaṇṭha's younger brother, Śaṅkara, and several anonymous pupils and commentators.[15] In addition, one Nārāyaṇa of a family named Mahiṣamaṅgala completed Śaṅkara's unfinished *Kriyā-kramakarī* on the *Līlāvatī*, as well as a *Līlāvatī*-commentary of his own and a few astronomical works, one of which mentions the year 1607. This Nārāyaṇa was the protégé of local mahārājas who ruled in the mid- and late sixteenth century and is much better known for his poetic and liturgical writings than for his achievements in *gaṇita*.[16] The Mādhava school's last identifiable member is another Ambalavāsi, Acyuta Piṣāraṭi, from Nīlakaṇṭha's village of Tṛkkaṇṭiyūr. Acyuta was a pupil of Jyeṣṭhadeva and a protégé of the local ruler Ravivarman of Veṭṭatunād (Prakāśa). His ten or so astronomical works include one composed in 1593, in which he refers to "the good teacher Jyeṣṭhadeva." Acyuta was also learned

[12] [Pin1970a], vol. 3, pp. 76–77, [Pin1985], [SarK1972], pp. 59–60, and [Srin2004b], Appendix 3. The *Yukti-bhāṣa* or *Gaṇita-Yukti-bhāṣa* has recently been edited and translated in [SarK2008], and discussed in [Div2007].

[13] [Pin1985], [SarK1972], pp. 58–59, [SarK1977b], pp. xliv–lxx.

[14] [Pin1970a], vol. 3, p. 47, [SarK1972], p. 57, [Hay1998].

[15] [SarK1972], pp. 57–61, [GolPi1991].

[16] [Pin1970a], vol. 3, pp. 150–151; [Pin1970a], vol. 4, p. 137; [SarK1975], pp. xxii–xxiv; [SarK1972], pp. 57–58.

in grammar, medicine, and poetics; he instructed the famous poet and grammarian Nārāyaṇa Bhaṭṭatiri, whose obituary verse on Acyuta gives a date for his death corresponding to 7 July 1621.[17] Many other Kerala scholars wrote on astronomy and mathematics during this period, but most of them cannot be definitely identified as part of the pedagogical family tree originating with Mādhava.[18]

7.3 INFINITE SERIES AND OTHER MATHEMATICS

The crest-jewel of the Kerala school is generally considered to be the infinite series for trigonometric quantities discovered by its founder, Mādhava. These are preserved in about a dozen verses explicitly attributed to Mādhava in various works by later members of the school, and translated in the following sections. Many of the accompanying rationales and explanations of these results are probably also originally due to Mādhava, even if not directly credited to him.

The quotations translated here are taken mostly from the commentaries *Yukti-dīpikā* and *Kriyā-kramakarī* by Śaṅkara: the *Yukti-dīpikā* on the first several verses of the second chapter of the *Tantra-saṅgraha* dealing with trigonometry, and the *Kriyā-kramakarī* on verse 199 of the *Līlāvatī*, where Bhāskara introduces the geometry of the circle. Some of the sources cited in the notes also draw on the Malayalam *Yukti-bhāṣa* of Jyeṣṭhadeva. For convenience, the topic headings in sections 7.3.1 and 7.3.2 identify their subjects in terms of modern trigonometric concepts, such as π and arctangent, and the names under which the results are most familiar in modern mathematics (e.g., "Leibniz's series," "Gregory's series").

7.3.1 Mādhava on the circumference and arcs of the circle

Mādhava's numerical value for π. Several verses ascribed to Mādhava deal with the computation of what we now call π, the ratio of the circumference C of a circle to its diameter D, and with the determination of arbitrary arcs from their Sines and Cosines. One of these verses, quoted in Śaṅkara's *Kriyā-kramakarī* on the *Līlāvatī*, contains a numerical value for the ratio C/D, encoded in the concrete number notation:

> The teacher Mādhava also mentioned a value of the circumference closer [to the true value] than that:
>
> "Gods [thirty-three], eyes [two], elephants [eight], serpents [eight], fires [three], three, qualities [three], Vedas [four], *nakṣatras* [twenty-seven], elephants [eight], arms [two] (2,827,433,388,233)—the wise

[17] [Pin1970a], vol. 1, pp. 36–38, [SarK1972], pp. 64–65.

[18] [SarK1972] contains biographical and bibliographical information on a number of these other Kerala scientists.

> said that this is the measure of the circumference when the diameter of a circle is nine *nikharva* [10^{11}]."[19]

Śaṅkara says here that Mādhava's value 2,827,433,388,233/900,000,000,000 is more accurate than "that," that is, more accurate than the traditional value for C/D of 355/113. (Mādhava's ratio corresponds to a value of π accurate to the eleventh decimal place, or 3.14159265359.)

Mādhava's derivation of the circumference by means of polygons. Śaṅkara then cites a set of four verses by Mādhava that prescribe a geometric method for computing the value of the circumference. This technique involves calculating the perimeters of successive regular circumscribed polygons, beginning with a square. The values of the polygonal sides in the first few cases are determined as follows:

> When in this way a very accurate circumference [is desired], if [one says], "how is it possible to calculate [that]?," Mādhava then said:
>
> "The square-root of an eighth part of the square of the side in a four-sided [figure] is the divisor. And [take] the quotient by the divisor from the product of the side [times] the difference of the divisor and one-fourth the side. Having extended [that] from the corner, the eight sides to be drawn are to be shown.
>
> "The square of half the side of the eight-sided [figure] is to be added to the square of the half-diameter. The square-root here is the hypotenuse. Divide by that the square of the half-diameter diminished by the square of half the side. Whatever is the result from that,
>
> "the hypotenuse diminished by that [result and] halved is called the divisor. And the multiplier is the hypotenuse minus the half-diameter. Having multiplied the half-side by this multiplier and divided [it] by the divisor, whatever is the quotient in this case too,
>
> "having extended that from the corner at the juncture of the sides, there should be in the intervening space here a sixteen-sided [figure]. And hence in this way there should be a tooth-[i.e., thirty-two-] sided [figure], and hence [the circumference of] the circle is to be determined."[20]

The beginning of Śaṅkara's commentary on these verses explains how to set up the construction for calculating the polygon sides:

---

[19] *Kriyā-kramakarī* on *Līlāvatī* 199, [SarK1975], p. 377. Translated in [IkeSIM], p. 4, and [Plo2007b], pp. 481–482. See also [Sara1979], p. 157.

[20] *Kriyā-kramakarī* on *Līlāvatī* 199, [SarK1975], pp. 377–378. See also [Srin2004b], Appendix 3; [Sara1979], pp. 156–157; and [SarK2008], vol. 1, sect. 6.2.

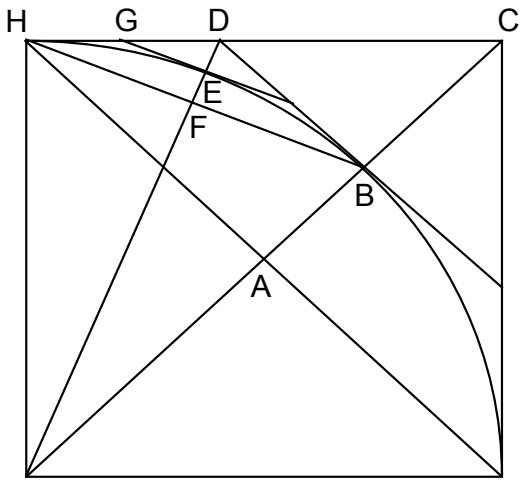

Figure 7.1 Mādhava's computation of the circumference by successive polygons.

> This is the meaning of those four verses: There at the beginning, having drawn a square figure [with side] equal to the desired diameter, [and] divided by eight the square of that side, [take] the square-root of whatever is the quotient. [Its] name is the "divisor."[21]

That is, when the circumference of a circle with radius R is sought, first circumscribe about it a square of side $s = 2R$ (one quadrant of which is shown in figure 7.1, where CH is equal to R, and B and E are points on the circle bisecting tangent line segments). Then compute the difference CD between R and the half-side DH of the circumscribed regular octagon, using Mādhava's "divisor" $\sqrt{s^2/8} = s/(2\sqrt{2})$. Mādhava says that CD is found from "the quotient by the divisor from the product of the side [times] the difference of the divisor and one-fourth the side." This is validated by the similarity of right triangles DBC and HAC, as follows:

$$CD = BC \cdot \frac{CH}{AC} = \left(\frac{s}{\sqrt{2}} - \frac{s}{2}\right) \cdot \frac{s/2}{s/(2\sqrt{2})} = \left(\frac{s}{2\sqrt{2}} - \frac{s}{4}\right) \cdot \frac{s}{s/(2\sqrt{2})}.$$

When this quantity is "extended from the corner" C, the remaining segment DH is the desired half-side of the octagon. This is then used to compute the difference DG between DH and the half-side GH of the circumscribed

[21] *Kriyā-kramakarī* on *Līlāvatī* 199, [SarK1975], p. 378.

sixteen-sided polygon. Mādhava defines the following quantities to derive it:

$$\sqrt{DH^2 + R^2} = \text{hypotenuse},$$
$$\frac{R^2 - DH^2}{\sqrt{DH^2 + R^2}} = \text{result},$$
$$\left(\sqrt{DH^2 + R^2} - \frac{R^2 - DH^2}{\sqrt{DH^2 + R^2}}\right) \cdot \left(\frac{1}{2}\right) = \frac{DH^2}{\sqrt{DH^2 + R^2}} = \text{divisor},$$
$$\sqrt{DH^2 + R^2} - R = \text{multiplier}, \quad \text{and}$$
$$DG = DH \cdot \frac{\text{multiplier}}{\text{divisor}} = DH \cdot \frac{\sqrt{DH^2 + R^2} - R}{DH^2/\sqrt{DH^2 + R^2}}.$$

Since in figure 7.1 the "multiplier" ($\sqrt{DH^2 + R^2} - R$) is DE, and the "divisor" $DH^2/\sqrt{DH^2 + R^2}$ is DF, it is clear from the similar right triangles GED and HFD that

$$DG = DE \cdot \frac{DH}{DF} = (\sqrt{DH^2 + R^2} - R) \cdot \frac{DH}{DH^2/\sqrt{DH^2 + R^2}},$$

which is equivalent to Mādhava's result. If DG is then "extended" from the "juncture" D, the remainder GH is the desired half-side of the regular hexadecagon. The procedure can then be repeated to compute the half-side of the 32-gon, and so forth to approximate the circumference as precisely as desired.

The Mādhava-Leibniz series for π. An alternative rule for the circumference is then described by Śaṅkara as "easier" than the foregoing. It corresponds to finding a sum C of some arbitrary number of terms in a series, plus a final correction term, depending on the diameter $D = 2R$ as follows:

$$C \approx \frac{4D}{1} - \frac{4D}{3} + \frac{4D}{5} - \ldots + (-1)^{n-1}\frac{4D}{2n-1} + (-1)^n \frac{4Dn}{(2n)^2 + 1}.$$

This method, too, is due to Mādhava, as Śaṅkara states:

> An easier way to get the circumference is mentioned by him (Mādhava). That is to say:
>
> "Add or subtract alternately the diameter multiplied by four and divided in order by the odd numbers like three, five, etc., to or from the diameter multiplied by four and divided by one.
>
> "Assuming that division is completed by dividing by an odd number, whatever is the even number above [next to] that [odd number], half of that is the multiplier of the last [term].
>
> "The square of that [even number] increased by 1 is the divisor of the diameter multiplied by 4 as before. The result from these two (the multiplier and the divisor) is added when [the previous term is] negative, when positive subtracted.

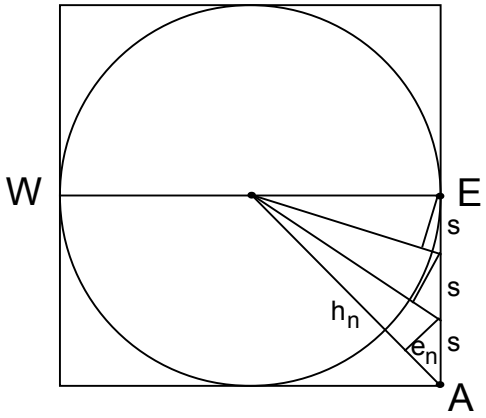

Figure 7.2 Subdivision of the quadrant for the circumference series

> "The result is an accurate circumference. If division is repeated many times, it will become very accurate."[22]

Śaṅkara lays out an elaborate geometrical rationale for the derivation of this series, starting with the procedure illustrated in figure 7.2. As before, it relies on the construction of a circle in a circumscribed square, which is here oriented to the cardinal directions (a nice use of the physical context of the problem as a sort of graphical symbolism to help explain the construction). The square's vertical half-side $EA = R$ in this case is subdivided into some number n of equal segments of length s. The endpoints of the segments are connected to the center of the circle by "hypotenuse-lines." The length of each "hypotenuse-line" h_i, and that of the perpendicular "edge" e_i dropped on h_i from the end of the previous hypotenuse-line are then to be determined. As Śaṅkara puts it,

> For demonstrating this, having placed a square whose four sides are equal to a desired diameter, draw a circle in it by means of a compass [whose opening] is equal to half of the diameter and draw the east-west and north-south line(s). Then the circumference touches the middle of each of the four sides of the square in all four directions. Having placed as many dots as you wish with equal intervals at that end of the eastern side which is south from the end of the east-west line, draw lines from the center of the circle to the dots.
>
> Then add the square of the distance between the end of that line and the end of the east-west line to the square of the half-diameter, and take the square root of each. The measure of

[22] *Kriyā-kramakarī* on *Līlāvatī* 199, [SarK1975], p. 379; the same quoted verses appear in *Yukti-dīpikā* 271–274 on *Tantra-saṅgraha* 2.1, [SarK1977b], p. 101. Translated in [IkeSIM], p. 11, and [Plo2007b], p. 482. See also, for the following derivation, [Sara1979], pp. 159–166, and [Srin2004b], Appendix 3.

> the length of the hypotenuse-line is produced.... That [distance] passing from the end of the left line to the hypotenuse [which is] the southern line, is placed with its [perpendicular] direction opposite to its (the hypotenuse's). This is the edge of the segment.... This figure is similar in shape to the figure whose hypotenuse is the southern line and edge is the half-diameter.[23]

In other words, the hypotenuse-line h_i is found from R and the vertical segment is by the Pythagorean theorem. Then the edge e_i is known from the pair of similar right triangles with hypotenuses h_i and s. That is,

$$h_i = \sqrt{R^2 + (is)^2} \quad \text{and} \quad e_i = \frac{s \cdot R}{h_i}.$$

Śaṅkara's next step is to compute the value of each small arc c_i of the circumference, which he takes to be equal to its corresponding Sine g_i, if the arcs are small enough. As illustrated in figure 7.3, g_i is derived in terms of the known edge e_i from similar right triangles contained between each pair of adjacent hypotenuses. It is then approximated in terms of the "southern" hypotenuse h_i, after substituting in the above expression for e_i:

> Having multiplied the distance derived in this way by the half-diameter, divide by [the length of] the left one of two adjacent lines. The result is the Sine of the part of the circumference between the lines. [That] is also a part of the circumference when the segment is small. Therefore, multiply each segment by the square of the half-diameter.... There, though it should be divided by the product of the two adjacent lines, even if it is divided by the square of one of those two, the error will not be large in the calculation of a part of the circumference.... Therefore [this] is said assuming the division of the parts multiplied by the square of half of the diameter by the squares of the southern lines.[24]

That is,

$$c_i \approx g_i = \frac{e_i \cdot R}{h_{i-1}} = \frac{sR^2}{h_i h_{i-1}} \approx \frac{sR^2}{{h_i}^2},$$

since ${h_i}^2$ is assumed to be close to $h_i h_{i-1}$.

The following very condensed remarks give a rough sketch of Śaṅkara's (presumably originally Mādhava's) line of argument for the remainder of his lengthy proof. The basic strategy is to find infinite series expressions for the individual c_i, and a way of adding them up term by term to produce a single series for the total arc of the half-quadrant $C/8$. Śaṅkara says:

[23] *Kriyā-kramakarī* on *Līlāvatī* 199, [SarK1975], p. 380; translated in [IkeSIM], pp. 12–13, and [Plo2007b], pp. 483–484.

[24] *Kriyā-kramakarī* on *Līlāvatī* 199, [SarK1975], p. 380; translated in [IkeSIM], p. 13, and [Plo2007b], p. 484.

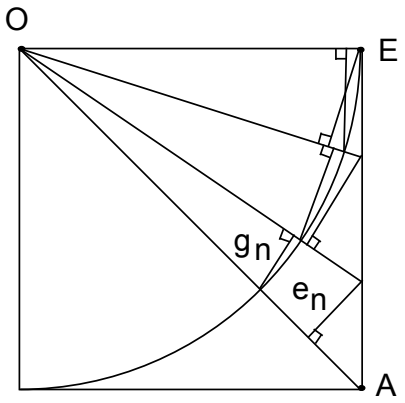

Figure 7.3 Detail of the subdivided quadrant.

If the multiplicands and the divisors were of one kind, then, after multiplying [the multiplicands] by the sum of the multipliers and dividing by the divisor once, the sum of the quotients would result. Therefore a method for deriving divisors of one kind should be sought.... There in the calculation of the first results multiply [the multiplicand] by the difference between the square of each line and the square of the half-diameter and divide by the square of the half-diameter. [Then] make the sum of the quotients....

Here in the calculation of the sum of each result the sum of the multipliers is assumed to be the multiplier.... Therefore the method of calculating the sum of the squares, the sum of the squares of squares, etc., should be investigated here....

[H]alf of [the product of] the half-diameter multiplied by the number of the sides increased by 1 is the sum of the sides.... The product of the portion and the number of the sides increased by 1 is greater than the half-diameter by a portion. Only when the portion is small will the result be accurate. In such a case half of [the product of] the half-diameter multiplied by the half-diameter is the product of the portion and the sum of the sides.

Now the calculation of the sum of the squares [of the sides] should be investigated.... Here, however, only one multiplier is equal to the half-diameter, and the others are somewhat less.... The sum of the sides multiplied by these in order should be subtracted from the sum [of the sides] multiplied by the half-diameter in order to arrive at the sum of the squares.

Here whatever are the products of the shortened parts and the successively previous sides equal to the radius diminished by its [portions] taken in reversed order beginning with the last term, the sum of these is the sum of the sums multiplied by

> the portion.... The sum multiplied by the half-diameter is the sum of the squares and the sum of half of the squares.... Therefore, it is arrived at that a third of the cube of the half-diameter is the sum of the squares multiplied by the portion....
>
> In like manner, each successive [result] from the successive powers involving the half-diameter divided by each of the successive divisors increased by [its own] third, fourth, etc., is the sum of the cubes, etc., multiplied by the portion. Therefore divisors are increased by one at a time.... Therefore just the half-diameter should be divided separately by odd numbers like three, five, etc. The results should be subtracted from or added to the half-diameter in order.[25]

First, the expression for each small arc c_i in the half-quadrant $C/8$ is rewritten so that the divisors are "of one kind," instead of each depending on a different h_i. The resulting expression involves a series of even powers of "each line" is:

$$c_i \approx \frac{sR^2}{{h_i}^2} = s - \frac{s({h_i}^2 - R^2)}{R^2} + \frac{s({h_i}^2 - R^2)^2}{R^4} - \frac{s({h_i}^2 - R^2)^3}{R^6} + \dots$$
$$= s - \frac{s((is)^2)}{R^2} + \frac{s((is)^4)}{R^4} - \frac{s((is)^6)}{R^6} + \dots$$

This is simply a special case of a more general infinite series for any expression involving some quantity a, a "multiplier" p, and a "divisor" q:

$$\frac{ap}{q} = a + \frac{a(p-q)}{p} + \frac{a(p-q)^2}{p^2} + \frac{a(p-q)^3}{p^3} + \dots.$$

The above series is derived by recursively rewriting its successive terms as follows:

$$\begin{aligned}\frac{ap}{q} &= a + \frac{a(p-q)}{q} \\ &= a + \frac{a(p-q)}{p} \cdot \frac{p}{q} \\ &= a + \frac{a(p-q)}{p} + \frac{a(p-q)}{p} \cdot \frac{(p-q)}{q} \\ &= a + \frac{a(p-q)}{p} + \frac{a(p-q)^2}{p^2} \cdot \frac{p}{q},\end{aligned}$$

and so forth.

[25] *Kriyā-kramakarī* on *Līlāvatī* 199, [SarK1975], pp. 381–385; translated in [IkeSIM], pp. 15–23, and [Plo2007b], pp. 485–490.

Now we can add up all the corresponding terms in each of our expressions for c_i, arriving at "the sum of the multipliers" in each case.

$$\begin{aligned}
c_1 &\approx \frac{sR^2}{{h_1}^2} = s - \frac{s(1s)^2}{R^2} + \frac{s(1s)^4}{R^4} - \frac{s(1s)^6}{R^6} + \ldots, \\
c_2 &\approx \frac{sR^2}{{h_2}^2} = s - \frac{s(2s)^2}{R^2} + \frac{s(2s)^4}{R^4} - \frac{s(2s)^6}{R^6} + \ldots, \\
&\vdots \\
c_n &\approx \frac{sR^2}{{h_n}^2} = s - \frac{s(ns)^2}{R^2} + \frac{s(ns)^4}{R^4} - \frac{s(ns)^6}{R^6} + \ldots \\
\frac{C}{8} &= \sum_{i=1}^{n} c_i \approx R - \frac{s}{R^2}\sum_{i=1}^{n}(is)^2 + \frac{s}{R^4}\sum_{i=1}^{n}(is)^4 - \frac{s}{R^6}\sum_{i=1}^{n}(is)^6 + \ldots
\end{aligned}$$

As Śaṅkara says, this requires us to find more manageable ways to express the sums of even powers of successive integers. The "product of the portion [s] and the sum of the sides [is]" is easily approximated by means of the well-known rule for the sum of an arithmetic progression:

$$s \cdot \sum_{i=1}^{n} is = s \cdot \frac{s \cdot n(n+1)}{2} = \frac{ns \cdot s(n+1)}{2} \approx \frac{R^2}{2},$$

if s is small and n is large, because $R = ns$.

Computing "the sum of the squares multiplied by the portion" is somewhat trickier. The squares of the sides $(is)^2$ are first rewritten as products of each side is with its corresponding complementary segment $R - s(n-i)$. Then the sum of these products is broken out into the sum of the sides "multiplied by the half-diameter" minus the sum of successive sums of sides "multiplied by the portion":

$$\begin{aligned}
\sum_{i=1}^{n}(is)^2 &= \sum_{i=1}^{n}(is)(R - s(n-i)) \\
&= R\sum_{i=1}^{n} is - s\sum_{i=1}^{n}(is)(n-i)) \\
&= R\sum_{i=1}^{n} is - s\sum_{j=1}^{n-1}\sum_{i=1}^{j} is, \quad \text{and thus} \\
R\sum_{i=1}^{n} is &= \sum_{i=1}^{n}(is)^2 + s\sum_{j=1}^{n-1}\sum_{i=1}^{j} is.
\end{aligned}$$

We can then use our previous approximation for the portion times the sum of the sides to show that "the sum multiplied by the half-diameter is the

sum of the squares and the sum of half of the squares":

$$\begin{aligned} R\sum_{i=1}^{n} is &= \sum_{i=1}^{n}(is)^2 + s\sum_{j=1}^{n-1}\sum_{i=1}^{j} is \\ &\approx \sum_{i=1}^{n}(is)^2 + \sum_{j=1}^{n-1}\frac{(js)^2}{2} \\ &\approx \sum_{i=1}^{n}(is)^2 + \frac{1}{2}\sum_{i=1}^{n}(is)^2 \\ &= \frac{3}{2}\sum_{i=1}^{n}(is)^2. \end{aligned}$$

Hence "the sum of the squares multiplied by the portion" is "a third of the cube of the half-diameter":

$$\begin{aligned} s\sum_{i=1}^{n}(is)^2 &\approx s\cdot\frac{2}{3}\cdot R\sum_{i=1}^{n} is \\ &\approx \frac{2}{3}\cdot R\cdot\frac{R^2}{2} \\ &= \frac{R^3}{3}. \end{aligned}$$

We can use similar arguments to approximate the sums of other integer powers as "successive powers involving the half-diameter divided by each of the successive divisors increased by [its own] third, fourth, etc.":

$$s\cdot\sum_{i=1}^{n}(is)^k \approx \frac{R\cdot R^k}{k+k/k} = \frac{R^{k+1}}{k+1}.$$

Finally, these expressions for even powers are substituted for the corresponding terms in our series for $C/8$:

$$\begin{aligned} \frac{C}{8} &\approx R - \frac{s}{R^2}\sum_{i=1}^{n}(is)^2 + \frac{s}{R^4}\sum_{i=1}^{n}(is)^4 - \frac{s}{R^6}\sum_{i=1}^{n}(is)^6 + \dots \\ &\approx R - \frac{1}{R^2}\frac{R^3}{3} + \frac{1}{R^4}\frac{R^5}{5} - \frac{1}{R^6}\frac{R^7}{7} + \dots, \end{aligned}$$

which is equivalent to Mādhava's prescribed series for C excluding its correction term (discussed below).

The Mādhava-Gregory series for the arctangent. Śaṅkara goes on in the commentary *Kriyā-kramakarī* to quote another set of verses, also generally ascribed to Mādhava, generalizing this series result for an arbitrary arc of the circumference:

> Now, by just the same argument, the determination of the arc of a desired Sine can be [made]. That is as [follows]:
>
> "The first result is the product of the desired Sine and the radius divided by the Cosine. When one has made the square of the Sine the multiplier and the square of the Cosine the divisor,
>
> "now a group of results is to be determined from the [previous] results beginning with the first. When these are divided in order by the odd numbers 1, 3, and so forth,
>
> "and when one has subtracted the sum of the even[-numbered results] from the sum of the odd [ones], [that] should be the arc. Here, the *smaller* of the Sine and Cosine is required to be considered as the desired [Sine].
>
> "Otherwise there would be no termination of the results even if repeatedly [computed]."
>
> By means of the same stated argument, the circumference can be determined in another way too. That is as [follows]:
>
> "The first result should be the square root of the square of the diameter multiplied by twelve. From then on, the result should be divided by three, [in] each successive [case].
>
> "When these are divided in order by the odd numbers beginning with 1, and when one has subtracted the even [results] from the sum of the odd, [that] should be the circumference."[26]

The above sets of quoted verses also appear in Śaṅkara's *Yukti-dīpikā*, separated by a brief description that sets up their rationale:

> Whatever is the end of the desired arc [measured] from the direction-string in the circle, it is extended outside from the intersection with the Sine, [falling] upon the side of the quadrilateral [figure].
>
> The portion of the circumference [up to] that intersection is to be determined, [in accordance] with its being an arc, by means of results from the corresponding sum of squares and so forth, as previously stated.
>
> Now, determine such a portion of that circumference from the Sine of a [zodiacal] sign [i.e., 30° or $C/12$].[27]

The above formulas state that for an arbitrary arc θ whose Sine and Cosine are known (and assuming $\operatorname{Sin}\theta < \operatorname{Cos}\theta$),

$$\theta = \frac{R\operatorname{Sin}\theta}{1\operatorname{Cos}\theta} - \frac{R\operatorname{Sin}^3\theta}{3\operatorname{Cos}^3\theta} + \frac{R\operatorname{Sin}^5\theta}{3\operatorname{Cos}^5\theta} - \frac{R\operatorname{Sin}^7\theta}{3\operatorname{Cos}^7\theta} + \dots$$

[26] *Kriyā-kramakarī* on *Līlāvatī* 199, [SarK1975], pp. 385–386; the same quoted verses (with minor variants) appear in *Yukti-dīpikā* 206–209 and 212–214 on *Tantra-saṅgraha* 2.1, [SarK1977b], pp. 95–96. Also translated in [IkeSIM], pp. 25–26. See also [Sara1979], pp. 182–184; [Srin2004b], Appendix 3; and [SarK2008], vol. 1, sect. 6.6–6.7.

[27] *Yukti-dīpikā* 210–212 on *Tantra-saṅgraha* 2.1, [SarK1977b], p. 96. See also [Sara1979], pp. 183–184.

In particular,

$$C = \frac{\sqrt{12D^2}}{1} - \frac{\sqrt{12D^2}}{3 \cdot 3} + \frac{\sqrt{12D^2}}{5 \cdot 3^2} - \frac{\sqrt{12D^2}}{7 \cdot 3^3} + \dots$$
$$= \frac{12R}{1\sqrt{3}} - \frac{12R}{3\sqrt{3}^3} + \frac{12R}{5\sqrt{3}^5} - \frac{12R}{7\sqrt{3}^7} + \dots,$$

since the ratio of $\mathrm{Sin}(C/12)$ to $\mathrm{Cos}(C/12)$ is $1/\sqrt{3}$. And these results can be justified geometrically "as previously stated," just like our earlier series for the half-quadrant

$$\frac{C}{8} = R - \frac{1}{R^2}\frac{R^3}{3} + \frac{1}{R^4}\frac{R^5}{5} - \frac{1}{R^6}\frac{R^7}{7} + \dots$$

For an arbitrary arc θ, though, we would subdivide into portions not the entire half-side of the circumscribing square or R but rather a segment of it cut off by a radius "extended outside from the intersection with the Sine." This segment is the Tangent of θ, equal to $R\,\mathrm{Sin}\,\theta/\,\mathrm{Cos}\,\theta$. Therefore our corresponding expression for arbitrary θ will be

$$\theta = \frac{R\,\mathrm{Sin}\,\theta}{1\,\mathrm{Cos}\,\theta} - \frac{R^3\,\mathrm{Sin}^3\,\theta}{3R^2\,\mathrm{Cos}^3\,\theta} + \frac{R^5\,\mathrm{Sin}^5\,\theta}{3R^4\,\mathrm{Cos}^5\,\theta} - \frac{R^7\,\mathrm{Sin}^7\,\theta}{3R^6\,\mathrm{Cos}^7\,\theta} + \dots\ .$$

The correction term for Mādhava's π series. The final term $\frac{4Dn}{(2n)^2+1}$ in the series for C compensates for the inaccuracy caused by the slow convergence of its terms. Historians have suggested that Mādhava found this term by experimenting with "pulverizer-like" continued fraction approximations to the error produced by successive partial sums of the series, when compared to a recognized value of π such as $355/113$.[28] That is, if $C(k)$ represents the kth partial sum of the Mādhava-Leibniz infinite series for the circumference, then the errors produced by $C(k)$ as compared to $\pi/4$ for the first few values of k are the following:

[28]For a detailed explanation of the conjectured derivation of the various correction terms and also of Śaṅkara's rationale for them, see [HayKY1990]. [Gup1992c] argues for essentially the same reconstruction, but using a different standard value of π. See also [SarK2008], vol. 1, sect. 6.8–6.10.

$$\frac{C(1)}{4D} - \frac{1}{4}\cdot\frac{355}{113} = 1 - \frac{355}{452} = \frac{97}{452} = \cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{31}{33}}}} \approx \frac{1}{4+\cfrac{1}{1}},$$

$$\frac{C(2)}{4D} - \frac{1}{4}\cdot\frac{355}{113} = \left(1-\frac{1}{3}\right) - \frac{355}{452} = -\frac{161}{1356} \approx -\frac{1}{8+\cfrac{1}{2}},$$

$$\frac{C(3)}{4D} - \frac{1}{4}\cdot\frac{355}{113} = \left(1-\frac{1}{3}+\frac{1}{5}\right) - \frac{355}{452} = \frac{551}{6780} \approx \frac{1}{12+\cfrac{1}{3}}.$$

From such examples, apparently, Mādhava simply extrapolated the general error term $\dfrac{1}{4k+\cfrac{1}{k}} = \dfrac{k}{(2k)^2+1}$. When multiplied by $4D$ and made positive or negative depending on the parity of k, this becomes the correction term tacked on to the end of $C(k)$. If this reconstruction of Mādhava's reasoning is accurate, it indicates that he was as skilled in working with bold numerical intuitions as with painstaking geometric rationales.

Mādhava's arc-difference rule. Another procedure for finding an unknown arc from known Sines and Cosines is explained in the verses of Nīlakaṇṭha's second chapter of the *Tantra-saṅgraha* itself, along with a rule for computing the Sine and Cosine (discussed below). Both rules are directly attributed to Mādhava. The formula for the arc is as follows:

> "The divisor [derived] from the sum of the Cosines is divided by the difference of the two given Sines. The radius multiplied by two is divided by that [result]. That is the difference of the arcs."
>
> Thus is made the determination of the Sine and arc as stated by Mādhava.[29]

In other words, if the Sines and Cosines of the known arc θ and the unknown one $\theta + \Delta\theta$ are given, then their difference $\Delta\theta$ is found from

$$\Delta\theta \approx \frac{2R}{\dfrac{\operatorname{Cos}\theta + \operatorname{Cos}(\theta+\Delta\theta)}{\operatorname{Sin}(\theta+\Delta\theta) - \operatorname{Sin}\theta}}.$$

Śaṅkara remarks on this in his other *Tantra-saṅgraha* commentary, the *Laghu-vivṛti*, explaining why a similar-triangle relation justifies this approximation:

[29] *Tantra-saṅgraha* 2.14–15, [SarK1977b], pp. 112, 120, and [Pil1958], p. 21.

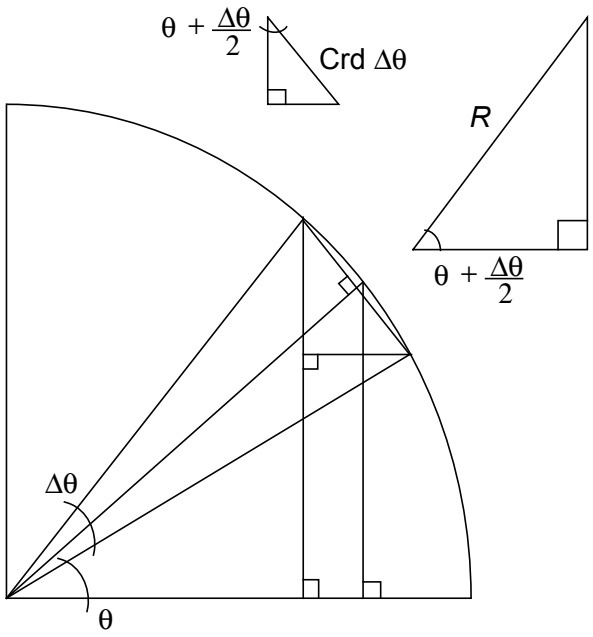

Figure 7.4 The rationale for the arc-difference rule.

> Here, where the divisor should be made from the Cosine of the medial arc [i.e., $(\theta+\Delta\theta/2)$], it is said [to be made] with the sum of the Cosines of both full [arcs], by assuming that that [sum] equals twice the medial Cosine. But in reality, the sum of the Cosines of the two full [arcs] is somewhat less than twice the medial Cosine. Because of the deficiency of that divisor, the result of that [is] somewhat too big. But actually, that is what is desired: for that result is really the chord [of $\Delta\theta$], which is a little less than its arc, so the arc is what is desired. So an excess of the result is [in fact] right. And the radius is said to be doubled because of the doubling of the Cosine forming the divisor. Again, as a rule there is equality of the Chord and its arc. Thus the determination of the Sine [and arc] is called accurate.[30]

It is evident from the similar triangles in figure 7.4 that

$$\frac{\operatorname{Crd}\Delta\theta}{R} = \frac{\operatorname{Sin}(\theta+\Delta\theta)-\operatorname{Sin}\theta}{\operatorname{Cos}(\theta+\Delta\theta/2)}, \quad \text{so} \quad \frac{\operatorname{Crd}\Delta\theta}{2R} = \frac{\operatorname{Sin}(\theta+\Delta\theta)-\operatorname{Sin}\theta}{2\operatorname{Cos}(\theta+\Delta\theta/2)}.$$

As Śaṅkara explains, the rationale for Mādhava's rule for $\Delta\theta$ depends on the fact that $\Delta\theta$ is a little larger than $\operatorname{Crd}\Delta\theta$. This justifies replacing "twice the medial Cosine," or $2\operatorname{Cos}(\theta+\Delta\theta/2)$, with the slightly smaller "sum of the Cosines of both full [arcs]," $\operatorname{Cos}\theta+\operatorname{Cos}(\theta+\Delta\theta)$, to more or less balance the substitution of $\Delta\theta$ for $\operatorname{Crd}\Delta\theta$ in the other ratio.

[30] *Laghu-vivṛti* on *Tantra-saṅgraha* 2.10–14, [Pil1958], pp. 21–21; translated in [Plo2001]. See also [Plo2005b].

7.3.2 Mādhava on the Sine and Cosine

Mādhava's "Taylor series approximations" for Sine and Cosine. In addition to working out rules for circumferences and arcs, Mādhava also apparently left a substantial body of work on determining Sines and Cosines, traces of which survive in later quotations. For example, just prior to quoting the abovementioned rule for $\Delta\theta$, Nīlakaṇṭha in the *Tantra-saṅgraha* states a formula for determining the Sine and Cosine of a given arc θ. This, the "determination of the Sine as stated by Mādhava," employs the Sine and Cosine of an arc α tabulated in a Sine table, as follows:

> "Having set down the two composite Sines [i.e., the Sine and Cosine of the tabulated arc α] closest to the arc [θ] whose Sine and Cosine are sought, one should compute the arc [of] deficiency or excess.
>
> And set down as a divisor 13751 divided by twice the arcminutes of that [difference arc], for the purpose of the mutual correction of those [quantities]. Having first divided [by that divisor] one [of the tabulated Sine or Cosine], add or subtract [the result] with respect to the other, according as the [difference] arc is excessive or deficient.
>
> Now in the same way, apply that [corrected quantity] times two to the other: this is the correction."[31]

The Sine and Cosine of α are "composite" in the sense that they are accumulated from tabulated Sine-differences. The "arc of deficiency or excess," $(\theta - \alpha)$ in arcminutes, is used to compute a "divisor" D:

$$D = \frac{13751}{2(\theta - \alpha)} = \frac{4 \cdot 3437;45}{2(\theta - \alpha)} \approx \frac{2R}{\theta - \alpha}$$

when the trigonometric radius R is taken to be $3437;45 \approx 360 \cdot 60/2\pi$, a more precise form of the traditional Indian value 3438. Then the known Sine and Cosine are "corrected" by each other:

$$\operatorname{Sin}\theta \approx \operatorname{Sin}\alpha + \frac{\operatorname{Cos}\alpha}{D}, \qquad \operatorname{Cos}\theta \approx \operatorname{Cos}\alpha - \frac{\operatorname{Sin}\alpha}{D},$$

and subsequently these "corrected" terms, multiplied by two, are applied "in the same way"—that is, divided by the divisor D and then added or

[31] *Tantra-saṅgraha* 2.10–13, [SarK1977b], p. 112; translated in [Plo2005b] and in [Gup1969], pp. 92–94. The same rule is cited in a different form by Parameśvara, including a variant with a third-order form ([Gup1969], pp. 94–96, [Gup1974], [Plo2001]). See also [SarK2008], vol. 1, sect. 7.4.3.

subtracted:

$$\begin{aligned}\operatorname{Sin}\theta &\approx \operatorname{Sin}\alpha + \left(\operatorname{Cos}\alpha - \frac{\operatorname{Sin}\alpha}{D}\right)\frac{2}{D} \\ &= \operatorname{Sin}\alpha + \frac{(\theta-\alpha)\operatorname{Cos}\alpha}{R} - \frac{(\theta-\alpha)^2\operatorname{Sin}\alpha}{2R^2}, \\ \operatorname{Cos}\theta &\approx \operatorname{Cos}\alpha - \left(\operatorname{Sin}\alpha + \frac{\operatorname{Cos}\alpha}{D}\right)\frac{2}{D} \\ &= \operatorname{Cos}\alpha - \frac{(\theta-\alpha)\operatorname{Sin}\alpha}{R} + \frac{(\theta-\alpha)^2\operatorname{Cos}\alpha}{2R^2},\end{aligned}$$

which is equivalent to a modern second-order Taylor series expansion.

The Mādhava-Newton series for the Sine and Cosine. Other often-cited verses by Mādhava encode in the *kaṭapayādi* alphanumeric notation coefficients for the first few terms of other power series for the Sine and Cosine. Śaṅkara includes them in his commentary on *Tantra-saṅgraha* 2:

> "*vidvāṃs tunnabalaḥ kavīśanicayaḥ sarvārthaśīlasthiro nirviddhāṅganarendraruṅ* ...
>
> "(The wise ruler whose army has been struck down gathers together the best of advisors and remains firm in his conduct in all matters; then he shatters the king whose army has not been destroyed). When these five [numbers] have been spoken in order, [starting] from the one which is at the bottom multiplied by the square of the given arc, when one has divided [by the square of the arc of 90 degrees], the quotient [of each] is to be subtracted, proceeding upwards, [but the quotient] of the last [is multiplied and divided] by the cube; [the quotient is to be subtracted] from the end on the arc.
>
> "*stenaḥ strīpiśunaḥ sugandhinaganud bhadrāṅgabhavyāsano mīnāṅgo narasiṃha ūnadhanakṛdbhūr eva* ...
>
> "([Viṣṇu is] the thief, the betrayer of women, the mover of the fragrant mountain, with shining limbs and auspicious pose, the fish-limbed, Narasiṃha, and the maker [?] of decrease and increase). When these six [numbers are known], [starting] from the one which is at the bottom multiplied by the square of the given arc, when one has divided [by the square of the arc of 90 degrees], the quotient [of each] is to be subtracted, proceeding upwards. Then the final result should be [the value] of the Versine."[32]

[32] *Yukti-dīpikā* 437–438 on *Tantra-saṅgraha* 2, [SarK1977b], pp. 117–118, [Pin1985]. [GolPi1991], p. 52, notes that the same verses are explicitly ascribed to Mādhava by Nīlakaṇṭha in *Āryabhaṭīya-bhāṣya* on *Āryabhaṭīya* 2.17.

In the words of the first Sanskrit sentence, the significant *kaṭapayādi* consonants (those immediately preceding vowels) correspond to the following numerical equivalents:

v	*v*	*t*	*n*	*b*	*l*	*k*	*v*	*ś*	*n*	*c*	*y*	*s*	*v*
4	4	6	0	3	3	1	4	5	0	6	1	7	4
th	*ś*	*l*	*th*	*r*	*n*	*v*	*dh*	*g*	*n*	*r*	*r*	*r*	
7	5	3	7	2	0	4	9	3	0	2	2	2	

This represents the following set of five coefficients for the Sine series: $[0;0,]44.[0;]33,06.\ 16;05,41.\ 273;57,47.\ 2220;39,40.$

In the second *kaṭapayādi* sentence, the consonants are numerically interpreted as follows:

t	*n*	*r*	*p*	*ś*	*n*	*s*	*g*	*dh*	*n*	*g*	*n*	*bh*	*r*	*g*	*bh*	*y*
6	0	2	1	5	0	7	3	9	0	3	0	4	2	3	4	1
s	*n*	*m*	*n*	*g*	*n*	*r*	*s*	*h*	*n*	*dh*	*n*	*k*	*bh*	*r*	*v*	
7	0	5	0	3	0	2	7	8	0	9	0	1	4	2	4	

These constitute a set of six Versine series coefficients, namely $[0;0,]06.[0;]05,12.\ 03;09,37.\ 071;43,24.\ 872;03,05.\ 4241;09,0.$

The application of these coefficients is explained by Śaṅkara in his subsequent verses describing how the terms of the desired series are to be constructed:

> Having multiplied the arc and the results of each [multiplication] by the square of the arc, divide by the squares of the even [numbers] together with [their] roots, multiplied by the square of the Radius, in order. Having put down the arc and the results one below another, subtract going upwards. At the end is the Sine; the epitome of that is made by means of [the verses] beginning "*vidvān*," etc.
>
> Having multiplied unity and the results of each [multiplication] by the square of the arc, divide by the squares of the even [numbers] minus their roots, multiplied by the square of the Radius, in order. But divide the first [instead] by twice the Radius. Having put down the results one below another, subtract going upwards. At the end is the Versine; the epitome of that is made by means of [the verses] beginning "*stena strī*," etc.[33]

Here Śaṅkara begins the computation for the Sine of an arc θ by computing a sequence of "results" of multiplication: the first is $\theta\cdot\theta^2$, the second $\theta\cdot\theta^2\cdot\theta^2$, the third $\theta\cdot\theta^2\cdot\theta^2\cdot\theta^2$, and so forth. For $i=1,2,3,\ldots$, the ith result is divided by R^2 times a factor equal to "the squares of the even numbers $[2i]$

[33] *Yukti-dīpikā* 440–444ab on *Tantra-saṅgraha* 2, [SarK1977b], p. 118, [GolPi1991]. See also, for these verses and the discussion below of their rationale, [Plo2005b], and [SarK2008], vol. 1, sect. 7.5.

minus their roots," or $(2i)^2 + 2i$. The resulting sequence of results will be

$$\frac{\theta^3}{R^2(2^2+2)} = \frac{\theta^3}{R^2 \cdot 3!},$$
$$\frac{\theta^5}{R^4(2^2+2)(4^2+4)} = \frac{\theta^5}{R^4 \cdot 5!},$$
$$\frac{\theta^7}{R^6(2^2+2)(4^2+4)(6^2+6)} = \frac{\theta^7}{R^6 \cdot 7!},$$

and so forth. Taking "the arc" as the first term, these results are then to be "put down one below another" and successively subtracted "going upwards": that is, each term is diminished by the difference of the two following terms to ultimately produce the Sine, as follows:

$$\operatorname{Sin}\theta = \theta - \left(\frac{\theta^3}{R^2 \cdot 3!} - \left(\frac{\theta^5}{R^4 \cdot 5!} - \left(\frac{\theta^7}{R^6 \cdot 7!} - \cdots\right)\right)\right).$$

In the same way, the series for the Versine is also determined by a recursive subtraction of the prescribed "results":

$$\begin{aligned}\operatorname{Vers}\theta &= \frac{\theta^2}{R \cdot 2} - \left(\frac{\theta^4}{R^3 \cdot 2(4^2-4)} - \left(\frac{\theta^6}{R^5 \cdot 2(4^2-4)(6^2-6)} - \cdots\right)\right) \\ &= \frac{\theta^2}{R \cdot 2!} - \left(\frac{\theta^4}{R^3 \cdot 4!} - \left(\frac{\theta^6}{R^5 \cdot 6!} - \cdots\right)\right).\end{aligned}$$

Each of Mādhava's *kaṭapayādi* coefficients, when multiplied by the appropriate power of the given arc expressed as $\theta^{(\prime)}/(90 \cdot 60)$, produces one of the terms in the above Sine or Versine series. For example,

$$2220;39,40 = \frac{5400^3}{3!R^2}, \quad 273;57,47 = \frac{5400^5}{5!R^2},$$

and so forth. These values imply that R is again taken approximately equal to $360 \cdot 60/2\pi$, or more precisely $3437;44,48$ arcminutes. The powers of 5400 in their numerators (requiring θ to be divided by 5400 to cancel them out) are presumably just scale factors to keep the coefficients at a convenient order of magnitude.

Śaṅkara's geometric rationale for these series depends ultimately on the same similar-triangle relationship, linking the difference of two consecutive Sines with the Cosine of the "medial" arc halfway between them, that we saw earlier in the explanation of Mādhava's arc-difference rule. By this relationship, Sine-differences depend on Cosines, and Cosine-differences (which are equivalent to Sine "second-differences," or the differences between consecutive Sine-differences) on Sines. (A modern mathematical parallel is the

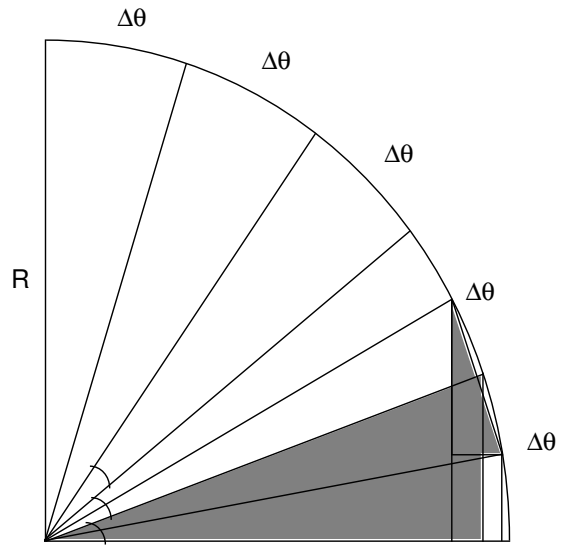

Figure 7.5 Similar triangles relating the Sine and Cosine and their differences.

concept of the cosine as the first derivative of the sine and vice versa, meaning that the sine is its own second derivative.)

More specifically, consider the quadrant in figure 7.5, divided into n unit arcs $\Delta\theta$, the Chord of which is denoted by Crd. Let the Sine and Cosine of the ith consecutive arc $i\Delta\theta$ be denoted Sin_i and Cos_i, and let the consecutive differences $(\mathrm{Sin}_i - \mathrm{Sin}_{i-1})$ and $(\mathrm{Cos}_{i-1} - \mathrm{Cos}_i)$ be denoted $\Delta\,\mathrm{Sin}_i$ and $\Delta\,\mathrm{Cos}_i$, respectively. Likewise, let the Sine and Cosine of each "medial" arc $(i\Delta\theta + \Delta\theta/2)$ be represented by $\mathrm{Sin}_{i.5}$ and $\mathrm{Cos}_{i.5}$, while $\Delta\,\mathrm{Sin}_{i.5} = \mathrm{Sin}_{i.5} - \mathrm{Sin}_{i.5-1}$ and $\Delta\,\mathrm{Cos}_{i.5} = \mathrm{Cos}_{i.5-1} - \mathrm{Cos}_{i.5}$ (always positive). From the pairs of similar right triangles such as the shaded triangles in figure 7.5, we have the following relationships between Sines, Cosines, and their differences:

$$\Delta\,\mathrm{Sin}_{i.5} = \mathrm{Cos}_i \cdot \frac{\mathrm{Crd}}{R}, \quad \Delta\,\mathrm{Cos}_{i.5} = \mathrm{Sin}_i \cdot \frac{\mathrm{Crd}}{R};$$

$$\Delta\,\mathrm{Sin}_{i+1} = \mathrm{Cos}_{i.5} \cdot \frac{\mathrm{Crd}}{R}, \quad \Delta\,\mathrm{Cos}_{i+1} = \mathrm{Sin}_{i.5} \cdot \frac{\mathrm{Crd}}{R}.$$

These proportions and their relationship to the second-differences are explained by Śaṅkara, roughly as follows:

> Divide the Sine produced to the junction of the first arc, multiplied by the Chord, by the Radius: [it is] the difference of the Cosines in the centers of those arcs. Having multiplied that [Cosine-difference, $\Delta\,\mathrm{Cos}_{1.5}$] by the Chord, divide again by the Radius. [That] should be the difference of the differences of the Sines produced to the junctions of the first and second [arcs]. Therefore, divide the first Sine, multiplied by the square of the

> Chord, by the square of the Radius. [The] quotient is the difference of the differences of the first and second [Sines]....
>
> In the same way, the division by the square of the Radius of every Sine multiplied by the square of the Chord should be similarly the difference of [its own difference and] the difference after that....
>
> However many differences of [Sine-]differences there are, when the sum of that many, beginning with the first, is subtracted from the first difference, the desired difference is obtained. When it is desired to obtain the total of the differences of [Sine-]differences, [the quotient] from the sum of Sines multiplied by the square of the Chord, divided by the square of the Radius, should be the sum of the differences of the differences. Therefore, the sum of the differences of the differences is [derived] from the sum of the differences of the Cosines produced to the centers of their own arc-portions.[34]

That is, if the second-difference $\Delta\Delta\,\mathrm{Sin}_i$ is defined as $\Delta\,\mathrm{Sin}_i - \Delta\,\mathrm{Sin}_{i+1}$, then from the above proportions,

$$\Delta\Delta\,\mathrm{Sin}_i = \Delta\,\mathrm{Sin}_i - \Delta\,\mathrm{Sin}_{i+1} = \mathrm{Cos}_{i.5-1}\cdot\frac{\mathrm{Crd}}{R} - \mathrm{Cos}_{i.5}\cdot\frac{\mathrm{Crd}}{R} = \Delta\,\mathrm{Cos}_{i.5}\cdot\frac{\mathrm{Crd}}{R}$$
$$= \mathrm{Sin}_i\cdot\frac{\mathrm{Crd}^2}{R^2}.$$

And since each Sine-difference is the difference of the preceding Sine-difference and their second-difference,

$$\begin{aligned}
\Delta\,\mathrm{Sin}_{i+1} &= \Delta\,\mathrm{Sin}_i - \Delta\Delta\,\mathrm{Sin}_i \\
&= \Delta\,\mathrm{Sin}_{i-1} - \Delta\Delta\,\mathrm{Sin}_{i-1} - \Delta\Delta\,\mathrm{Sin}_i \\
&\vdots \\
&= \Delta\,\mathrm{Sin}_1 - \Delta\Delta\,\mathrm{Sin}_1 - \ldots - \Delta\Delta\,\mathrm{Sin}_i \\
&= \Delta\,\mathrm{Sin}_1 - \sum_{k=1}^{i}\Delta\Delta\,\mathrm{Sin}_k \\
&= \Delta\,\mathrm{Sin}_1 - \sum_{k=1}^{i}\mathrm{Sin}_k\cdot\frac{\mathrm{Crd}^2}{R^2} \\
&= \Delta\,\mathrm{Sin}_1 - \sum_{k=1}^{i}\Delta\,\mathrm{Cos}_{k.5}\cdot\frac{\mathrm{Crd}}{R}.
\end{aligned}$$

The next step is to note that any desired Sine Sin_i is just a sum of cumulative Sine-differences that can be expressed as above. And since the Versine is just

[34] *Yukti-dīpikā* 352–364 on *Tantra-saṅgraha* 2, [SarK1977b], pp. 109–110.

R minus the Cosine, we can replace Cosine-differences in such expressions with the corresponding Versine-differences, since

$$\Delta \operatorname{Vers}_i = \operatorname{Vers}_i - \operatorname{Vers}_{i-1} = R - \operatorname{Cos}_i - (R - \operatorname{Cos}_{i-1}) = \operatorname{Cos}_{i-1} - \operatorname{Cos}_i = \Delta \operatorname{Cos}_i \,.$$

Consequently,

$$\begin{aligned}
\operatorname{Sin}_i &= \Delta \operatorname{Sin}_1 + \Delta \operatorname{Sin}_2 + \ldots + \Delta \operatorname{Sin}_i \\
&= i\Delta \operatorname{Sin}_1 - \sum_{k=1}^{i-1} \sum_{j=1}^{k} \operatorname{Sin}_j \cdot \frac{\operatorname{Crd}^2}{R^2} \\
&= i\Delta \operatorname{Sin}_1 - \sum_{k=1}^{i-1} \sum_{j=1}^{k} \Delta \operatorname{Cos}_{j.5} \cdot \frac{\operatorname{Crd}}{R} \\
&= i\Delta \operatorname{Sin}_1 - \sum_{k=1}^{i-1} \sum_{j=1}^{k} \Delta \operatorname{Vers}_{j.5} \cdot \frac{\operatorname{Crd}}{R}.
\end{aligned}$$

As Śaṅkara puts it,

> Therefore, many sums of Sines one below another, ending with the sum of the first and second, are to be made here successively. [The quotient] from the sum of all those [sums], multiplied by the square of the Chord, divided by the square of the Radius—which is the result produced from the sums of the Sines—is equal to the quotient [from division] by the Radius of the sum of [the successive sums of] the Versine-differences, times the Chord. However many differences are considered in the case of [any] desired arc, subtract that [sum of sums] from the first difference times that number of differences.[35]

The remaining steps in Śaṅkara's demonstration involve juggling various simplifying approximations and recursive corrections to them, as follows:

1. Assume that small quantities are equal to unity, in particular, that the unit arc $\Delta\theta$ equals its Chord and Sine, and approximate successive Sines or Versines by successive arcs or integers.

2. Find approximate expressions for sums of successive integers and their sums.

3. Recursively correct the approximations in step (1) to produce the successive terms of infinite series for the Sine and Versine.

The following brief excerpts from Śaṅkara's next fifty-odd verses indicate how these steps are followed. For step 1:

[35] *Yukti-dīpikā* 367–370 on *Tantra-saṅgraha* 2, [SarK1977b], p. 111.

> Because of the uniform smallness of the arc-portions, the first Sine-difference is assumed [instead].... Therefore, subtract from the desired arc the result produced as stated.... Whatever is the sum-of-sums of the arcs—owing to ignorance of [the values of] the Sines—is to be considered the sum [of the sums] of the Sines. But in that case, the last Sine should be the desired arc.... Because of the smallness [of $\Delta\theta$], those arc-portions are considered [to be composed] with unity [i.e., as integers]. So the integers in the desired arc are equal to that [arc].

> Therefore the sum of the Sines is assumed from the sum of the numbers having one as their first term and common difference. That, multiplied by the Chord [between] the arc-junctures, is divided by the Radius. The quotient should be the sum of the differences of the Cosines drawn to the centers of those arcs.... The other sum of Cosine-differences, [those] produced to the arc-junctures, [is] the Versine.[But] the two are approximately equal, considering the minuteness of the arc-division. On account of considering the arc-units [equal] to unity because of [their] minuteness, the Chord too is equal to that [unity], which makes no difference in the multiplier.[36]

In other words, if $\Delta \operatorname{Sin}_1 \approx \Delta\theta$, then $i\Delta \operatorname{Sin}_1$ can be replaced by the arc $i\Delta\theta$. And if $\Delta\theta$ is the unit arc, then $i\Delta\theta$ is just i. More radically, Śaṅkara suggests that a sum of i sums of arcs can be substituted for a sum of $(i-1)$ sums of the corresponding Sines. (He notes that those Sines can also be expressed in terms of medial Cosine- or Versine-differences, and owing to the "minuteness of the arc-division," medial differences can be approximated by the adjacent nonmedial ones.) Finally, $\operatorname{Crd}\Delta\theta$ too is assumed equal to unity.

[36] *Yukti-dīpikā* 375–385 on *Tantra-saṅgraha* 2, [SarK1977b], pp. 111–113.

So, starting from our immediately preceding expression for Sin_i in terms of a double sum of Sines, we modify it as follows:

$$\begin{aligned}
\mathrm{Sin}_i &= \Delta\,\mathrm{Sin}_1 + \Delta\,\mathrm{Sin}_2 + \ldots + \Delta\,\mathrm{Sin}_i \\
&= i\Delta\,\mathrm{Sin}_1 - \sum_{k=1}^{i-1}\sum_{j=1}^{k}\mathrm{Sin}_j \cdot \frac{\mathrm{Crd}^2}{R^2} \\
&\approx i\Delta\theta - \sum_{k=1}^{i}\sum_{j=1}^{k} j\Delta\theta \cdot \frac{\mathrm{Crd}^2}{R^2} \\
&\approx i - \sum_{k=1}^{i}\sum_{j=1}^{k} j \cdot \frac{\mathrm{Crd}^2}{R^2} \\
&\approx i - \sum_{k=1}^{i}\sum_{j=1}^{k}\Delta\,\mathrm{Vers}_j \cdot \frac{\mathrm{Crd}}{R} \\
&= i - \sum_{j=1}^{i}\mathrm{Vers}_j \cdot \frac{\mathrm{Crd}}{R} \\
&\approx i - \sum_{j=1}^{i}\mathrm{Vers}_j \cdot \frac{1}{R}.
\end{aligned}$$

Hence a double sum of Sines is essentially the same as a double sum of integers, which (times a factor of R) is the same as a sum of Versines. Now we need a more convenient way to express this double sum of integers, which brings us to step 2. First, nested sums are recast as products, which are then approximated as sums of integer powers:

> Whatever is the product of however many numbers beginning with the first-term and increasing [successively] by one, that [product] is divided by the product of that many numbers beginning with one and increasing by one. The results one after another are the sums-of-sums of those [numbers]....
>
> Half that square of the arc should be the sum of the [consecutive] arcs. Because half the product of the first-term and the first-term plus one is a sum: therefore, from the cube and squared-square [etc.] of the desired arc, divided by the product of numbers beginning with one and increasing by one, there are many resulting sums one after another.[37]

So if we have q recursive sums of integers beginning with 1, and the last term of the qth sum is p, then they can be expressed in terms of products of

[37] *Yukti-dīpikā* 386–391 on *Tantra-saṅgraha* 2, [SarK1977b], p. 113.

$(q+1)$ successive integers:

$$\sum_{j_q=1}^{p} \cdots \sum_{j_2=1}^{j_3} \sum_{j_1=1}^{j_2} j_1 = \frac{p \cdot (p+1) \cdot \ldots \cdot (p+q)}{1 \cdot 2 \cdot \ldots \cdot (q+1)} = \frac{(p+q)!}{(p-1)!(q+1)!}.$$

Now we assume that a single sum or arithmetic progression of i integers can be approximated by half the square of i. And the same approximation gives us corresponding expressions for the double sum, triple sum, and so on, which can also be considered single sums of successive powers of integers, as in our previous derivation of the Mādhava-Leibniz series:

$$\sum_{j_1=1}^{i} j_1 = \frac{i(i+1)}{1 \cdot 2} \approx \frac{i^2}{2},$$

$$\sum_{j_2=1}^{i} \sum_{j_1=1}^{j_2} j_1 = \frac{i(i+1)(i+2)}{1 \cdot 2 \cdot 3} \approx \frac{i^3}{6} \approx \sum_{j=1}^{i} \frac{j^2}{2},$$

$$\sum_{j_3=1}^{i} \sum_{j_2=1}^{j_3} \sum_{j_1=1}^{j_2} j_1 = \frac{i(i+1)(i+2)(i+3)}{1 \cdot 2 \cdot 3 \cdot 4} \approx \frac{i^4}{24} \approx \sum_{j=1}^{i} \frac{j^3}{6},$$

and so on.

The final step of the rationale uses these approximations to modify our expressions for the Sine and Versine derived in step 1. These involve deriving successive terms of their series from mutual corrections to their previous terms. As Śaṅkara explains,

> The difference of the desired Sine and [its] arc is from the sum of the sum of the Sines. [Half] the square of the desired arc should be divided by the Radius; thence is [found] the Versine. But thence whatever [results] from the cube of the same desired arc [divided] by the square of the Radius, from that the quotient with six is approximately inferred [to be] the difference of the arc and Sine...
>
> To remove the inaccuracy [resulting] from producing [the Sine and Versine expressions] from a sum of arcs [instead of Sines], in just this way one should determine the difference of the [other] Sines and [their] arcs, beginning with the next-to-last. And subtract that [difference each] from its arc: [those] are the Sines of each [arc]. Or else therefore, one should subtract the sum of the differences of the Sines and arcs from the sum of the arcs. Thence should be the sum of the Sines. From that, as before, determine the sum of the Versine-differences...
>
> [W]hatever sum-of-sums is inferred from the determination of the Versine, the difference of the Sine and [its] arc is deduced from the sum-of-sums after that [one]. Thence in this case, the quotient from the product of the cube and square of the desired arc,

> [divided] by whatever is the product of five numbers beginning with one and increasing by one, [is] also divided by the squared-square of the Radius. The difference of the Sine and [its] arc determined in this way should be more accurate.. . .
>
> The sum of Sines is the Versine. So the [new] difference of Sine and arc [is produced] from the [sum-of-sums] after that. And thus in this way it is made from the sum-of-sums after that, again and again.[38]

From our earlier expression of Sin_i as the difference between i and a double sum of integers or a sum of Versines, we may write equivalent expressions involving sums of integer powers:

$$i - \mathrm{Sin}_i \approx \sum_{k=1}^{i} \sum_{j=1}^{k} j \cdot \frac{1}{R^2} \approx \sum_{j=1}^{i} \mathrm{Vers}_j \cdot \frac{1}{R} \approx \sum_{j=1}^{i} \frac{j^2}{2} \cdot \frac{1}{R} \approx \frac{i^3}{6} \cdot \frac{1}{R^2}.$$

But since each Vers/R term initially represented a sum of Sin/R^2 terms which we approximated by a sum of arcs, each Vers/R term is off by an amount corresponding to the difference between the arc and Sine. So we can "remove the inaccuracy" by subtracting out just this difference:

$$\mathrm{Vers}_i \approx \sum_{j=1}^{i} \left(j - \frac{j^3}{6R^2} \right) \cdot \frac{1}{R} = \sum_{j=1}^{i} \frac{j}{R} - \sum_{j=1}^{i} \frac{j^3}{6R^3} \approx \frac{i^2}{2R} - \frac{i^4}{24 \cdot R^3}.$$

Then this modified expression for the Versine can be substituted back into the approximation for the difference between the arc and Sine:

$$\begin{aligned} \mathrm{Sin}_i &\approx i - \sum_{j=1}^{i} \mathrm{Vers}_j \cdot \frac{1}{R} \approx i - \sum_{j=1}^{i} \left(\frac{j^2}{2R} - \frac{j^4}{4!R^3} \right) \cdot \frac{1}{R} \\ &= i - \sum_{j=1}^{i} \frac{j^2}{2R^2} + \sum_{j=1}^{i} \frac{j^4}{4!R^4} \\ &\approx i - \frac{i^3}{3!R^2} + \frac{i^5}{5!R^4}. \end{aligned}$$

If we had used this more accurate expression for the difference between arc and Sine, instead of the previous one, to correct the Versine, it would produce a more accurate sixth-order expression for the Versine, which in turn could be used to produce a still more accurate difference of the arc and Sine, and so back and forth indefinitely to give the terms of the infinite series for Sine and Versine.

These series were evidently used by Mādhava to produce more accurate values of Sines, encoded in *kaṭapayādi* notation in the following collection of mnemonic phrases that total six verses:

[38] *Yukti-dīpikā* 401–403, 406–408, 417–419, 424 on *Tantra-saṅgraha* 2, [SarK1977b], pp. 114–116.

śreṣṭhaṃ nāma variṣṭhānāṃ himādrir vedabhāvanaḥ |
tapano bhānuḥ sūktajño madhyamaṃ viddhidohanam ||

dhigājyo nāśanaṃ kaṣṭaṃ channabhogāśayāmbikā |
mṛgāhāro nareśo 'yaṃ vīro raṇajayotsukaḥ ||

mūlaṃ viśuddhaṃ nāĺasya gāneṣu viraĺā narāḥ |
aśuddhiguptā coraśrīḥ śaṅkukarṇo nageśvaraḥ ||

tanujo garbhajo mitraṃ śrīmān atra sukhī sakhe |
śaśī rātrau himāhāro vegajñaḥ pathisindhuraḥ ||

chāyālayo gajo nīlo nirmalo nāsti satkule |
rātrau darpaṇam abhrāṅgaṃ nagas tuṅganakho balī ||

dhīro yuvā kathā lolaḥ pūjyo nārījanair bhagaḥ |
kanyagāre nāgavallī deyo viśvasthalī bhṛguḥ ||[39]

The meaning of these phrases as ordinary Sanskrit verse is secondary, and not totally coherent. The first one and a half lines might be translated as the following rather epigrammatic remarks:

The name of the most excellent ones is best.
The creator of knowledge is the Himalaya mountain.
The knower of the Vedic hymns is the burning sun.

A truly learned commentator could doubtless provide a meaningful literary interpretation for the entire verse collection, but a reader concerned primarily with its mathematical meaning would probably not care.

Each line or half-verse consists of two quarter-verses with eight syllables (and eight *kaṭapayādi* consonants) apiece. Each quarter-verse represents one of the twenty-four Sines, with the special semivocalic consonant represented here by $\acute{l}$ taken to signify 9. Thus the six verses state the following Sine values accurate to the second sexagesimal place:

1	0224;50,22	2	0448;42,85	3	0670;40,16	4	0889;45,15
5	1105;01,39	6	1315;34,07	7	1520;28,35	8	1718;52,24
9	1909;54,35	10	2092;46,03	11	2266;39,50	12	2430;51,15
13	2584;38,06	14	2727;20,52	15	2858;22,55	16	2977;10,34
17	3083;13,17	18	3176;03,50	19	3255;18,22	20	3320;36,30
21	3371;41,29	22	3408;20,11	23	3430;23,11	24	3437;44,48

These values are not a very remarkable improvement over those derived from Govindasvāmin's modified Sine-differences that we examined in section 4.3.3; they differ by at most half an arcminute or so. Their chief significance appears to be not so much in their superior precision as in the insightful rationales by which they were derived.

[39] Quoted in Nīlakaṇṭha's commentary on *Āryabhaṭīya* 2.12, [SasK1930], p. 55.

7.3.3 Other mathematical investigations

The above highlights from Mādhava's own work and its exegetical tradition, profound and complex as they are, represent only a fraction of the mathematical studies of Mādhava's school. The same interest in analysis and demonstration that produced exceptional discoveries in trigonometry was also applied to other mathematical topics. The remainder of this section touches on a couple of the salient features of these achievements.

Commentaries and rationales. The members of the Mādhava school carefully studied earlier mathematical works, as is clear from the various commentaries and supercommentaries that they wrote on them. Earlier works doubtless provided some of the inspiration for their own detailed analyses: consider, for example, the arguments involving small subdivisions of a sphere's surface that Bhāskara II used to demonstrate his formulas for its surface area and volume, as we saw in section 6.2.3. It is not such a long step from Bhāskara's arguments about adding up small pieces of a sphere's surface to Śaṅkara's (and presumably Mādhava's) about adding up small pieces of a circle's circumference.

But some of the Kerala school texts are particularly remarkable not just for the profundity but also for the breadth with which they apply these approaches to demonstration. The commentary of Nīlakaṇṭha on the *gaṇita* chapter of the *Āryabhaṭīya*, for example, provides detailed rationales for all its rules, many of them geometrically very ingenious. The work of Śaṅkara's teacher Citrabhānu, mentioned in section 7.2, does the same for algebraic rules for solution of equations. (Interestingly, although geometric reasoning is used extensively to demonstrate algebraic rules, the reverse does not seem to be true: the Kerala mathematicians apparently did not express elaborate geometric problems by equations and manipulate them algebraically to derive their solutions.) The most comprehensive of these works on rationales is the Malayalam *Yukti-bhāṣa* of Jyeṣṭhadeva, which explains and demonstrates the mathematics in Nīlakaṇṭha's *siddhānta*, the *Tantra-saṅgraha*, from the eight fundamental operations of arithmetic up through the spherical geometry of the moon's phases.[40]

It is evident that in Mādhava's school, explanations and rationales were considered valid subjects for mathematical creativity in their own right, not just concessions to the "slow-witted" or verifications to dispel doubt about a rule. As we have seen in our quotations from Śaṅkara, some rationales were evidently even deemed important enough to embody in verse, like the formulas of a base text.

Numerical methods. A subject in which demonstrations were not employed

[40] A couple of Nīlakaṇṭha's demonstrations for Āryabhaṭa's rules on summation of series are translated and explained in [Plo2007b], pp. 493–498. His interpretation and demonstration of Āryabhaṭa's Sine-difference rule as a trigonometrically accurate formula rather than a numerical approximation (see section 5.1.1, *Āryabhaṭīya* 2.12) is discussed in [Hay1997a]. The rationales of Citrabhānu are presented in [Hay1998], and the *Yukti-bhāṣa* with English translation and commentary in [SarK2008].

but that seems nonetheless to have inspired considerable effort and ingenuity was that of "not-just-once" computations, or iterative approximations. Parameśvara was the most active member of the Kerala school in this field, devising more general versions of several of the iterative rules of earlier authors, as well as coming up with new ones. He seems to have been interested in what are today called convergence conditions for iterative algorithms, namely, the situations when they will fail to produce successively better approximations to the desired exact value, or when the successive approximations will converge to it only slowly. His work in this area includes various ways to improve convergence for unreliable iterative rules. Like all other Indian mathematicians, Parameśvara appears to have worked with iterative methods purely empirically; no rationales for their operation are offered.[41]

7.4 ASTRONOMY AND SCIENTIFIC METHODOLOGY

Mādhava's school introduced new developments of existing traditions in astronomy as well as in mathematics. In addition to the standard *pakṣas* of Sanskrit astronomy, Kerala astronomers were familiar with some variant systems. One of these is the so-called *vākya* or "sentence" system mentioned in section 4.3.1, which consists essentially of versified tables of periodically recurring true positions of the celestial bodies, within which the user can interpolate to find true positions for any desired time. The tables are called "sentences" because they are encoded in *kaṭapayādi* notation and thus, like Mādhava's Sine values and coefficients that we saw in section 7.3.2, can have a literary meaning as well as a numerical one. Some of the surviving astronomical texts of Mādhava himself belong to this genre.[42]

Another South Indian astronomical system was adopted and modified by Parameśvara in a work called the *Dṛg-gaṇita*, or "Observation and computation." Although Indian astronomers routinely invoked the desirability of having calculations agree with observations, Parameśvara's work is one of the few places where we see direct evidence of an observational record being constructed.[43]

Parameśvara recorded a number of his own eclipse observations at the end of the *Siddhānta-dīpikā*, a supercommentary he wrote on Govindasvāmin's

[41] Some of Parameśvara's work on iterative approximations is discussed in [Plo1996a] and [Plo2004].

[42] [Pin1970a], vol. 4, pp. 414–415. A different Parameśvara, also writing on astronomy in Kerala in the fifteenth century, composed a *Haricarita*, or "Deeds of Hari (Viṣṇu)," that is simultaneously an account of the life of the divine hero Kṛṣṇa (an incarnation of Viṣṇu) and a set of *vākyas* for lunar motion: [Pin1970a], vol. 4, p. 192.

[43] This raises further questions about the applicability of the statistical analysis evaluations of Billard discussed in section 4.6.2, since Billard concludes that Parameśvara's parameters are "not only speculative but of very mediocre convergence," [Bil1971], p. 156. If systematic observations are the foundation of numerous sets of quite accurate parameters in Indian astronomy, then why would Parameśvara, the most systematic observer attested in the Indian tradition, have produced parameters that are only mediocre?

commentary on Bhāskara I's *Mahā-bhāskarīya*. In these observations (at least some of which were apparently made by assistants or with their help), Parameśvara notes the location, the date, and the altitude of the eclipsed body, as the following example shows:

> When the *ahar-gaṇa* [accumulated days from the start of the Kaliyuga] measured 1,655,662, the sun was seen eclipsed slightly on the banks of the Nilā river by keen observers.
>
> At the time of first contact, the "foot-shadow" [*padabhā*] measured 15. At the time of release, 10 less a half.
>
> On that day, the two eyes of the men observing the disc of the sun were not especially injured; therefore at this time a dullness of the rays is to be assumed.[44]

This corresponds to the partial solar eclipse (magnitude 0.07) visible at Ālattūr (presumably observed at Parameśvara's own *illam*) on 1 February 1432. Identifying the locations and dates of Parameśvara's recorded observations makes it possible to compute the corresponding altitudes predicted by modern astronomy for the bodies at these times. (They indicate that Parameśvara's unusual altitude measure, the so-called "foot-shadow," is equivalent to the shadow of a gnomon six feet high. This peculiar quantity is further discussed in section 7.5.) But as far as we can tell at present, the observations and astronomical parameters recorded by Mādhava, Parameśvara, and their successors did not require any extraordinary precision for values of π or Sines and Cosines, whose brilliant refinements in the Kerala school seem to have been motivated instead by pure mathematical curiosity.

Other astronomical innovations of a different sort followed in the work of Nīlakaṇṭha. In his treatises *Jyotir-mīmāṃsā* (roughly, "Analysis of astronomy") and *Siddhānta-darpaṇa* ("Mirror of *siddhāntas*"), Nīlakaṇṭha makes an extremely intriguing attempt to explain astronomical investigations metaphysically, and to synthesize *siddhānta* practices into a coherent mathematical model. As we have seen in sections 4.3.6 and 6.4, Indian astronomers frequently strove to reconcile or balance arguments from textual authority, often based on a text's status as divine revelation, with the requirements of mathematical consistency or observational accuracy. Nīlakaṇṭha, on the other hand, explicitly advocates collecting observational data to refine and modify the models and parameters of an astronomical system, irrespective of whether they agree with established tradition. Moreover, he appeals to the traditional Sanskrit discipline of *mīmāṃsā*, or the philosophy of scriptural exegesis, to analyze the concept of textual authority. In the following passage, he compares a naive view of scriptural authority with the actual, gradual, process of creating a scientific tradition:[45]

[44] *Siddhānta-dīpikā* 74–76 on *Mahā-bhāskarīya* 5.77, [Will2005], sect. 4.14.4, p. 263; [Kup1957], p. 330.

[45] *Jyotir-mīmāṃsā*: [SarK1977a], pp. 2–3. On this subject see also [Srin2004a] and [NarR2007].

> "Pleased by [feats of] asceticism, Brahman taught to Āryabhaṭa the various numbers [of] the revolutions and circumferences, etc., existing for the computational determination of the planets. Āryabhaṭa in turn combined all that [knowledge], just as [it was] taught, with ten verses [in the chapter on parameters of the *Āryabhaṭīya*]"—this [is what] some think. "Because of Brahman's omniscience, freedom from passion and hatred etc., and certainty of truth, how [can there be] criticism [or testing] of that?" Stupid! [It is] not thus. The favor of the deity is just the cause of mental clarity. Neither Brahman nor the Sun-god [but Āryabhaṭa] himself taught it....
>
> > "[There is] agreement between [the positions of] the moon etc. determined by computation and observation [for one] at a particular place; then [for] another with a fixed opinion, [it is] a teaching confirmed by computation; then [for] one understanding the received teaching, [it is] an authoritative tradition, [which was] to another agreement [with observation], a teaching, [etc.]. Thus authority [is established] from continuity of tradition." (from a commentary on *Mīmāṃsā-sūtra*)

This bold approach to the construction and revision of mathematical models allowed Nīlakaṇṭha to make some ingenious modifications to the Āryapakṣa system. Most strikingly, he changed the arrangement of planetary orbital circles to something that was computationally equivalent to a quasi-heliocentric system similar to the later one of Tycho Brahe, where most of the planets orbit the sun, which in turn orbits the earth. He reversed the usual order of deferents and epicycles so that the center of a planet's own orbit traveled on its *manda*-epicycle (see section 4.3.3), whose center in turn was located at the sun's longitude on the *śīghra*-epicycle, which was centered on the earth. Nīlakaṇṭha fully recognized certain physical implications of this approach, noting, for example:[46]

> Mercury completes its own revolution-circle in eighty-eight days.... And this is not suitable, because its [sidereal] revolution is perceived in one year, not in eighty-eight days.... This statement is made: the earth is not encompassed by the revolution-circle of [either of] those two [Mercury and Venus]. The earth is always outside it.

But although Nīlakaṇṭha's innovations made the star-planets' actual revolutions no longer geocentric, it is not clear that he intended them to be literally heliocentric. He appears to have made them circle about points having the

[46] Commentary on *Āryabhaṭīya* 4.3, [Pil1957], pp. 8–9. Based on the translation in [SarK2008], vol. 2, Epilogue.

same longitude as the sun, but at different distances from the earth.[47] In any case, Nīlakaṇṭha's system was an ingeniously radical rethinking that brilliantly resolved computational inconsistencies in the earlier models; it demonstrates once more, as we saw in section 4.6.3, the advantages that Indian astronomers could reap from not being locked into rigid physical preconceptions about the geometry of the cosmos.

7.5 QUESTIONS OF TRANSMISSION

We know that Greco-Islamic mathematics had a well-established tradition of detailed geometric proof. And we know that the school of Mādhava used detailed geometric rationales in brilliant ways to deduce results about trigonometry, based on manipulations of indefinitely small quantities very reminiscent of the infinitesimals and power series of later European mathematics. Bearing in mind these facts and the multicultural, cosmopolitan nature of the population that flourished on the Malabar coast, it is natural to wonder about possible connections between the Kerala, Islamic, and European mathematical traditions.

For example, it is tempting to read into some of the innovations in the work of the Mādhava school, such as their elaborate geometric demonstrations, their complicated trigonometric formulas, the observational records of Parameśvara, and the boldly rationalist methodology of Nīlakaṇṭha, traces of the Greco-Islamic mathematical science that was presumably familiar to some of their Muslim neighbors on the Kerala coast. In particular, one strong suggestion of (possibly indirect) technological influence from the Islamic tradition appears in the eclipse observations recorded by Parameśvara. Recollect that the quantity that Parameśvara uses to indicate solar altitude, the *padabhā*, is equivalent to the shadow cast by a gnomon measured in feet. Such measures are typical for the altitude scales or "shadow squares" inscribed on the back of Islamic astrolabes, suggesting that Parameśvara was reading his altitude measurements off an instrument of this type.[48] On the other hand, the use of an Islamic astrolabe, which was quite common among Indian astronomers in the second millennium (see section 8.3.1), need not indicate any acquaintance with or interest in Islamic astronomy or mathematics per se.

As for the more abstract resemblances between mathematical interests and outlooks in the two traditions, it must be borne in mind that they are unsupported by any definite evidence for Islamic influence on the content

[47] The similarities between Nīlakaṇṭha's system and a Tychonic model are noted and explained in [Ram1994] and [SarK2008], vol. 2, Epilogue; some of his other innovations, as well as arguments against the claim that he actually espoused physical heliocentrism, are discussed in [Pin2001].

[48] See [SarS2008], p. 246. A comparison of some features of Kerala and Islamic trigonometry, without inferring any influence from the latter on the former, is given in [Plo2002a].

or methodology of Kerala mathematics. On the contrary, the work of the Mādhava school is very deeply rooted in earlier Sanskrit learning: consider, for example, the classic similar-right-triangle proportions and sums of series in the *gaṇita* tradition that contributed to the development of the rationales for trigonometric infinite series, or the *mīmāṃsā* texts that Nīlakaṇṭha used to buttress his defense of revising astronomical models. Moreover, it is far from clear how such a hypothetical Islamic-Kerala transmission would have taken place across ethnic and cultural boundaries, with no obvious social channel of communication between Arab merchant society in the cosmopolitan port cities and the rural *illams* of the scholar-priests.

A similar struggle between speculation and prudence is provoked by the fascinating conceptual similarities between the Mādhava school's methods in infinite series and early modern European infinitesimal calculus techniques. Some scholars have proposed that the former could have been the ultimate source of the latter.[49] In broad outline, this hypothesis suggests that Jesuit missionaries in the vicinity of Cochin in the second half of the sixteenth century sought improved trigonometric and calendric methods in order to solve problems of navigation. Finding the Sine methods described in works of the Kerala school useful for that purpose, they transmitted this knowledge to their correspondents in Europe, whence it was disseminated via informal scholarly networks to the early developers of European calculus methods.

Although this idea is intriguing, it does not seem at present to have moved beyond speculation. There are no known records of sixteenth- or seventeenth-century Latin translations or summaries of these mathematical texts from Kerala. Nor do the innovators of infinitesimal concepts in European mathematics mention deriving any of them from Indian sources. It is true that historical theories about mathematical transmission in antiquity have sometimes been accepted primarily on the basis of such conceptual similarities rather than of (unavailable) documentary evidence. And the historiographic question thus raised is an interesting one: what are or should be the criteria for accepting a hypothesis of cross-cultural transmission as plausible, and are those criteria culturally dependent? However, it seems reasonable to argue that a mathematical culture as heavily documented as early modern Europe's is better mapped historically than most ones in the ancient world, and thus requires stronger evidence if it is to be significantly revised.

Other features of this hypothesis also are less than convincing. First and foremost, how would Sanskrit texts on infinitesimal methods for trigonometry and the circle, transmitted no earlier than the late sixteenth century, have inspired the quite different European infinitesimal techniques used at the start of the seventeenth century? Mathematicians like Kepler and Cavalieri focused instead on mechanical questions such as centers of gravity and areas and volumes of revolution, as well as general problems of quadrature

[49] A strong form of this view is argued in, for example, [Raju2007], pp. 267–374. Some of the similarities between Kerala school techniques and early modern calculus are examined in [Bre2002] and [Kat1995].

explicitly associated with the work of Hellenistic forerunners and of Renaissance geometers such as Nicholas of Cusa. The infinite series that Mādhava discovered did not appear in Latin mathematics until late in the seventeenth century, as suggested by the names—Newton, Leibniz, Gregory—attached to them by earlier historians.

In short, as brilliant and fascinating as the accomplishments of the Kerala school were, there is as yet no persuasive reason to think that our understanding of them will end up substantially changing the history of later European mathematics. So far, all the known similarities between the Kerala school and European works, as well as those between Islamic science and the Kerala school, can be more economically explained as cases of parallel evolution than as evidence of direct transmission.

However, that conclusion should not be taken to mean that medieval Indian mathematics as a whole was isolated or uninfluential with regard to other mathematical cultures. We turn now to the verified transmissions of mathematical ideas to and from India, particularly in its connections with the medieval Islamic world.

Paper 49

David Mumford (2010)

Review of "*Mathematics in India*" by Kim Plofker

Notices of the American Mathematical Society, Vol. 57, No. 3, March 2010, pp. 385–390.

David Mumford

David Mumford is a Fields medal winning American mathematician known for his work in algebraic geometry. It was in this article that Mumford described the work of the Kerala School as the "crown jewel" of Indian mathematics. The chapter of Kim Plofker's book discussing the work of the Kerala School has also been included in the present collection (see p. 648).

Source: American Mathematical Society

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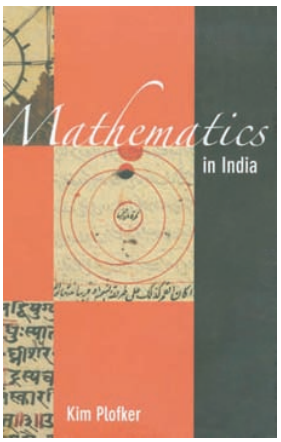
Book Review

Mathematics in India

Reviewed by David Mumford



Mathematics in India
Kim Plofker
Princeton University Press, 2008
US$39.50, 384 pages
ISBN-13: 978-0691120676

Did you know that Vedic priests were using the so-called Pythagorean theorem to construct their fire altars in 800 BCE?; that the differential equation for the sine function, in finite difference form, was described by Indian mathematician-astronomers in the fifth century CE?; and that "Gregory's" series $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$ was proven using the power series for arctangent and, with ingenious summation methods, used to accurately compute π in southwest India in the fourteenth century? If any of this surprises you, Plofker's book is for you.

Her book fills a huge gap: a detailed, eminently readable, scholarly survey of the full scope of Indian[1] mathematics and astronomy (the two were inseparable in India) from their Vedic beginnings to roughly 1800. There is only one other survey, Datta and Singh's 1938 *History of Hindu Mathematics*, recently reprinted but very hard to obtain in the West (I found a copy in a small specialized bookstore in Chennai). They describe in some detail the Indian work in arithmetic and algebra and, supplemented by the equally hard to find *Geometry in Ancient and Medieval India* by Sarasvati Amma (1979), one can get an overview of most topics.[2] But the drawback for Westerners is that neither gives much historical context or explains the importance of astronomy as a driving force for mathematical research in India. While Western scholars have been studying traditional Indian mathematics since the late eighteenth century and Indian scholars have been working hard to assemble and republish surviving Sanskrit manuscripts, a widespread appreciation of the greatest achievements and the unique characteristics of the Indian approach to mathematics has been lacking in the West. Standard surveys of the history of mathematics hardly scratch the surface in telling this story.[3] Today, there is a resurgence of activity in this area both in India and the West. The prosperity and success of India has created support for a new generation of Sanskrit scholars to dig deeper into the huge literature still hidden in Indian libraries. Meanwhile the shift in the West toward a multicultural perspective has allowed us Westerners to shake off old biases and look more clearly at other traditions. This book will go a long way to opening the eyes of all mathematicians and historians of mathematics to the rich legacy of mathematics to which India gave birth.

The first episode in the story of Indian mathematics is that of the *Śulba-sūtras*, "The rules of the cord", described in section 2.2 of Plofker's book.[4]

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[1] *The word "India" is used in Plofker's book and in my review to indicate the whole of the Indian subcontinent, including especially Pakistan, where many famous centers of scholarship, e.g., Takshila, were located.*

[2] *For those who might be in India and want to find copies, Datta and Singh's book is published by Bharatiya Kala Prakashan, Delhi, and Amma's book by Motilal Banarsidass, Delhi. An excellent way to trace the literature is through Hayashi's article "Indian mathematics" in the AMS's CD* History of Mathematics from Antiquity to the Present: A Selective Annotated Bibliography *(2000).*

[3] *The only survey that comes close is Victor Katz's* A History of Mathematics.

[4] *There are multiple ways to transcribe Sanskrit (and Hindi) characters into Roman letters. We follow the precise scholarly system, as does Plofker (cf. her Appendix A) which uses diacritical marks: (i) long vowels have a bar over them; (ii) there are "retroflex" versions of t, d, and n where the tongue curls back, indicated by a dot beneath the letter; (iii) h, as in th, indicates aspiration, a breathy sound, not the English "th"; and (iv) the "sh" sound is written either as ś or as ṣ (the two are distinguishable to Indians but not native English speakers).*

These are part of the "limbs of the Vedas", secular compositions[5] that were orally transmitted, like the sacred verses of the Vedas themselves. The earliest, composed by Baudhāyana, is thought to date from roughly 800 BCE. On the one hand, this work describes rules for laying out with cords the sacrificial fire altars of the Vedas. On the other hand, it is a primer on plane geometry, with many of the same constructions and assertions as those found in the first two books of Euclid. In particular, as I mentioned above, one finds here the earliest explicit statement of "Pythagorean" theorem (so it might arguably be called Baudhāyana's theorem). It is completely clear that this result was known to the Babylonians circa 1800 BCE, but they did not state it as such—like all their mathematical results, it is only recorded in examples and in problems using it. And, to be sure, there are no justifications for it in the *Śulba-sūtras* either—these sutras are just lists of rules. But Pythagorean theorem was very important because an altar often had to have a specific area, e.g., two or three times that of another. There is much more in these sutras: for example, Euclidean style "geometric algebra", very good approximations to $\sqrt{2}$, and reasonable approximations to π.

Another major root of Indian mathematics is the work of Pāṇini and Piṅgala (perhaps in the fifth century BCE and the third century BCE respectively), described in section 3.3 of Plofker's book. Though Pāṇini is usually described as the great grammarian of Sanskrit, codifying the rules of the language that was then being written down for the first time, his ideas have a much wider significance than that. Amazingly, he introduced abstract symbols to denote various subsets of letters and words that would be treated in some common way in some rules; and he produced rewrite rules that were to be applied recursively in a precise order.[6] One could say without exaggeration that he anticipated the basic ideas of modern computer science. One wishes Plofker had described Pāṇini's ideas at more length. As far as I know, there is no exposition of his grammar that would make it accessible to the non-linguist/Sanskrit scholar. P. P. Divakaran has traced the continuing influence of the idea of recursion on Indian mathematics,[7] leading to the thesis that this is one of the major distinctive features of Indian mathematics.

Piṅgala, who came a few centuries later, analyzed the prosody of Sanskrit verses. To do so, he introduced what is essentially binary notation for numbers, along with Pascal's triangle (the binomial coefficients). His work started a long line of research on counting patterns, including many of the fundamental ideas of combinatorics (e.g., the "Fibonacci" sequence appears sometime in 500-800 CE in the work of Virahānka). There is an interesting treatment of this early period of Indian mathematics in Frits Staal's excellent recent book *Discovering the Vedas*,[8] ch.14. For example, Staal traces recursion back to the elaborate and precise structure of Vedic rituals.

After this period, unfortunately, one encounters a gap, and very little survives to show what mathematicians were thinking about for more than 500 years. This was the period of Alexander's invasion, the Indo-Greek Empire that existed side by side with the Mauryan dynasty including Aśoka's reign, and the Indo-Scythian and Kushan empires that followed. It was a period of extensive trade between India and the West, India and China. Was there an exchange of mathematical ideas too? No one knows, and this has become a rather political point. Plofker, I believe, does a really good job discussing the contentious issues, stating in section 4.6 the "consensus" view but also the other points of view. She states carefully the arguments on both sides and lets the reader take away what he or she will. She deals similarly with the early influences from the Middle East in section 2.5 and of the exchanges with the Islamic world in Chapter 8.

For my part, I follow my late colleague David Pingree, who trained a whole generation of scholars in ancient mathematics and astronomy. He argues that the early version of Greek astronomy, due to Hipparchus, reached India along with Greek astrology. The early Indian division of the ecliptic into twenty-eight *Nakṣatras*, (the moon slept with a different wife every night in each trip around the ecliptic) was replaced by the Greek zodiac of twelve solar constellations and—more to the point—an analysis of solar, lunar, and planetary motion based on epicycles appears full-blown in the great treatise, the *Āryabhaṭīya* of Āryabhaṭa, written in 499 CE. But also many things in the Indian treatment are totally different from the Greek version. Their treatment of spherical trigonometry is based on three-dimensional projections, using right triangles *inside* the sphere,[9] an approach

[5] *Technically, they are called* smṛti *("remembered text") as opposed to* śruti *("heard", i.e., from divine sources).*

[6] *To get a glimpse of this, see Plofker, p. 54; F. Staal, "Artificial languages across sciences and civilization",* J. Indian Philosophy, *pp. 89–141 (esp. sections 11–12), 2006; or B. Gillon, "Aṣṭādhyāyī and linguistic theory",* J. Indian Philosophy, *pp. 445–468, 2007.*

[7] *"Notes on Yukti-Bhāṣā: Recursive methods in Indian mathematics", forthcoming in a book entitled* Studies in the History of Mathematics in India.

[8] *Penguin Books, 2008.*

[9] *A basic formula in the Gola section of the* Āryabhaṭīya *is that if P is a point on the ecliptic with longitude λ, then the declination δ of P is given by* $\sin(\delta) = \sin(\lambda).\sin(i)$, *$i$ the inclination of the ecliptic. If I understand it right, later writings suggest this was proven by considering the planar right triangle given by P, P_1, P_2, where P_1 is the orthogonal projection of P onto the plane of the equator (inside the sphere!) and P_2 is its projection onto the line*

which I find much simpler and more natural than Ptolemy's use of Menelaus's theorem. Above all, as mentioned above, they found the finite difference equation satisfied by samples $\sin(n.\Delta\theta)$ of sine (see Plofker, section 4.3.3). This seems to have set the future development of mathematics and astronomy in India on a path totally distinct from anything in the West (or in China).

It is important to recognize two essential differences here between the Indian approach and that of the Greeks. First of all, whereas Eudoxus, Euclid, and many other Greek mathematicians created pure mathematics, devoid of any actual numbers and based especially on their invention of indirect *reductio ad absurdum* arguments, the Indians were primarily applied mathematicians focused on finding algorithms for astronomical predictions and philosophically predisposed to reject indirect arguments. In fact, Buddhists and Jains created what is now called Belnap's four-valued logic claiming that assertions can be true, false, neither, or both. The Indian mathematics tradition consistently looked for *constructive arguments and justifications and numerical algorithms.* So whereas Euclid's *Elements* was embraced by Islamic mathematicians and by the Chinese when Matteo Ricci translated it in 1607, it simply didn't fit with the Indian way of viewing math. In fact, there is no evidence that it reached India before the eighteenth century.

Secondly, this scholarly work was mostly carried out by Brahmins who had been trained since a very early age to memorize both sacred and secular Sanskrit verses. Thus they put their mathematics not in extended treatises on parchment as was done in Alexandria but in very compact (and cryptic) Sanskrit verses meant to be memorized by their students. What happened when they needed to pass on their sine tables to future generations? *They composed verses of sine differences*, arguably because these were much more compact than the sines themselves, hence easier to set to verse and memorize.[10] Because their tables listed sines every 3.75 degrees, these first order differences did not closely match the sine table read backward; but the second differences were almost exactly a small negative multiple of the sines themselves, and this they noticed.

There are several excellent recent books that give more background on these early developments. The mathematical sections of the *Āryabhaṭīya* with the seventh century commentary on it by Bhāskara (I) and an extensive modern commentary, all entitled *Expounding the Mathematical Seed*, has been published[11] in English by Agathe Keller. One hopes she will follow this with an edition of the astronomical chapters. And Glen van Brummelen has written a cross-cultural study of the use of trigonometry, entitled *The Mathematics of the Heavens and the Earth*,[12] which compares in some detail Greek and Indian work.

Chapters 5 and 6 of Plofker's book, entitled *The Genre of Medieval Mathematics* and *The Development of "Canonical" Mathematics*, are devoted to the sixth through twelfth centuries of Indian mathematical work, starting with Āryabhaṭa and ending with Bhāskara (also called Bhāskara II or Bhāskaracharya, distinguishing him from the earlier Bhāskara). This was a period of intense mathematical-astronomical activity from which many works have survived, and I want to touch on some of its high points. We find already in the seventh century the full arithmetic of negative numbers in Brahmagupta's *Brāhmasphuṭa-siddhānta* (see Plofker, p. 151). This may sound mundane but, surprisingly, nothing similar appears in the West until Wallis's *Algebra* published in 1685.[13] And in the *Bakhshālī* manuscript, an incredibly rare birch bark manuscript unearthed by a farmer's plow in 1881, we find algebraic equations more or less in the style of Viète, Fermat, and Descartes. It is incomplete and neither title nor author survives, but paleographical evidence suggests that it was written between the eighth and twelfth centuries, and Hayashi argues that its rules and examples date from the seventh century.[14] The manuscript puts equations in boxes, like our displayed formulas. On p. 159 of Plofker's book, she gives the example from bark fragment 59. The full display in the original is below.

0	5	yu	mū	0		sā	0	7	+	mū	0
1	1			1			1	1			

Here the 0's (given by solid dots in the manuscript) stand for unknowns, the 1's (given by the sigma-like subscripted symbols in the manuscript)

through Υ*, the intersection of the equator and the ecliptic. It is immediate to derive the formula using this triangle.*

[10] *Using* $R = 3438$*, the number of minutes in a radian to the nearest integer, and* $\Delta\theta = 3.75$ *degrees, they calculated* $R \cdot \sin(n\Delta\theta)$ *to the nearest integer.*

[11] *Springer-Verlag, 2008, for an obscene price of $238!!*

[12] *Princeton University Press, 2009.*

[13] *For example, both Cardano and Harriot were unsure whether to make* $(-1) \cdot (-1)$ *equal to* -1 *or* $+1$*.*

[14] *See the fully edited and commented edition by Takao Hayashi,* The Bakhshali Manuscript: An Ancient Indian Mathematical Treatise*, John Benjamin Pub. Co., 1995.*

are just denominators, the *yu* means 5 is added to the unknown on its left, + sign signifies that 7 is subtracted, *mū* means square root, and *sā* is a pronoun indicating that the 1st and 3rd unknowns are equal. The whole thing has the modern equivalent:

$$\sqrt{x+5} = w, \quad \sqrt{x-7} = z.$$

This is to be solved in integers, giving $x = 11$. Note, however, that the Bakhshālī manuscript does not solve its problems using manipulations of its equations. In Brahmagupta's treatment of algebra in the *Brāhma-Sphuṭa-Siddhānta*, distinct colors are used to represent distinct unknowns (see Plofker's discussion of his Chapter 18, pp. 149–157).

The use of negative numbers to represent points on a line to the left of a base point appears in Bhāskara's *Līlāvatī*. This twelfth-century book, described in section 6.2.1 of Plofker's book, is arguably the most famous of all Indian texts on mathematics. Given the fact that *Līlāvatī* literally means "beautiful" or "playful" and that many verses are addressed to "the fawn-eyed one", the conjecture made by a Persian translator that the book was written to explain mathematics to Bhāskara's daughter seems quite reasonable.

Another basic tool which appears in all Indian manuscripts is what they called the pulverizer. This is an extension of the Euclidean algorithm, the idea of starting with two positive integers and repeatedly subtracting the lesser from the greater. They go further than Euclid in using this to systematically write down all solutions of first-order integer equations $ax + by = c$. It seems unlikely that the Greek algorithm, embedded in the *Elements* in highly abstract form, was transmitted to India, hence more likely that the idea was discovered independently in India. In fact, Indian astronomers had a very pressing application for this algorithm. Although they had, in fact, abandoned almost all of the ancient Vedic astronomy, they were not happy doing this and they retained one startling idea from that tradition: the vast epochs into which the past was divided, the yugas, all had to begin with one spectacular conjunction of the sun, the moon, and all the planets. To ascertain when the present yuga began and thus put future predictions on a sound basis, they had to solve such integer equations involving the periods of the heavenly bodies.

There are other high points of the work of this period. One of them is Brahmagupta's formula for the area of a quadrilateral inscribed in a circle. How he discovered this is a fascinating question. No justification has been found in any manuscripts earlier than the Kerala work (see below). It can be derived from Pythagorean theorem and simple geometry but only with substantial algebraic computation. Did Brahmagupta use algebra, manipulations of algebraic equations, to find it or not? That he gives many quite complex auxiliary results on cyclic quadrilaterals suggests he played with such quadrilaterals extensively.[15]

Indian work on Pell's equation in the general form $x^2 - Ny^2 = c$ also goes back to Brahmagupta. He discovered its multiplicative property—solutions for c_1 and c_2 can be "multiplied" to give one for c_1c_2 (Plofker, pp. 154-156). A complete algorithm, known as the "cyclic method", for constructing a solution to the basic equation $x^2 - Ny^2 = 1$ was discovered by Jayadeva, whose work is dated indirectly to the eleventh century (Plofker, pp. 194-195). Note again the emphasis on construction instead of indirect proofs of existence, which are the staple of our treatment of the subject.[16] Why such a focus on this equation? One idea is that if x, y is a solution, then x/y is a good approximation to $\sqrt{N}$.

The discovery of the finite difference equation for sine led Indian mathematicians eventually to the full theory of calculus for polynomials and for sine, cosine, arcsine, and arctangent functions, that is, for everything connected to the circle and sphere that might be motivated by the applications to astronomy. This work matured over the thousand-year period in which the West slumbered, reaching its climax in the work of the Kerala school in the fourteenth to sixteenth centuries. I won't describe the full evolution but cannot omit a mention of the discovery of the formula for the area and volume of the sphere by Bhāskara II. Essentially, he rediscovered the derivation found in Archimedes' *On the Sphere and the Cylinder I*. That is, he sliced the surface of the sphere by equally spaced lines of latitude and, using this, reduced the calculation of the area to the integral of sine. Now, he knew that cosine differences were sines but, startlingly, he integrates sine by summing his tables! He seems well aware that this is approximate and that a limiting argument is needed but this is implicit in his work. My belief is that, given his applied orientation, this was the more convincing argument. In any case, the argument using the discrete fundamental theorem of calculus is given a few centuries later by the Kerala school, where one also finds explicit statements on the need for a limiting process, like: "The greater the number [of subdivisions of an arc], the more accurate the circumference [given by the length of the inscribed polygon]" and "Here the arc segment has to be imagined to be as small as one wants... [but] since one has to explain [it] in a certain [definite] way, [I] have said [so far] that a quadrant has twenty-four chords."

[15] *Added in proof: I just received a copy of S. Kichenassamy's article "Brahmagupta's derivation of the area of a cyclic quadrilateral",* Historia Mathematica, *2009.*

[16] *See, e.g., Artin's* Algebra, *Prentice-Hall, 1991, pp. 434-437.*

Chapter 7 of Plofker's book is devoted to the crown jewel of Indian mathematics, the work of the Kerala school. Kerala is a narrow fertile strip between the mountains and the Arabian Sea along the southwest coast of India. Here, in a number of small villages, supported by the Maharaja of Calicut, an amazing dynasty[17] of mathematicians and astronomers lived and thrived. A large proportion of their results were attributed by later writers to the founder of this school, Madhava of Sangamagramma, who lived from approximately 1350 to 1425. It seems fair to me to compare him with Newton and Leibniz. The high points of their mathematical work were the discoveries of the power series expansions of arctangent, sine, and cosine. By a marvelous and unique happenstance, there survives an informal exposition of these results with full derivations, written in Malayalam, the vernacular of Kerala, by Jyeṣṭhedeva perhaps about 1540. This book, the *Gaṇita-Yukti-Bhāṣā*, has only very recently been translated into English with an extensive commentary.[18] As a result, this book gives a unique insight into Indian methods. Simply put, these are recursion, induction, and careful passage to the limit.

I want to give one example in more detail, the derivation of the power series expansion for sine. It seems most transparent to explain the idea of the proof in modern form and then to indicate how Jyeṣṭhadeva's actual derivation differed from this. The derivation is based on the integral equation for sine:

$$\begin{aligned}\theta - \sin(\theta) &= \int_0^\theta (1-\cos(\beta))d\beta \\ &= \int_0^\theta \left(\int_0^\beta \sin(\alpha)d\alpha\right) d\beta = (K * \sin)(\theta) \\ &\qquad \text{where } K(x,y) = \max(0, x-y).\end{aligned}$$

Jyeṣṭhadeva uses a finite difference form of this equation using discrete samples of sine: he subdivides the arc $[0,\theta]$ into n "arc-bits" of size $\Delta\theta = \theta/n$ and, choosing a big radius R (like 3438, see above), he works with sampled "Rsines" $B_k = R \cdot \sin(k\Delta\theta)$ and also the "full chord" of the arc-bit: $2R\sin(\Delta\theta/2)$. Then, based on the formula for the second difference of sines which goes back to Aryabhata, he derives:

$$\begin{aligned}\theta - \sin(\theta) \approx bB_1 - B_n &= (2\sin(\Delta\theta/2))^2 \\ &\cdot ((B_1 + \cdots + B_{n-1}) + (B_1 + \cdots + B_{n-2}) \\ &\qquad + \cdots + (B_2 + B_1) + B_1)\,.\end{aligned}$$

Note that the right-hand side is exactly the finite difference version of the double integral in the calculus version. Here is how Jyeṣṭhadeva expressed this formula (p. 97 of Sarma's translation; here "repeated summation" stands for the sum of sums on the right):

> Here, multiply the repeated summation of the Rsines by the square of the full chord and divide by the square of the radius. ... In this manner we get the result that when the repeated summation of the Rsines up to the tip of a particular arc-bit is done, the result will be the difference between the next higher Rsine and the corresponding arc. Here the arc-bit has to be conceived as being as minute as possible. Then the first Rsine difference will be the same as the first arc-bit. Hence, if multiplied by the desired number, the result will certainly be the desired arc.

He is always clear about which formulas are exact and which formulas are approximations. In the modern approach, one has the usual iterative solution to the integral equation $\theta - K*\theta + K*K*\theta - \cdots$, and you can work out each term here using the indefinite integrals $\int_0^y x^n dx = \frac{y^{n+1}}{n+1}$ resulting in the power series for sine. Jyeṣṭhadeva does the same thing in finite difference form, starting with $\sin(\theta) \approx \theta$, and recursively improving the estimate for sine by resubstitution into the left-hand side of the above identity. Instead of the integral of powers, he needs the approximate sum of powers, i.e., $\sum_{k=1}^n k^p = \frac{k^{p+1}}{p+1} + O(k^p)$, and he has evaluated these earlier (in fact, results like this go way back in Indian mathematics). Then repeated resubstitution gives the usual power series $\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots$ for sine. I consider this argument to be completely correct, but I am aware that it is *not* a rigorous proof by modern standards. It can, however, be converted into such a proof by anyone with basic familiarity with ϵ, δ-techniques. I hope I have whetted your appetite enough so you will want to feast on the riches laid out in Plofker's book and the *Gaṇita-Yukti- Bhāṣā* itself.

It is very tempting to read the history of mathematics as a long evolution toward the present state of deep knowledge. I see nothing wrong with understanding the older discoveries in the light of what we know now—like a contemporary metallurgist analyzing ancient swords. Needham, the great scholar of Chinese science, wrote "To write the History of Science we have to take modern science as the yardstick—that is the only thing we can do—but modern science will change and the end is not yet." Nevertheless, it is much more satisfying, when reading ancient works, to know as

[17] *The names we know form an essentially linear sequence of teacher and student (sometimes son).*

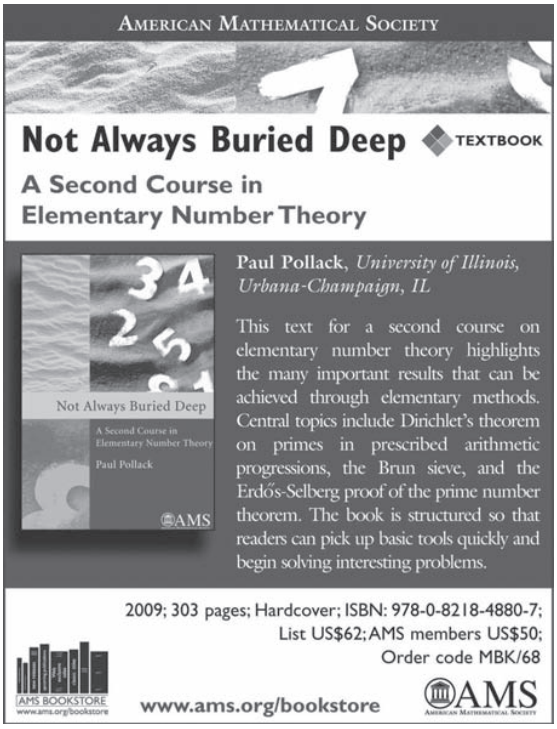
[18] *Translated and edited by the late K. V. Sarma, with notes by K. Ramasubramanian, M. D. Srinivas, and M. S. Sriram, and published in India in 2008 by the Hindustan Book Agency at a price of $31 and distributed in the West by Springer for $199, a 640% markup.*

much as possible about the society in which these mathematicians worked, to know what mathematics was used for in their society, and how they themselves lived.

Chapter 1 in Plofker's book is an extensive introduction that gives vital background on the history and traditions from which all the Indian work sprang. A series of extremely helpful appendices provide basic facts about Sanskrit, a glossary of Sanskrit terms, and a list of the most significant Indian mathematicians with whatever basic facts about them are known (often distressingly little). In places she gives some literal translations such as those of numbers in the colorful concrete number system that uses a standard list of sets with well-known cardinalities, e.g., "In a *kalpa*, the revolutions of the moon are equal to five skies [0], qualities [3], qualities, five, sages [7], arrows [5]", which means 5,753,300,000 lunar months with the digits described backwards starting from the one's place. It is becoming more and more recognized that, for a good understanding of ancient writings, one needs *both* an extremely literal translation and one paraphrased so as to be clear in modern terms. In Sanskrit a literal translation of the complex compound words sometimes gives additional insight into the author's understanding (as well as the ambiguities of the text), so one wishes there were more places in this book where Plofker gave such literal translations without using modern expressions.

I have not touched on the astronomical side of the story. Suffice it to say that almost all treatises from the sixth century on deal with both astronomy and mathematics. To follow these, one needs a bit of a primer in geocentric astronomy, a vanishing specialty these days, and Plofker provides a very handy introduction in section 4.1. Just as in Indian mathematics, there is a steady increase in sophistication over the centuries, culminating in dramatic advances in Kerala. Most strikingly, Nīlakaṇṭha in the fifteenth century proposed a model in which the planets were moving in eccentric and inclined circles with respect to the mean sun moving in its ecliptic orbit—a "virtually" heliocentric model remarkably better than Ptolemy's.

It is high time that the full story of Indian mathematics from Vedic times through 1600 became generally known. I am not minimizing the genius of the Greeks and their wonderful invention of pure mathematics, but other peoples have been doing math in different ways, and they have often attained the same goals independently. Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge. In India, where concrete applications were never far from theory, justifications were more informal and mostly verbal rather than written. One should also recall that the European Enlightenment was an orgy of correct and important but semirigorous math in which Greek ideals were forgotten. The recent episodes with deep mathematics flowing from quantum field and string theory teach us the same lesson: that the muse of mathematics can be wooed in many different ways and her secrets teased out of her. And so they were in India: read this book to learn more of this wonderful story!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Paper 50

Ranjan Roy (2011)

Power Series in Fifteenth-Century Kerala

Chapter 1 in *Sources in the Development of Mathematics - Infinite Series and Products from Fifteenth to the Twenty-first Century*, University Press, Cambridge, 2011, pp. 1-15.

Ranjan Roy

For a biographical note on Ranjan Roy, see p. 445.

Source: Cambridge University Press

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1

Power Series in Fifteenth-Century Kerala

1.1 Preliminary Remarks

The Indian astronomer and mathematician Madhava (c. 1340–c. 1425) discovered infinite power series about two and a half centuries before Newton rediscovered them in the 1660s. Madhava's work may have been motivated by his studies in astronomy, since he concentrated mainly on the trigonometric functions. There appears to be no connection between Madhava's school and that of Newton and other European mathematicians. In spite of this, the Keralese and European mathematicians shared some similar methods and results. Both were fascinated with transformation of series, though here they used very different methods.

The mathematician-astronomers of medieval Kerala lived, worked, and taught in large family compounds called illams. Madhava, believed to have been the founder of the school, worked in the Bakulavihara illam in the town of Sangamagrama, a few miles north of Cochin. He was an Emprantiri Brahmin, then considered socially inferior to the dominant Namputiri (or Nambudri) Brahmin. This position does not appear to have curtailed his teaching activities; his most distinguished pupil was Paramesvara, a Namputiri Brahmin. No mathematical works of Madhava have been found, though three of his short treatises on astronomy are extant. The most important of these describes how to accurately determine the position of the moon at any time of the day. Other surviving mathematical works of the Kerala school attribute many very significant results to Madhava. Although his algebraic notation was almost primitive, Madhava's mathematical skill allowed him to carry out highly original and difficult research.

Paramesvara (c.1380–c.1460), Madhava's pupil, was from Asvattagram, about thirty-five miles northeast of Madhava's home town. He belonged to the Vatasreni illam, a famous center for astronomy and mathematics. He made a series of observations of the eclipses of the sun and the moon between 1395 and 1432 and composed several astronomical texts, the last of which was written in the 1450s, near the end of his life. Sankara Variyar attributed to Paramesvara a formula for the radius of a circle in terms of the sides of an inscribed quadrilateral. Paramesvara's son, Damodara, was the teacher of Jyesthadeva (c. 1500–c. 1570) whose works survive and give us all the surviving proofs of this school. Damodara was also the teacher of Nilakantha (c. 1450–c. 1550)

who composed the famous treatise called the *Tantrasangraha* (c. 1500), a digest of the mathematical and astronomical knowledge of his time. His works allow us determine his approximate dates since in his *Aryabhatyabhasya*, Nilakantha refers to his observation of solar eclipses in 1467 and 1501. Nilakantha made several efforts to establish new parameters for the mean motions of the planets and vigorously defended the necessity of continually correcting astronomical parameters on the basis of observation. Sankara Variyar (c. 1500–1560) was his student.

The surviving texts containing results on infinite series are Nilakantha's *Tantrasangraha*, a commentary on it by Sankara Variyar called *Yuktidipika*, the *Yuktibhasa* by Jyesthadeva and the *Kriyakramakari*, started by Variyar and completed by his student Mahisamangalam Narayana. All these works are in Sanskrit except the *Yuktibhasa*, written in Malayalam, the language of Kerala. These works provide a summary of major results on series discovered by these original mathematicians of the indistinct past:

A. Series expansions for arctangent, sine, and cosine:

   1. $\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$,
   2. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$,
   3. $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$,
   4. $\sin^2\theta = \theta^2 - \frac{\theta^4}{(2^2-2/2)} + \frac{\theta^6}{(2^2-2/2)(3^2-3/2)} - \frac{\theta^8}{(2^2-2/2)(3^2-3/2)(4^2-4/2)} + \cdots$.

   In the proofs of these formulas, the range of θ for the first series was $0 \le \theta \le \pi/4$ and for the second and third was $0 \le \theta \le \pi/2$. Although the series for sine and cosine converge for all real values, the concept of periodicity of the trigonometric functions was discovered much later.

B. Series for π:

   1. $\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n} \pm f_i(n+1), \quad i = 1, 2, 3,$ where

      $$f_1(n) = 1/(2n), \quad f_2(n) = n/(2(n^2+1)),$$

      and

      $$f_3(n) = (n^2+4)/(2n(n^2+5));$$

   2. $\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots$;
   3. $\frac{\pi}{4} = \frac{4}{1^5+4\cdot1} - \frac{4}{3^5+4\cdot3} + \frac{4}{5^5+4\cdot5} - \cdots$;
   4. $\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{3\cdot3} + \frac{1}{5\cdot3^2} - \frac{1}{7\cdot3^3} + \cdots$;
   5. $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{(2\cdot2^2-1)^2-2^2} + \frac{1}{(2\cdot4^2-1)^2-4^2} + \frac{1}{(2\cdot6^2-1)^2-6^2} + \cdots$;
   6. $\frac{\pi-2}{4} \approx \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \cdots \mp \frac{1}{n^2-1} \pm \frac{1}{2\big((n+1)^2+2\big)}$.

These results were stated in verse form. Thus, the series for sine was described:

> The arc is to be repeatedly multiplied by the square of itself and is to be divided [in order] by the square of each even number increased by itself and multiplied by the square of the radius. The arc and the terms obtained by these repeated operations are to be placed in sequence in a column,

and any last term is to be subtracted from the next above, the remainder from the term then next above, and so on, to obtain the jya (sine) of the arc.

So if r is the radius and s the arc, then the successive terms of the repeated operations mentioned in the description are given by

$$s \cdot \frac{s^2}{(2^2+2)r^2}, \quad s \cdot \frac{s^2}{(2^2+2)r^2} \cdot \frac{s^2}{(4^2+4)r^2}, \ldots$$

and the equation is

$$y = s - s \cdot \frac{s^2}{(2^2+2)r^2} + s \cdot \frac{s^2}{(2^2+2)r^2} \cdot \frac{s^2}{(4^2+4)r^2} - \cdots$$

where $y = r \sin(s/r)$. Nilakantha's *Aryabhatiyabhasya* attributes the sine series to Madhava. The *Kriyakramakari* attributes to Madhava the first two cases of B.1, the arctangent series, and series B.4; note that B.4 can be derived from the arctangent by taking $\theta = \pi/6$. The extant manuscripts do not appear to attribute the other series to a particular person. The *Yuktidipika* gives series B.6, including the remainder; it is possible that this series is due to Sankara Variyar, the author of the work. We can safely conclude that the power series for arctangent, sine, and cosine were obtained by Madhava; he is, thus, the first person to express the trigonometric functions as series. In the 1660s, Newton rediscovered the sine and cosine series; in 1671, James Gregory rediscovered the series for arctangent.

The series for $\sin^2\theta$ follows directly from the series for $\cos\theta$ by an application of the double angle formula, $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$. The series for $\pi/4$ (B.1) has several points of interest. When $n \to \infty$, it is simply the series discovered by Leibniz in 1673. However, this series is not useful for computational purposes because it converges extremely slowly. To make it more effective in this respect, the Madhava school added a rational approximation for the remainder after n terms. They did not explain how they arrived at the three expressions $f_i(n)$ in B.1. However, if we set

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n} \pm f(n), \tag{1.1}$$

then the remainder $f(n)$ has the continued fraction expansion

$$f(n) = \frac{1}{2} \cdot \frac{1}{n+} \frac{1^2}{n+} \frac{2^2}{n+} \frac{3^2}{n+} \cdots, \tag{1.2}$$

when $f(n)$ is assumed to satisfy the functional relation

$$f(n+1) + f(n-1) = \frac{1}{n}. \tag{1.3}$$

The first three convergents of this continued fraction are

$$\frac{1}{2n} = f_1(n), \quad \frac{n}{2(n^2+1)} = f_2(n), \quad \text{and} \quad \frac{1}{2}\frac{n^2+4}{n(n^2+5)} = f_3(n). \tag{1.4}$$

Although this continued fraction is not mentioned in any extant works of the Kerala school, their approximants indicate that they must have known it, at least implicitly. In fact, continued fractions appear in much earlier Indian works. The *Lilavati* of Bhaskara (c. 1150) used continued fractions to solve first-order Diophantine equations and Variyar's *Kriyakramakari* was a commentary on Bhaskara's book.

The approximation in equation B.6 is similar to that in B.1 and gives further evidence that the Kerala mathematicians saw a connection between series and continued fractions. If we write

$$\frac{\pi-2}{4} = \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \cdots \pm \frac{1}{n^2-1} \pm g(n+1), \text{ then} \tag{1.5}$$

$$g(n) = \frac{1}{2n} \cdot \frac{1}{n+} \frac{1\cdot 2}{n+} \frac{2\cdot 3}{n+} \frac{3\cdot 4}{n+} \cdots, \text{ and} \tag{1.6}$$

$$g_1(n) = \frac{1}{2n}, \quad g_2(n) = \frac{1}{2(n^2+2)}. \tag{1.7}$$

Newton, who was very interested in the numerical aspects of series, also found the $f_1(n) = 1/(2n)$ approximation when he saw Leibniz's series. He wrote in a letter of 1676 to Henry Oldenburg:

> By the series of Leibniz also if half the term in the last place be added and some other like device be employed, the computation can be carried to many figures.

Though the accomplishments of Madhava and his followers are quite impressive, the members of the school do not appear to have had any interaction with people outside of the very small region where they lived and worked. By the end of the sixteenth century, the school ceased to produce any further original works. Thus, there appears to be no continuity between the ideas of the Kerala scholars and those outside India or even from other parts of India.

1.2 Transformation of Series

The series in equations B.2 and B.3 are transformations of

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

by means of the rational approximations for the remainder. To understand this transformation in modern notation, observe:

$$\frac{\pi}{4} = (1 - f_1(2)) - \left(\frac{1}{3} - f_1(2) - f_1(4)\right) + \left(\frac{1}{5} - f_1(4) - f_1(6)\right) - \cdots. \tag{1.8}$$

The $(n+1)$th term in this series is

$$\frac{1}{2n+1} - f_1(2n) - f_1(2n+2) = \frac{1}{2n+1} - \frac{1}{4n} - \frac{1}{4(n+1)} = \frac{-1}{(2n+1)^3 - (2n+1)}. \tag{1.9}$$

Thus, we arrive at equation B.2. Equation B.3 is similarly obtained:

$$\frac{\pi}{4} = (1 - f_2(2)) - \left(\frac{1}{3} - f_2(2) - f_2(4)\right) + \left(\frac{1}{5} - f_2(4) - f_2(6)\right) - \cdots, \tag{1.10}$$

and here the $(n+1)$th term is

$$\frac{1}{2n+1} - \frac{n}{(2n)^2+1} - \frac{n+1}{(2n+2)^2+1} = \frac{4}{(2n+1)^5 + 4(2n+1)}. \tag{1.11}$$

Clearly, the nth partial sums of these two transformed series can be written as

$$s_i(n) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \mp \frac{1}{2n-1} \pm f_i(2n), \quad i = 1, 2. \tag{1.12}$$

Since series (1.8) and (1.10) are alternating, and the absolute values of the terms are decreasing, it follows that

$$\frac{1}{(2n+1)^3 - (2n+1)} - \frac{1}{(2n+3)^3 - (2n+3)} < \left|\frac{\pi}{4} - s_1(n)\right| < \frac{1}{(2n+1)^3 - (2n+1)^3}. \text{ Also,} \tag{1.13}$$

$$\frac{4}{(2n+1)^5 + 4(2n+1)} - \frac{4}{(2n+3)^5 + 4(2n+3)} < \left|\frac{\pi}{4} - s_2(n)\right| < \frac{4}{(2n+1)^5 + 4(2n+1)}. \tag{1.14}$$

Thus, taking fifty terms of $1 - \frac{1}{3} + \frac{1}{5} - \cdots$ and using the approximation $f_2(n)$, the last inequality shows that the error in the value of π becomes less than 4×10^{-10}. The Leibniz series with fifty terms is normally accurate in computing π up to only one decimal place; by contrast, the Keralese method of rational approximation of the remainder produces numerically useful results.

1.3 Jyesthadeva on Sums of Powers

The Sanskrit texts of the Kerala school with few exceptions contain merely the statements of results without derivations. It is therefore extremely fortunate that Jyesthadeva's Malayalam text *Yuktibhasa*, containing the methods for obtaining the formulas, has survived. Sankara Variyar's *Yuktidipika* is a modified Sanskrit version of the *Yuktibhasa*. It seems that the *Yuktibhasa* was the text used by Jyesthadeva's students at his illam. From this, one may surmise that Variyar, a student of Nilakantha, also studied with Jyesthadeva whose illam was very close to that of Nilakantha.

A basic result used by the Kerala school in the derivation of their series is that

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n} k^p = \frac{1}{p+1}. \tag{1.15}$$

This relation has a long history; sums of powers of integers have been used in the study of area and volume problems at least since Archimedes. Archimedes summed

$$S_n^{(p)} = \sum_{k=1}^{n} k^p$$

for $p = 1$ and $p = 2$. For $p = 2$, he proved the more general result: If $A_1, A_2, \ldots, A_n$ are n lines (we may take them to be numbers) forming an ascending arithmetical progression in which the common difference is equal to A_1 (the least term), then

$$(n+1)A_n^2 + A_1(A_1 + A_2 + \cdots + A_n) = 3(A_1^2 + A_2^2 + \cdots + A_n^2). \tag{1.16}$$

This implies that

$$3(1^2 + 2^2 + \cdots + n^2) = n^2(n+1) + (1 + 2 + 3 + \cdots + n). \tag{1.17}$$

Archimedes used this formula in his work on spirals and in computing the volume of revolution of a segment of a parabola about its axis. The celebrated Arab mathematician al-Haytham (c.965–1039), known also as Alhazen, generalized Archimedes's formula to find the volume of revolution of segment of a parabola about its base. The calculation involved sums of cubes and fourth powers of integers. Al-Haytham proved his generalization by means of a diagram; it can be expressed in modern notation by

$$(n)S_n^{(p)} = S_n^{(p)} + S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_{n-1}^{(p-1)}. \tag{1.18}$$

It is interesting that the statement of Jyesthadeva's first lemma leading to the proof of (1.15) is a restatement of al-Haytham's formula; Jyesthadeva's result was stated:

> Whenever we wish to obtain the sum (sankalitam) of any given powers [say the pth powers of natural numbers, up to an assigned limit n], we multiply the sankalitam of the next lower powers [that is, $(p-1)$th powers, up to the given limit n] by the limit [n]. The result will contain the required sankalitam and also the sankalitam of all the sankalitams of all lower powers up to various limits.

Jyesthadeva's next lemma stated:

> Multiply the lower [power] sankalitam [up to the limit of n] by the limit [n]. Subtract from this product the result of dividing the product by an integer one more than the given power [p]. The result will be [asymptotically equal to] the desired sankalitam.

Thus

$$nS_n^{(p-1)}\left(1 - \frac{1}{p+1}\right) \sim S_n^{(p)} \text{ as } n \to \infty. \tag{1.19}$$

Jyesthadeva proved this result inductively, but he did not perform the induction completely. It is easy to see that (1.19) is equivalent to (1.15) and thus Jyesthadeva assumed that

$$S_n^{(p-1)} \sim n^p/p,$$

which is certainly true for $p = 1$. From this it can be deduced that

$$S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_n^{(p-1)} \sim \frac{1^p + 2^p + \cdots + n^p}{p} = \frac{S_n^{(p)}}{p} \text{ as } n \to \infty.$$

Jyesthadeva asserted this but verified it only for $p = 2$ and 3. But once we fill in the gap by proving this for all p, equation (1.18) implies that

$$(n+1)S_n^{(p-1)} \sim S_n^{(p)} + \frac{S_n^{(p)}}{p} \text{ as } n \to \infty.$$

Hence by the inductive hypothesis it follows that

$$S_n^{(p)} \sim \frac{n^{p+1}}{p+1} \text{ as } n \to \infty.$$

This was Jyesthadeva's argument for (1.15).

1.4 Arctangent Series in the *Yuktibhasa*

The following derivation of the arctangent series, attributed to Madhava, boils down to the integration of $1/(1+x^2)$, as do the methods of Gregory and Leibniz.

In Figure 1.1, AC is a quarter circle of radius one with center O; $OABC$ is a square. The side AB is divided into n equal parts of length δ so that $n\delta = 1$ and $P_{k-1}P_k = \delta$. EF and $P_{k-1}D$ are perpendicular to OP_k. Now, the triangles OEF and $OP_{k-1}D$ are similar, implying that

$$\frac{EF}{OE} = \frac{P_{k-1}D}{OP_{k-1}} \quad \text{or} \quad EF = \frac{P_{k-1}D}{OP_{k-1}}.$$

The similarity of the triangles $P_{k-1}P_kD$ and OAP_k gives

$$\frac{P_{k-1}P_k}{OP_k} = \frac{P_{k-1}D}{OA} \quad \text{or} \quad P_{k-1}D = \frac{P_{k-1}P_k}{OP_k}.$$

Thus,

$$EF = \frac{P_{k-1}P_k}{OP_{k-1}OP_k} \simeq \frac{P_{k-1}P_k}{OP_k^2} = \frac{P_{k-1}P_k}{1+AP_k^2} = \frac{\delta}{1+k^2\delta^2}.$$

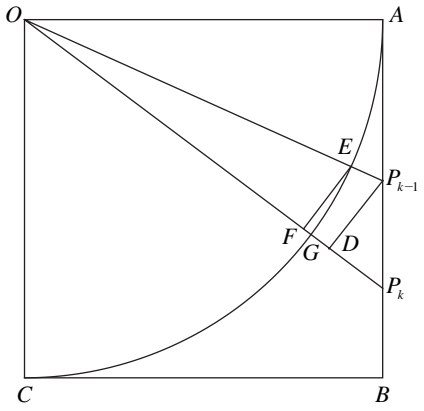

Figure 1.1. Rectifying a circle by the arctangent series.

Now

$$\text{arc } EG \simeq EF \simeq \frac{\delta}{1+k^2\delta^2},$$

and if we write $AP_k = x = \tan\theta$, where $\theta = A\widehat{O}P_k$, then

$$\arctan x = \lim_{k\to\infty} \sum_{j=1}^{k} \frac{\delta}{1+j^2\delta^2}. \tag{1.20}$$

To compute this limit, Jyesthadeva expanded $\frac{1}{1+j^2\delta^2}$ as a geometric series. He derived the series by an iterative procedure:

$$\frac{1}{1+x} = 1 - x\left(\frac{1}{1+x}\right) = 1 - x\left(1 - x\left(\frac{1}{1+x}\right)\right).$$

Thus, (1.20) is converted to

$$\begin{aligned}
\arctan x &= \lim_{k\to\infty}\left(\delta\sum_{j=1}^{k} 1 - \delta^3\sum_{j=1}^{k} j^2 + \delta^5\sum_{j=1}^{k} j^4 - \cdots\right)\\
&= \lim_{k\to\infty}\left(\frac{x}{k}\sum_{j=1}^{k} 1 - \frac{x^3}{k^3}\sum_{j=1}^{k} j^2 + \frac{x^5}{k^5}\sum_{j=1}^{k} j^4 - \cdots\right)\\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.
\end{aligned}$$

The last step follows from (1.15). Note that this is the Madhava–Gregory series for $\arctan x$ and the series for $\pi/4$ follows by taking $x = 1$.

1.5 Derivation of the Sine Series in the *Yuktibhasa*

Once again, Madhava's derivation of the sine series has similarities with Leibniz's derivation of the cosine series. In Figure 1.2, suppose that $A\widehat{O}P = \theta$, $OP = R$, P is the midpoint of the arc $P_{-1}P_1$, and PQ is perpendicular to OA, where O is the origin of the coordinate system. Let $P = (x, y)$, $P_1 = (x_1, y_1)$, and $P_{-1} = (x_{-1}, y_{-1})$. From the similarity of the triangles $P_{-1}Q_1P_1$ and OPQ, we have

$$\frac{P_{-1}P_1}{OP} = \frac{x_{-1} - x_1}{y} = \frac{y_1 - y_{-1}}{x}. \tag{1.21}$$

For a small arc $P_{-1}P = \Delta\theta/2$, identified by Jyesthadeva with the line segment $P_{-1}P$, we can write (1.21) as

$$\cos\left(\theta + \frac{\Delta\theta}{2}\right) - \cos\left(\theta - \frac{\Delta\theta}{2}\right) = -\sin\theta\,\Delta\theta \text{ and} \tag{1.22}$$

$$\sin\left(\theta + \frac{\Delta\theta}{2}\right) - \sin\left(\theta - \frac{\Delta\theta}{2}\right) = \cos\theta\,\Delta\theta. \tag{1.23}$$

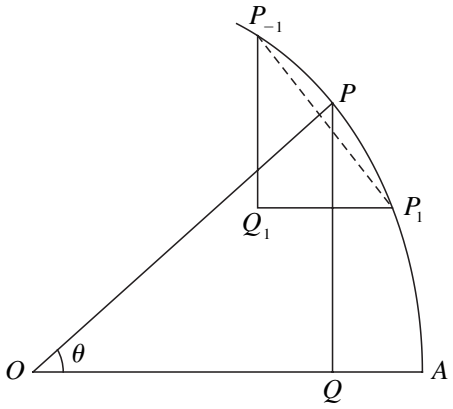

Figure 1.2. Derivation of the sine series.

In fact, Bhaskara earlier stated this last relation and proved it in the same way; he applied it to the discussion of the instantaneous motion of planets. Interestingly, in the 1650s, Pascal used a very similar argument to show that $\int \cos\theta\, d\theta = \sin\theta$ and $\int \sin\theta\, d\theta = -\cos\theta$.

From (1.22) and (1.23) Jyesthadeva derived the result, given in modern notation:

$$\sin\theta - \theta = -\int_0^\theta \int_0^t \sin u\, du dt. \tag{1.24}$$

We also note that Leibniz found the series for cosine using a similar method of repeated integration. In Jyesthadeva, the integrals are replaced by sums and double integrals by sums of sums. The series is then obtained by using successive polynomial approximations for $\sin\theta$. For example, when the first approximation $\sin u \approx u$ is used in the right-hand side of (1.24), the result is

$$\sin\theta - \theta \sim -\frac{\theta^3}{3!} \quad \text{or} \quad \sin\theta \sim \theta - \frac{\theta^3}{3!}.$$

When this approximation is employed in (1.24), we obtain

$$\sin\theta - \theta \sim -\frac{\theta^3}{3!} + \frac{\theta^5}{5!}.$$

Briefly, Jyesthadeva arrived at the sums approximating (1.24) by first dividing AP into n equal parts using division points $P_1, P_2, \ldots, P_{n-1}$. Denote the midpoint of the arc $P_{k-1}P_k$ as $P_{k-1/2}$. Then by (1.21)

$$x_{k+1/2} - x_{k-1/2} = -\frac{\Delta\theta}{2R} y_k, \quad k = 1, 2, \ldots, n-1. \tag{1.25}$$

We also have

$$(y_{k+1} - y_k) - (y_k - y_{k-1}) = \frac{\Delta\theta}{2R}\left(x_{k+1/2} - x_{k-1/2}\right), \quad k = 1, \ldots, n-1 \text{ or} \tag{1.26}$$

$$y_{k+1} - 2y_k + y_{k-1} = -\left(\frac{\Delta\theta}{2R}\right)^2 y_k, \quad k = 1, 2, \ldots, n-1. \tag{1.27}$$

Now start with $k = n-1$ and multiply the equations by $1, 2, \ldots, n-1$ respectively and sum up the resulting equations. We then have

$$\begin{aligned} y_n - n y_1 &= -\left(\frac{\Delta\theta}{2R}\right)^2 (y_{n-1} + 2y_{n-2} + \cdots (n-1)y_1) \\ &= -\left(\frac{\Delta\theta}{2R}\right)^2 (y_1 + (y_1 + y_2) + \cdots + (y_1 + y_2 + \cdots y_{n-1}))\,. \end{aligned} \tag{1.28}$$

This is the result corresponding to (1.24). To obtain the successive polynomial approximations, Jyesthadeva had to work with sums of powers of integers; in order to deal with these sums, he applied the same lemma (1.15) he had used for the arctangent series.

1.6 Continued Fractions

The noted twelfth-century Indian mathematician Bhaskara, who lived and worked in the area now known as Karnataka, used continued fractions in his c. 1150 *Lilavati*. The Kerala school was certainly familiar with Bhaskara's work, since they commented on it. It is therefore possible that they were aware of the specific continued fractions (1.2) and (1.6) for the error terms, even though they mentioned only the first few convergents of these fractions. They did not indicate how they obtained these convergents. Some historians have suggested that Madhava may have found the approximations for the error term, without knowing the continued fractions, by comparing the first few partial sums of the series with a known rational approximation of π. Others speculate that Madhava may have used a method of Wallis.

Whether or not Madhava knew it, Wallis's technique can be used to derive the continued fractions of which the Kerala school gave the convergents; this may be of interest. Start with the functional equation (1.4) for $f(n)$,

$$f(n+1) + f(n-1) = \frac{1}{n}. \tag{1.29}$$

It is obvious that a first approximation for $f(n)$ is given by $f(n) \approx \frac{1}{2n}$. As a first step toward the continued fraction for $f(n)$, set

$$f(n) = \frac{1}{2r_n^{(0)}} \quad \text{and} \quad r_n^{(0)} = n + \frac{1}{r_n^{(1)}}. \tag{1.30}$$

It follows from (1.29) that $r_n^{(0)}$ satisfies

$$\left(2r_{n+1}^{(0)} - n\right)\left(2r_{n-1}^{(0)} - n\right) = n^2. \tag{1.31}$$

From (1.30)

$$2r_{n+1}^{(0)} - n = n + 2 + \frac{2}{r_{n+1}^{(1)}},$$

and a similar relation holds for $r_{n-1}^{(0)}$. When these values are substituted in (1.31), some calculation gives us

$$\left(2r_{n+1}^{(1)} - (n-2)\right)\left(2r_{n-1}^{(1)} - (n+2)\right) = n^2. \tag{1.32}$$

Once again, $r_n^{(1)} \approx n$. So assume $r_n^{(1)} = n + 1/s_n^{(2)}$ and substitute in (1.32) to get, after simplification,

$$16s_{n-1}^{(2)}s_{n-1}^{(2)} - 2(n+4)s_{n+1}^{(2)} - 2(n-4)s_{n-1}^{(2)} - 4 = 0.$$

To obtain an equation such as (1.31) or (1.32), multiply this last equation by 4, set

$$s_n^{(2)} = r_n^{(2)}/2^2,$$

and add n^2 to both sides to get

$$\left(2r_{n+1}^{(2)} - (n-4)\right)\left(2r_{n-1}^{(2)} - (n+4)\right) = n^2. \tag{1.33}$$

We then have $r_n^{(1)} = n + 2^2/r_n^{(2)}$. A similar calculation shows that

$$r_n^{(2)} = n + 3^2/r_n^{(3)}$$

satisfies the equation

$$\left(2r_{n+1}^{(3)} - (n-6)\right)\left(2r_{n-1}^{(3)} - (n+6)\right) = n^2. \tag{1.34}$$

Inductively, it can be shown that if

$$r_n^{(k-1)} = n + k^2/r_n^{(k)}, \text{ and}$$

$$\left(2r_{n+1}^{(k-1)} - (n-2(k-1))\right)\left(2r_{n-1}^{(k-1)} - (n+2(k-1))\right) = n^2, \text{ then}$$

$$\left(2r_{n+1}^{(k)} - (n-2k)\right)\left(2r_{n-1}^{(k)} - (n+2k)\right) = n^2.$$

It follows that $f(n)$ has the continued fraction expansion (1.2). In a similar way, we may obtain the continued fraction (1.6) for $g(n)$ if we start with the functional relation

$$g(n-1) + g(n+1) = \frac{1}{n^2-1}.$$

It may be instructive to consider another method for finding the continued fractions of the Kerala school, also a method for obtaining the successive convergents. There is certainly no clear indication that this method was discovered before Gauss did so in his work on approximate quadrature, published in 1813. Start with a series of the form

$$f(n) = \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots. \tag{1.35}$$

Note that it is always possible to associate a continued fraction with (1.35) by applying successive division. Write (1.35) as

$$f(n) = \frac{1}{n/a_1 - a_2/a_1^2 + t_1(n)}. \tag{1.36}$$

From this we can see that

$$t_1(n) = \frac{b_1}{n} + \frac{b_2}{n^2} + \frac{b_3}{n^3} + \cdots,$$

a series of the same kind as (1.35). So the process can be continued, and the result is a continued fraction for $f(n)$. To find the numbers $a_1, a_2, a_3, a_4, \ldots$, substitute (1.35) in (1.29). The first four values are

$$a_1 = 1/2,\ a_2 = 0,\ a_3 = -1/2,\ a_4 = 0.$$

From these values, we obtain the second convergent of the continued fraction for $f(n)$ by applying the process described in (1.36):

$$\frac{1}{2n + \frac{2}{n}} = \frac{n}{2(n^2+1)}.$$

By also using $a_5 = 5/2$ and $a_6 = 0$, we obtain the third convergent: $(n^2+4)/(2n(n^2+5))$. One problem that arises in the computation of a_1, a_2, a_3, etc., is finding the series expansions of $1/(n+1)^2$, $1/(n-1)^2$, $1/(n+1)^3$, etc. Although this may appear to require knowledge of the binomial theorem for negative integer powers, observe that the series may be obtained by repeatedly multiplying the geometric series by itself. In our chapter on the binomial theorem, we see that Newton verified the correctness of his binomial theorem by multiplying series.

1.7 Exercises

1. Prove that if C is the circumference and D the diameter of a circle, then

$$C = 3D + 6D\left(\frac{1}{(2\cdot 2^2-1)^2-2^2} + \frac{1}{(2\cdot 4^2-1)^2-4^2} + \frac{1}{(2\cdot 6^2-1)^2-6^2} + \cdots\right).$$

   This result (equivalent to B.5) is easily derived from series B.2; it is contained in the *Karanapaddhati*, by an unknown author from the Putumana illam in Sivapur, Kerala. The result is described: "Six times the diameter is divided separately by the square of twice the squares of even integers minus one, diminished by the squares of the even integers themselves. The sum of the resulting quotients increased by thrice the diameter is the circumference." See Bag (1966). Also see Srinivasienger (1967), p. 149.

2. Compute

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{149} - f_3(150)$$

where f_3 is defined in (1.4). This gives π correct to eleven decimal places. In one of his astronomical works, Madhava gave a value of π: "For a circle of diameter 9×10^{11} units, the circumference is 2,827,433,38,233 units." This gives the approximate value of π as 3.14159265359, correct to eleven decimal places. The *Sadratnamala* by Sankara Verman of unknown date gives π to seventeen decimal places. See Parameswaran (1983), p. 194, and Srinivasiengar (1967).

3. Prove al-Haytham's formula (1.18).
4. This exercise outlines the proof of Paramesvara's formula for the radius of the circle circumscribing a cyclic quadrilateral, as given in the *Kriyakramakari*. First, prove that the product of the flank sides of any triangle divided by the diameter of its circumscribed circle is equal to the altitude of the triangle. This result follows from a rule given by Brahmagupta (c. 628) in an astronomical work, the *Brahmasphutasiddhanta*.

   Next, prove that the area of the cyclic quadrilateral is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = (a+b+c+d)/2$$

   and a, b, c, d are the lengths of the sides of the quadrilateral. This was also stated by Brahmagupta. The *Yuktibhasa* contains a complete proof. See also Kichenassamy (2010), who convincingly argues that Brahmagupta had a proof and reconstructs it from indications in *Brahmasphutasiddhanta.*

   Then, let $ABCD'$ be the quadrilateral obtained from $ABCD$ by interchanging the sides AD and CD, so that $AD' = CD = c$ and $CD' = AD = d$. Show that if x, y, z denote the three diagonals AC, BD, BD', respectively, then

$$yz = ab + cd, zx = bc + da, xy = ca + bd.$$

   This is, of course, Ptolemy's theorem. Ptolemy's formula is equivalent to the addition formula for the sine function; his *Almagest*, containing this relation, is heavily indebted to the *Chords in a Circle* of Hipparchus. Bhaskara defined the three diagonals in his *Lilavati*. See Boyer and Merzbach (1991) and Maor (1998), pp. 87–94. Finally, prove that the radius of the circle circumscribing the cyclic quadrilateral is

$$r = \sqrt{\frac{(ad+bc)(ac+bd)(ab+cd)}{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+d-d)}}.$$

   This is Paramesvara's formula, sometimes attributed to S. A. J. L'Huillier who published it in 1782. See Gupta (1977).
5. Use Archimedes's formula (1.17) to show that

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

6. Use al-Haytham's formula (1.19) to obtain

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \text{ and}$$

$$\sum_{k=1}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$$

7. Prove that with the notation as in (1.18),

$$\frac{S_n^{(p-1)}}{n^p} \sim \frac{1}{p} \text{ implies that } \frac{S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_n^{(p-1)}}{1^p + 2^p + \cdots + n^p} \sim \frac{1}{p} \text{ as } n \to \infty.$$

This proves the missing step in the *Yuktibhasa*.

1.8 Notes on the Literature

The Weierstrass quotation at the beginning of the preface is a translation by the author of a passage in a letter to Casorati; see Neuenschwander (1978b), p. 73. It seems that the work of Madhava and his followers on series became known outside India only when a British civil servant and Indologist, Charles M. Whish, wrote a paper on the subject, posthumously published in the *Transactions of the Royal Asiatic Society of Great Britain and Ireland* in 1835. This journal was founded by British Indologists in the early 1830s, though Sir William Jones had first conceived the idea about fifty years earlier. Unfortunately, Whish's paper had little impact. Interest in the Kerala school was renewed in the twentieth century by the efforts of C. Rajagopal and his associates, who published several papers on the topic. See Rajagopal (1949), Rajagopal and Aiyar (1951), Rajagopal and Venkataraman (1949), Rajagopal and Rangachari (1977), and Rajagopal and Rangachari (1986). The translation of the verse enunciating the series for sine is taken from Rajagopal and Rangachari (1977), p. 96. The translations of verses in the *Yuktibhasa* concerning sums of powers are contained in Rajagopal and Aiyar (1951), p. 70. Newton's letter is quoted from Newton (1959–60), p. 140. We discussed two methods for deriving continued fractions for the remainder term; for further details, see Srinivasiengar (1967), pp. 149–151 and Rajagopal and Rangachari (1977). The latter paper makes use of Whiteside's (1961) reconstruction of Wallis's incomplete derivation of Brouncker's continued fraction for π.

The *Yuktibhasa* of Jyesthadeva and the *Tantrasangraha* of Nilakantha have recently been published with commentaries in English by Sarma (1977) and (2008). Sarma (2008) was posthumously published with additional notes by Ramasubramanian, Srinivasa, and Sriram. This two-volume translation with extensive and informative commentary contains both the mathematical and astronomical portions; the original Malayalam text extends to 300 pages. Sarma (1972) also discusses the Kerala school, but from the astronomical point of view. Biographical information on the members of the Kerala school, as well as numerous other ancient and medieval Indian astronomers and mathematicians, can be found in David Pingree's five-volume work (1970–1994).

Readers who wish to read more on the Indian work on series, but with modern notation, may consult Roy (1990), Katz (1995), and Bressoud (2002). These papers are conveniently available in Anderson, Katz, and Wilson (2004). Also see the papers by Parameswaran (1983) on Madhava, Bag (1966) on the *Karanapaddhati*, Gupta (1977) on Paramesvara's rule for radius of the cyclic quadrilateral, and Sarma and Hariharan (1991) on the *Yuktibhasa*. Plofker (2009) presents a scholarly, detailed, and readable discussion of Kerala mathematics, with several excerpts on π translated from Sankara Variyar's *Kriyakramakari*. She also presents the derivation of the sine series with translations from the *Yuktidipika* and describes Takao Hayashi's suggested reconstruction of Madhava's remainder term results. In order to derive the continued fraction, Hayashi and his collaborators have compared the values of partial sums of Madhava's series for π with the then-known rational approximations for π. Van Brummelen (2009), on the history of trigonometry, discusses the contributions of the Kerala school and relates them to the astronomical work of medieval India. In the context of the development of astronomy, Van Brummelen (2009) presents the *Yuktibhasa* derivation of the sine series. This accessible presentation is very helpful, since the mathematics of the Kerala school was largely motivated by an interest in astronomy.

Paper 51

Sreeramula Rajeswara Sarma (2012)

The Kaṭapayādi System of Numerical Notation and its Spread outside Kerala

Revue d'histoire des mathématiques, Vol. 18, 2012, pp. 37–36.

Notes

This is a very scholarly account of the Kaṭapayādi system of numerical notation. This may be read in conjunction with C. M. Whish's article on the same topic which has also been included in this collection (see p. 109).

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THE *KAṬAPAYĀDI* SYSTEM OF NUMERICAL NOTATION AND ITS SPREAD OUTSIDE KERALA

Sreeramula Rajeswara Sarma

Abstract. — While the study of the transmission of scientific ideas from and to India has its own importance, it is also necessary to examine the transmission of ideas within India, from one region to another, from Sanskrit to regional languages and vice versa. This paper attempts to map the spread of the *Kaṭapayādi* system of numerical notation, widely popular in Kerala, to other parts of India, and shows that this very useful tool of mathematical notation, though well known in northern India, was rarely employed there.

The paper further refutes the contention of Bibhutibhusan Datta and Avadhesh Narayan Singh that there existed four distinct variants of the *Kaṭapayādi* system and shows that there were only two genuine variants of the system, one of these being limited to just one single text of unknown provenance.

Résumé (Le système de notation numérique *Kaṭapayādi* et sa diffusion en dehors du Kérala)

Alors que l'étude de la transmission des idées scientifiques entre l'Inde et le reste du monde a sa propre importance, il est également nécessaire d'examiner la transmission de celles-ci à l'intérieur du pays, entre les régions, de la langue sanskrite vers les langues régionales et vice-versa. Cet article se propose de cartographier la diffusion du système *Kaṭapayādi* de notation numérique, largement répandu au Kérala, aux others parties de l'Inde, et montre que ce très important outil de notation mathématique, alors que bien connu en Inde du Nord, y a été rarement employé.

D'autre part, cet article refute l'affirmation de Bibhutibhusan Datta et d'Avadhesh Narayan Singh sur l'existence de quatre variantes distinctes du

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système *Kaṭapayādi* et montre qu'en réalité il n'y en avait que deux, l'une d'entre elles étant limitée à un seul texte, d'origine inconnue.

INTRODUCTION

Indian astronomers/mathematicians developed two major strategies[1] for expressing large numbers consisting of several digits in the metrical pattern of Sanskrit verses with which they composed their works. In the first method, commonly known as *Bhūtasaṃkhyā* or word numerals, the digits 1 to 9 and zero were expressed by certain significant words.[2] Thus, for example, "moon" stands for 1 as there is only one moon; words like "eye" or "hand" represent 2, because "eyes" or "hands" occur always in pairs. In the second system called *Kaṭapayādi*, the digits are represented by the consonants of the Sanskrit alphabet. For example, 1 is denoted by *k*, *ṭ*, *p*, or *y*. Both the methods operate in the decimal place value system, and in both cases, the enumeration commences with the units' place and proceeds to the next higher powers of ten.

This may be illustrated as follows. In the *Bhūtasaṃkhyā* system, the year 2010 can be expressed as *ākāśa-vasudhā-nabhas-kara*. Here *ākāśa* = sky = 0, *vasudhā* = earth = 1, *nabhas* = sky = 0, *kara* = hand = 2. Since the enumeration begins with the units' place, the order of the digits has to be reversed, and the numerical equivalent of *ākāśa-vasudhā-nabhas-kara* is 2010. With its great wealth of synonyms, Sanskrit is capable of expressing numbers in many ways according to the exigencies of the verse meter; thus 2010 can also be expressed as *ananta-soma-kha-hasta* or *viyac-candra-gagana-nayana* and so on.

In the *Kaṭapayādi* system, 2010 can be denoted by *naṭanara* (*na* = 0, *ṭa* = 1, *na* = 0, *ra* = 2; reading backwards 2010). Here the consonants are chosen in such a way that their combination has some meaning and is not a mere jumble of sounds; *naṭanara* means "actor-man" or "a man who is an actor". Since three or four consonants are available for each digit (there are, however, only two consonants for the zero), and since any vowel can be added to a consonant, a number can be expressed in a variety of ways.

---

1 In his *Āryabhaṭīya*, Āryabhaṭa I employs an alphabetical notation, which is distinct from the *Bhūtasaṃkhyā* and *Kaṭapayādi*; cf. [Āryabhaṭa I 1976, 1.1, p. 3–5].

2 This system provides symbolic words for a few two-digit numbers as well, such as *dik* (10), *rudra* (11), *tithi* (15), and so on; cf. [Sarma 2003, p. 61–62].

Of these two systems, the *Bhūtasaṃkhyā* is older[3] and was used throughout India and also in South-East Asia, while the latter is essentially a feature of Kerala. In this paper I propose to survey the current state of research on the *Kaṭapayādi* system and discuss some hitherto unknown cases of its spread outside Kerala.

1. THE *KAṬAPAYĀDI* SYSTEM IN KERALA

The *Kaṭapayādi* system is an alphabetical notation where each consonant of the Sanskrit alphabet is given a numerical value.[4] The system is described in an anonymous line thus: *kādi nava, ṭādi nava, pādi pañca, yādy aṣṭau*, "the nine [consonants] starting with *ka*, the nine starting with *ṭa*, the five starting with *pa* and the eight starting with *ya* [respectively denote the numbers 1 to 9]." But it does not say how the zero is to be represented. The *Sadratnamālā* ("Garland of Pure Gems") by Śaṅkaravarman, composed in 1819 AD, gives a comprehensive definition by stating that "*na*, *ña* and the vowels [standing alone denote] zeros; [the consonants] beginning with *ka*, *ṭa*, *pa* and *ya*, [denote severally] the numbers [1 to 9]; in a conjunct [consonant, only] the consonant in the penultimate [place, which is followed by a vowel, denotes a] number; and a consonant not attached to a vowel should not be considered [as a number]."[5] This may be graphically shown in the following table:

1	2	3	4	5	6	7	8	9	0
ka	*kha*	*ga*	*gha*	*ṅa*	*ca*	*cha*	*ja*	*jha*	*ña*
ṭa	*ṭha*	*ḍa*	*ḍha*	*ṇa*	*ta*	*tha*	*da*	*dha*	*na*
pa	*pha*	*ba*	*bha*	*ma*					
ya	*ra*	*la*	*va*	*śa*	*ṣa*	*sa*	*ha*		unattached vowels

[3] Symbolic words representing numbers are used sporadically in the *Śrauta-sūtras* and *Vedāṅga-jyotiṣa*. Pingala's *Chandaḥsūtra*, which is generally placed in the second century BC, is the earliest text where these are employed rather extensively, with about a hundred occurrences; cf. [Sarma 2009b].

[4] The important literature on the *Kaṭapayādi* system is the following, arranged in chronological order: [Warren 1825, p. 335], [Whish 1827], [Jacquet 1835, p. 122–130], [Ojha 1971, p. 123], [Fleet 1911], [Fleet 1912], [Datta & Singh 1962, Part 1, p. 69–72], [Raja 1963], [Sarma 1972, p. 6–8], [Subbarayappa & Sarma 1985, p. 47–48], [Sarma 2003], [Yano 2006, p. 150–153], [Plofker 2009, p. 75–77].

[5] [Śaṅkaravarman 2001, 3.3, p. 22]:
nañāv acaś ca śūnyāni saṃkhyāḥ kaṭapayādayaḥ |
miśre tūpāntahal saṃkhyā na ca cintyo hal asvaraḥ ||
Though the *Kaṭapayādi* system was followed extensively for a very long time in Kerala, it is strange that there are no definitions earlier than this one in the *Sadratnamālā* of the nineteenth century! Perhaps it was thought unnecessary to define it as it was universally known.

Though neither of the definitions expressly states it, the numerals represented in this system are read from the right to the left.[6] This is a neat and elegant method of expressing long numbers in astronomical texts, more so because the numerical phrases, apart from the numerical value they represent, are made up of recognizable words with some meaning. An oft-quoted example is that of Melpatūr Nārāyaṇa Bhaṭṭatiri, a great Sanskrit poet of Kerala, closing his devotional poem *Nārāyaṇīyam* with the expression *āyur-ārogya-saukhyam*.[7] On the one hand it is a benediction for longevity (*āyur*), health (*ārogya*) and happiness (*saukhya*), for himself and for the readers of his work; on the other it is a chronogram indicating the date of completion of the work, viz. 17,12,210 civil days from the beginning of the present Kali era,[8] which translates to 8 December 1586 AD (Gregorian).

What is the antiquity of this system? What is the geographical extent of its use? As with many things Indian, there is no lack of efforts to ascribe "hoary" antiquity to this system, to a period prior to Piṅgala[9] or to the time of the *Mahābhārata*.[10] If that were so, there ought to be abundant examples

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6 Following the anonymous maxim *aṅkānāṃ vāmato gatiḥ*, "the digits proceed [from the right] to the left." This is true also of the word notation or the *Bhūtasaṃkhyā* system. On this convention of right-to-left enumeration, see [Sarma 2009b].

7 The final verse of this poem reads as follows [Nārāyaṇa, 100.11]:
ajñātvā te mahattvaṃ yad iha nigaditaṃ viśvanātha kṣamethāḥ
stotraṃ caitat sahasrottaram adhikataraṃ tvatprasādāya bhūyāt |
dvedhā nārāyaṇīyaṃ śrutiṣu ca januṣā stutyatāvarṇanena
sphītaṃ līlāvatārair idam iha kurutām āyurārogyasaukhyam ||
"Forgive, o lord of the universe, whatever I have said here without comprehending your greatness. May this hymn of praise, in [stanzas] more than a thousand, be the source of your grace! [The name of this hymn] *Nārāyaṇīyam* (lit. of / belonging to Nārāyaṇa) [is justified] in two ways: by birth [for it was created by me with the name Nārāyaṇa] and by its praise [of you with the name Nārāyaṇa] in accordance with the scriptures. May this hymn, full of the descriptions of your playful incarnations, confer long life, good health and happiness!"

8 Cf. [Sarma 2003, p. 41]. Here the number given is 17,12,211, obviously a printer's devil.

9 In a highly speculative exercise, [Kak 2000] postulates that the *Kaṭapayādi* (KTPY) notation existed even before the time of Piṅgala who "was the younger brother of Pāṇini" and concludes that his exercise "takes the history of the KTPY mapping more than a thousand years earlier than has been supposed."

10 [Madhavan 1991] is said to have argued that the *Kaṭapayādi* system was used in the *Upadeśasūtra* or *Jaiminisūtra* on astrology by Jamini who is identical with the author of the *Mīmāṃsāsūtra* and therefore "the system probably existed even as early as the fifth century B. C." and even earlier since "The *Mahābharata* is called *Jaya* because *jaya* in *Kaṭapayādi* system means eighteen". [Sarma 2003, p. 43], who refers to this view, adds that it is unacceptable.

of its use from all over India. As will be shown in the following pages, this system of notation was confined only to Kerala, from where it spread sporadically to other parts of India.

The earliest text to employ the *Kaṭapayādi* system is said to be the *Candra-vākyas* ("Moon-sentences") which are attributed to Vararuci. Vararuci is a legendary figure of Kerala who is supposed to have lived in the fourth century AD. But there is no firm evidence to show that he did indeed live in the fourth century AD.[11] According to David Pingree, "the earliest attested epoch of the lunar vākyas is 1184."[12] In his commentary on the *Āryabhaṭīya*, Śūryadeva Yajvan (b. 1191) asserts that Āryabhaṭa I must have known the *Kaṭapayādi* system,[13] implying thereby that the system was already prevalent in the fifth century AD. However, the first positive and datable occurrence is in the *Graha-cāra-nibandhana* which was composed in 683 AD in Kerala by Haridatta, who promulgated through this work the *Parahita* system of astronomy.[14] It cannot be said whether Haridatta himself invented the *Kaṭapayādi* system, but it has been widely

11 [Raja 1963, p. 122], after citing the legend about Vararuci belonging to the fourth century, cautions us, saying that "much credence cannot be given to such stories."

12 [Pingree 1981, p. 47].

13 Commenting on *Āryabhaṭīya*, *Gītikāpāda*, 2 [Āryabhaṭa I 1976, 1.2], where Āryabhaṭa I describes his own system of alphabetical notation, Sūryadeva opines that in the definition of his system of alphabetical notation Āryabhaṭa used the expression *kāt* (i.e. consonants starting from *ka*, and not from *ṭa*, *pa* or *ya*), in order expressly to distinguish his method from the prevailing *Kaṭapayādi* system where consonants severally starting from *ka*, *ṭa*, *pa* and *ya* successively represent the digits from 1 to 9. Cf. [Sūryadeva 1976, p. 10]: *nanu vargākṣarāṇāṃ svarūpavat kakārāditvasyāpi lokasiddhatvāt "kāt" ity etad anarthakam. na. vargākṣarāṇāṃ saṃkhyāpratipādane kaṭapayāditvaṃ nañayoś ca śūnyatvam api prasiddham. tannirāsārthaṃ "kād"-grahaṇam. "kāt" prabhṛty eva vargākṣarāṇāṃ saṃkhyā, na ṭakārāt pakārāc ca prabhṛti. "kāt" prabhṛti sarvāṇi saṃkhyāṃ pratipādayanti, na tu ñakāra-nakārayoś ca śūnyatvam ity arthaḥ.* "[If somebody were to say] 'Now the term *kāt* is redundant because the grouping of the consonants as *vargas* is well known, so too the nature of the consonants *ka* and others', [then our reply is] 'not so'. It is well known that the consonants beginning with *ka*, *ṭa*, *pa*, *ya* in the grouping called *varga* denote the numbers [1 to 9], and *na* and *ña* [denote] zero [in the *Kaṭapayādi* system]. In order to avoid the [possibility that his new system may be confused with the *Kaṭapayādi* system], the term *kāt* has been used by [Āryabhaṭa]. [It means that in Āryabhaṭa's system] the counting of *varga* consonants begins from *ka* only, and not from the letter *ṭa* or *pa*. From *ka* onwards, all [the *varga* consonants] denote numbers [from 1 to 25]; *ña* and *na* do not denote zero."

14 [Datta & Singh 1962, part I, p. 71] say "Its first occurrence known to us is found in *Laghubhāskarīya* of Bhāskara I (522)," and refer to the passage 1.18. However, [Shukla 1976, p. xx-xxii] conclusively demonstrated that Datta & Singh's conjecture of 522 being the date of Bhāskara I and the statement that he used the *Kaṭapayādi* notation are based on a spurious verse found only in one manuscript but not in others.

popular in Kerala at least from his time. Sundararāja (ca. 1475/1500), in his commentary *Laghuprakāśikā* on the *Vākyakaraṇa*, expressly associates the *Kaṭapayādi* system with Haridatta.[15]

The designation of the system as *Kaṭapayādi* (lit. that which begins [severally] with *ka*, *ṭa*, *pa* and *ya*) is attested at least from the ninth century. Kunjunni Raja states that Śaṅkaranārāyaṇa (ca. 825–900) mentions this name in his commentary on the *Laghubhāskarīya* of Bhāskara I.[16] The system is also referred to by its generic name as *Akṣarasaṃjñā* (lit. "alphabetical notation") as will be shown below.

In Malayalam this system is known also as *Paralpperu* where *paral* means "sea shell" and *peru* "name". This expression does not designate the alphabetical notation as such, but merely refers to its application in the astronomical computations performed traditionally in Kerala, using sea shells in this process. Lieutenant Colonel John Warren (1769–1830), [17] an officer in the service of the East India Company and for some time the acting director of the Madras Observatory, was the first European to describe how astronomers performed computations by placing sea shells at appropriate places, while reciting the relevant mnemonic lines with the numerical expressions in the *Kaṭapayādi* notation. The beginning of his description [Warren 1825, p. 334] is cited below. The chapter, which he calls "Fragment IV", has the following heading: *Computation of an Eclipse of the Moon by means of certain memorial and artificial words, and of shells in lieu of figures… By Sami Naden Sashia [=Svāmināthan Śeṣayya], a Kalendar maker residing in Pondicherry.*

> I often read and heard of the singular process by means of which the common Indian Almanac makers computed Eclipses; scoring their quantities with shells, instead of writing them in figures; and dispensing with the use of Tables, by means of certain artificial words, and syllables; which recalled the required

15 Cf. [Sundararāja 1962, p. 37]: *punar apīha … lāghavena vyavahārāya …haridattādibhir aṅgīkṛta-parahitādi-śāstrokta-nyāyena kaṭapayādibhir eva saṃjñābhidhīyate.* "Again here [in this *Vākyakaraṇa*], for the sake of ease in computation, the numbers are denoted by the *Kaṭapayādi* system only, which was adopted (*aṅgīkṛta*) by Haridatta and others and was employed in texts (*śāstra*) like the *Parahita* (i.e., *Grahacāra-nibandhana*) and others."

16 [Raja 1963, p. 129, n. 5], cites [Śaṅkaranārāyaṇa 1949, p. 38]: *kollapuryāṃ viṣuvacchāyāyāḥ pañcadaśasaṃkhyā-saṃpādita-rāśipramāṇāḥ kaṭapayādy-akṣara-baddhāḥ paṭhyante.* "The measures of the equinoctial shadow (*viṣuvacchāyā*) and measures of the zodiac signs, obtained with the number fifteen, in Kollāpurī will be presented in letters beginning with *ka*, *ṭa*, *pa* and *ya*."

17 On Warren, see [Kochhar 1991, p. 98], who likes to narrate that John Warren "was so popular that when his second daughter got married in 1829, the Hindus wanted to pay the expenses of the festivities."

> numbers and Equations to their recollection, and was long desirous to obtain a positive proof of the truth of that report, which I always suspected to be much exaggerated. After a long search for one of these mechanical computers, a person was introduced to me by my venerable friend Abbe *Mottet* (one of the Missionaries of the Institution *de Propaganda Fide* in this part of India), and found the *Sashia* thus introduced to me, competent to my object, for (as I wished) he did not understand a word of the theories of Hindu Astronomy, but was endowed with a retentive memory, which enabled him to arrange very distinctly his operations in his mind, and on the ground.
>
> This person, whose name was *Sami Naden Sashia*, computed before me the Lunar Eclipse which forms the subject of the present Fragment; and after a due examination of his process, I concluded (as I indeed had expected) that the artificial words which were supposed to elicit results, were only designed as vehicles for finding the arguments of the four Vakiam Tables published in this collection, and of some others not included therein, without which it would have been impossible for him to perform his task.[18]

About the geographical extent of its use, Raja [Raja 1963, p. 122] observes as follows: "The *Kaṭapayādi* system is well known only in South India and is most popular in Kerala. [...] It is generally believed to be one of the major contributions of Kerala to Indian mathematics." That it is employed in Kerala very widely, not only in works on astronomy and mathematics, but in non-scientific works also, not only in Sanskrit writings but in Malayalam as well, is now quite well established [Sarma 2003]. Indeed, in order to facilitate its use in Malayalam texts, the purely Dravidian consonant *ḷ* was also incorporated into the system with the numerical value of 7 [Sarma 1972, p. 6–8]. Even in non-scientific works, the authors preferred to give the date of composition in the form of *Kali-ahargaṇa* (i.e., the number of the civil days elapsed from the beginning of the Kali) in the *Kaṭapayādi* system. Since this number is rather large, the *Kaṭapayādi* system is more convenient than the word numerals; the added advantage being that the poet so chooses his chronogram that it yields, besides the numerical data, some significant or charming meaning as well.

Indeed, it seems that in traditional Kerala, children were taught the *Kaṭapāyadi* system immediately after they had learnt the alphabet. In his memoirs entitled *Ente Smaraṇakal*, Kāṇippayyūr Śaṅkaran Nampūtiripāṭu (1891–1981) narrates as follows: "The next lesson after writing in sand was *akṣarasaṃkhyā* (literally 'letter number'), which was a useful way of remembering the veritable universe of numerical formulae in astrological calculation, in the days when books and tables either containing or

18 In Kerala there are still astronomers/astrologers who can compute in this manner with the aid of sea-shells. Their performances deserve to be video-graphed and preserved for posterity.

making unnecessary such formulae, were not published. Each syllable was taken to represent a digit (according to the predominant letter in it) and hence each word, taken as a sequence of syllables, would represent a sequence of digits and thus a number. [...] Hence numerical formulae could be represented in memorized verses, as in some of the astrological verses I learned later on in my primary education."[19]

This system of notation proved to be very useful in Kerala. There it helped the development of a uniform mode of dating events in terms of the *Kali-ahargaṇa*, i.e. the number of civil days from the commencement of the present Kali era at the sunrise on Friday, 18 February 3102 BC (or −3101). *The Kali-ahargaṇa* is akin to, and an early prototype of, the modern method of dating in terms of "Julian days" from the noon of 1 January 4713 BC (−4712),[20] and it is less complicated and more accurate than other calendrical systems.[21] Moreover, the first step in determining the planetary positions on any given day is to compute the number of civil days up to that day from a certain epoch. If the date is given in any other calendrical system, it has to be converted into *ahargaṇa* from the commencement of Kali era or any other epoch. In the case of the *Kali-ahargaṇa* such conversion is not necessary; it can directly be employed in computing the planetary positions.

Likewise, this system facilitated the development of mnemonic tables like the *Candra-vākyas*, which in their turn contributed to the further advances of the Kerala school of mathematics.[22] The *Candra-vākyas* are 248 pithy phrases which state the daily angular motion of the Moon in signs, degrees and minutes. The first phrase *gīr naḥ śreyaḥ*, though it literally means "our welfare [lies in our] speech", gives the parameter 12°03′. The second phrase *dhenavaḥ śrīḥ*, "cows [are] prosperity," 24°09′. While the true moon

19 Translated and cited by [Wood 1985, p. 37]. It has not been possible for me to find out what sort of mnemonic verses were taught to children so that they became proficient in the *Kaṭapayādi* system.

20 This will be clear by the following example. On 1 January 2010, the *Kali-ahargaṇa* is 1,866,732 and the Julian day number at noon is 2,455,198. The difference between these two numbers, i.e. 588,466, remains constant. In the Islamic world, there is an analogous method, known as *ḥisāb al-jummal*, in which dates are expressed by means of chronograms made up of the *abjad* alpha-numerals [*Encyclopædia of Islam*, vol. 3, 1971, p. 468].

21 In the other systems of dating several variables have be taken into account, such as the different eras, lapsed or current years, solar or lunar months, in the latter case, whether ending in the full-moon (*pūrṇimānta*) or in the new-moon (*amānta*), and so on.

22 On Kerala Mathematics, see, among others, [Joseph 1991, p. 286–294] and [Plofker 2009, p. 217–253, esp. 245–246].

is calculated with these moon-sentences, the true planets are computed with the help of 2075 sentences of the anonymous *Vākyakaraṇa* which was composed in about 1300 [Pingree 1981, p. 48] [Sundararāja 1962].

2. THE SPREAD OF THE *KAṬAPAYĀDI* SYSTEM

From Kerala, the *Kaṭapayādi* system spread to the neighbouring Tamilnadu, where its popularity is fairly well attested. K. V. Sarma who wrote extensively on astronomy and mathematics in Kerala, and also on the history of the *Kaṭapayādi* system, enumerates several cases [Sarma 2003, p. 44]. Sūryadeva Yajvan, who hailed from Gangaikonda-Colapuram, also in Tamilnadu, mentions in one of his works the date of his birth as *vijñāna-dīptāśayaḥ* (lit. "one whose aspirations are illumined by his knowledge"), numerically 15,68,004 civil days from the commencement of the Kali era.[23] We have also seen from John Warren's testimony that the method of astronomical computations of Kerala was being practiced in Pondicherry in Tamil country still at the beginning of the nineteenth century, using *vākyas* that contain the *Kaṭapayādi* notation.

Outside the texts on mathematics/astronomy, one discipline where the *Kaṭapayādi* system was meaningfully employed in Tamilnadu is the science of music. The musicologist Veṅkaṭamakhin classified the *rāgas* according to their pitch content. In his *Caturdaṇḍi-prakāśikā*, composed in 1660 for the court of Tanjore, he listed all the 72 scales possible in the 12-tone division of the octave. Employing the *Bhūtasaṃkhyā* system, he named the 12 divisions (*cakras*) as *indu* (1), *netra* (2), *agni* (3), *veda* (4), *bāṇa* (5), *ṛtu* (6), *ṛṣi* (7), *vasu* (8), *brahmā* (9), *dik* (10), *rudra* (11) and *āditya* (12). For the individual names of the 72 *rāgas*, he made use of the *Kaṭapayādi* notation in such a way that the first two letters of each name indicate the serial number of the *rāga*. The remaining letters in the name indicate other features of the *rāga*. Thus, e.g. *Kanakāṅgī* shows the serial number 1, *Rūpavatī* 12, *Ṣaṇmukhapriyā* 56, or *Rasikapriyā* 72.[24]

In Karnataka, the *Kaṭapayādi* notation was cultivated by the Jain scholars. In particular, Nemicandra Siddhāntacakravartin (10th century AD) employs it occasionally in his writings, in addition to the ordinary mode

23 Cf. [Sūryadeva 1976, p. xxviii, n.1]. The date translates to 1192 February 4 Tuesday (Gregorian).

24 Cf. [Sarma 2003, p. 40, 43]; [Randel 2003, p. 813]; Wikipedia, s. v. Katapayadi system, accessed on 17 April 2011, contains an interesting chart which shows graphically the 72 *rāgas* along with their attributes.

of expressing numbers. Interestingly enough, Nemicandra uses the letter-numerals rather ambidextrously, that is to say, sometimes in the standard right-to-left mode of enumeration and sometimes in the non-standard left-to-right mode, as shown in the following examples:

2.1. *Ordinary mode of expressing numbers*

Gommaṭasāra, Jīvakāṇḍa, verse 350 [Nemicandra 1974a, p. 426]:

bāruttara-saya-koḍī tesīdī tahaya hoṃti lakkhāṇaṃ ।
aṭṭhāvaṇṇa-sahassā paṃceva padāṇi aṃgāṇaṃ ॥

"The number of words in the [twelfth] *aṅga* [of the Jain canon] are one hundred and twelve ten-millions (*koḍi*), eighty-three hundred-thousands (*lakkha*), fifty-eight thousands (*sahassa*) and five," i.e. 1,128,358,005.

2.2. *Letter numerals in the standard right-to-left enumeration*

Gommaṭasāra, Jīvakāṇḍa, verse 158 [Nemicandra 1974a, p. 221] [Datta 1930–31, p. 78, n. 4]:

ta-la-lī-na-ma-dhu-ga-vi-ma-laṃ dhū-ma-si-lā-gā-vi-co-ra-bha-ya-me-rū ।
ta-ṭa-ha-ri-kha-jha-sā hoṃti hu māṇusa-pajjatta-saṃkhaṃkaṃ ॥

Here the number of a class of human beings called *pajjatta* is stated to be
79,228,162,514,264,337,593,543,950,336.

2.3. *Letter numerals in non-standard left-to-right enumeration*

Trilokasāra, verse 98 [Nemicandra 1974b, p. 92] [Datta 1930–31, p. 78, n. 4]:

va-ṭa-la-va-ṇa-ro-ca-gā-na-ga-na-ja-ra-na-gaṃ kā-sa-sa-sa-gha-dha-ma-pa-ra-ka-dha-raṃ ।
viguṇa-na-suṇṇa-sahidaṃ pallassa roma-parisaṃkhā ॥

"The number of hairs [filling an enormous cosmic space called] *palla* are 413,452,630,308,203,177,749,512,192, followed by twice nine zeros," i.e. 413,452,630,308,203,177,749,512,192,000,000,000,000,000,000

Here it is not expressly stated that the digits are to be read from left to right. But the preceding verse 91 [Nemicandra 1974b, p. 92] states that this number is the product of four factors, viz. *ekkaṭṭhī* (18,446,744,073,709,551,616) × *paṇṇaṭṭhī* (65,536) × *uṇavīsa* (19) ×

aṭṭhārasa (18), and twice nine zeros. Therefore, only the left-to-right enumeration gives the correct product.

The fact that Nemicandra employs the *Kaṭapayādi* notation in both directions suggests that the system is of course known in Karnataka, but not quite well established in the tenth century.

In the Andhra region, however, no scientific or other kind of text appears to have employed this system of notation.

The level of popular usage can be gauged by inscriptions. It is well known that the *Bhūtasaṃkhyā* system of word numerals is widely employed in inscriptions all over India and also in South-East Asia in the chronograms expressing the dates. [Sircar 1965, p. 228–233]. But no comprehensive study has been undertaken so far about the use of the *Kaṭapayādi* system in inscriptions. Gaurishankar Hirachand Ojha lists just five Sanskrit inscriptions that use the *Kaṭapayādi* system, but he does not properly situate the inscriptions as to their time and provenance.[25] An examination of the inscriptions themselves shows that these are rather late records belonging to the 14–16th centuries and emanating from what are today Kerala, Tamilnadu and Andhra Pradesh.

Three of these inscriptions are from Kerala: (i) *rākāloke śakābde*, i.e. Śaka year 1312 = 1389–90 AD; (ii) *śrīmatkolambavarṣe bhavati* denotes the year 644 of the Kolam era, which corresponds to 1468–69 AD; (iii) a*bde kolambavarṣe viśati gavi gurau*, where the chronogram *viśati* has the numerical value of 654, of the Kolam era, which equals 1478–79 AD. [Varmâ 1873].

Two inscriptions are from Tamilnadu. One of these is from the Kāmākṣī Temple of Kāñcīpuram and records the gift of a copper door by Harihara II in *śaktyāloke śakābde*, i.e., Śaka year 1315 (= 1392–93 AD).[26] The other inscription is from Satyamangalam in Vellore district and was issued by Devarāya II. The date is mentioned thus in lines 39–40 [Hultzsch 1894–1895, p. 38]:

> *tattvāloke śakasyābde krodhisaṃvatsare śubhe* |
> *āṣāḍhāmātithau puṇye somavāravirājite* ||

"On Monday, on Amāvāsyā of Āṣāḍha month, in the Jovian year Krodhin, in the Śaka year 1346 (*tattvāloke*)." This corresponds to Monday, 24 July 1424 (Gregorian).

Only one inscription is noticed from the Telugu region. It is incised in Telugu characters on a stone pillar at a temple at Mangalagiri in Guntur

25 Cf. [Ojha 1971, p. 123]; the same is repeated by [Datta & Singh 1962, p. 70–71].

26 It may be noted that in the conjunct consonant *ktyā*, only the final consonant *y* has numerical value.

district and was issued by Kr̥ṣṇadevarāya [Lüders 1900–1901]. It contains several dates expressed in word numerals, occasionally also in ordinary numerals. In two cases, the *Kaṭapayādi* system is used. But the noteworthy feature is that into this Sanskrit inscription a Telugu gloss is inserted which explains that the chronogram is in alphabetical notation. Thus verse 13 which mentions the Śaka year 1437 in word notation is followed by the prose line in Telugu *sāl̥uvāṅka akṣarasaṃjña* | 1437 *śakavarṣālu*, "*sāl̥uvāṅka* is alphabetical notation. [It denotes] 1437 Śaka years."[27] In verse 29, there is again a *Kaṭapayādi* chronogram, followed by a Telugu gloss:

rāghavāya-gaṇite śakavarṣe rāghavāya racitācalapuryām |
vapragopuramayair nava[ha]rmyair gopamantritilakena saparyā ||
rāghavāya 1442 *akṣarasaṃjña*

"In the Śaka year counted by *rā* (2) *gha* (4) *vā* (4) *ya* (1), the illustrious minister Gopa performed the service for Rāghava (god Rāma), in Kondavīdu (*acalapurī*) with new shrines, together with the surrounding walls and high gate towers." [gloss in Telugu]. "*rāghavāya* is the alphabetical notation for 1442".

Does this mean that in the Śaka year 1442 (= 1520 AD), alphabetical notation was still so novel in Andhra region that a gloss in Telugu was thought necessary? Note also that it was called just by the generic term *akṣarasaṃjña* ("alphabetical notation") and not by the specific name *Kaṭapayādi*. Moreover, the last three inscriptions were issued by the kings of Vijayanagara dynasty: Harihara II (Śaka 1315), Devarāya II (Śaka 1346) and Kr̥ṣṇadevarāya (Śaka 1442). D. C. Sircar mentions yet another inscription issued by another king of this dynasty, viz. Sadāśivarāya, in Śaka 1467.[28] It is tempting to think that kings of this dynasty were particularly fond of the *Kaṭapayādi* notation and popularized it throughout their extensive realm which stretched over the modern states Tamilnadu, Karnataka and Andhra Pradesh. One may also conclude that, even though this system may have existed before Āryabhaṭa I in the fifth century AD and even though it was positively attested from the time it was used by Haridatta in 683 AD, it did not become popular enough to be employed in inscriptions until the fourteenth century.

But the inscriptional material gleaned so far is too meager to admit any conclusion. Since the time Ojha collected the five cases of occurrence in

27 [Lüders 1900–1901] notes that here the Dravidian retroflex *ḷ* does not denote 7. Clearly it was treated as identical with ordinary *l* which represents 3.

28 [Sircar 1965, p. 222, n. 1]. The chronogram is *setuvandya*. The inscription is published in *Epigraphia Indica*, vol. 29, p. 71 ff.

1884, no fresh attempt was made during these 126 years to scan the vast corpus of inscriptions for the use of the *Kaṭapayādi* notation, nor, for that matter, were literary works of the regions explored. It is essential to do so in order to clearly map the spread of this system.

There is some evidence to show that this alphabetical notation was known in Orissa. The Orissa State Museum has four manuscripts obviously dealing with this system: No. 2455 *Kaṭapayā* by Vrajasundara Paṭṭanāyaka; No. 2456 *Kaṭapayā, utkalānuvādasametā*, by Kālidāsa Vanamālī; No. 2457 *Kaṭapāyā*; No. 2458 *Kaṭapāyāsūtram* [Misra 1973]. These works appear to be in Sanskrit, and one of them (No. 2456) is said to contain a translation in the Oriya language.

Towards the middle of the tenth century Āryabhaṭa II employed in his astronomical work *Mahāsiddhānta* [1910] [Sarma 1966] a modified version of the *Kaṭapayādi* system [Fleet 1912] [Sarma 1966, part I, p. xx-xxii]. At the very beginning of the work, he defines the new version as follows: "The consonants starting from *ka*, *ṭa*, *pa*, *ya* represent the numerals from one (*rūpa*) [in succession] in the order of the consonants, *ña* and *na* [denoting] the zero." [29] This definition does not fully explain the system; one has to see the examples to understand how this version differs from the standard system. In this version, the consonants have the same value as in the standard system and the vowels—even when standing alone—have no numerical significance. In a conjunct consonant, each member has a numerical value and not just the last consonant as in the standard system. Finally, the digits are read from the left to the right as is the rest of the text, while in the standard *Kaṭapayādi* system the digits are read from the right to the left. Moreover, unlike in the standard system, Āryabhaṭa II makes no attempt to phrase the numerical expressions in such a way that they have some meaning. Consequently his number expressions turn out to be unpronounceable jumbles of sounds.[30] It is not clear why he reversed the general practice of enumerating the digits from the units' place onwards in a right-to-left direction. But the fact that he begins his work with a definition (albeit somewhat deficient) of his new system shows that he is consciously modifying the standard *Kaṭapayādi* system.

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29 [1910, 1.2]:
rūpāt kaṭapayapūrvā varṇā varṇakramād bhavanty aṅkāḥ |
ñnau śūnyaṃ prathamārthe ā chede ai tṛtīyārthe ||

30 Consider, for example, [1910, 2.7], which enumerates the revolutions of the Sun, Moon and Mars in a Kalpa respectively as *gha-ḍa-phe-na-ne-na-na-nū-nī-nā* = 4,320,000,000; *ma-tha-tha-ma-ga-g-la-bha-na-nu-nā* = 57,753,334,000; *kha-kha-jha-ta-jo-gī-pa-nī-ne-nā* = 2,296,831,000.

However, in Chapter 15 on arithmetic and geometry, he employs the word numerals, which he calls "well known symbols" (*prasiddha-saṃjñā*).[31] But the instances where he uses the word numerals are very few, and here he follows the standard practice of enumerating the digits from the units' place onwards. It is not clear either why he makes a special mention that he would use the well known symbols of word numerals in this chapter. He could easily have done with the alpha-numerals as in the rest of the chapters. It is not possible to solve this conundrum with any degree of certainty because, unlike the majority of the Sanskrit writers on astronomy, he does not mention his date, nor does he give any inkling of the region to which he belonged. Because he is mentioned by name by Bhāskara II in his *Siddhāntaśiromaṇi* 1150, he is generally placed in the middle of the tenth century [Pingree 1992]. In any case, no other writer after Āryabhaṭa II followed his modified version of the *Kaṭapayādi* notation.

The only writer outside the south India proper who appears to have employed the standard *Kaṭāpayādi* system in a scientific work is Bhāskara II (b. 1114) of Maharashtra, whose works on astronomy and mathematics attained a near-canonical status [Plofker 2009, p. 182 ff.]. In all his major works, viz. *Līlāvatī, Bījagaṇita, Siddhāntaśiromaṇi* and *Karaṇakutūhala* he makes use of the *Bhūtasaṃkhyā* system, but in his commentary on the *Śiṣyadhīvṛddhida* [1981], he uses the *Kaṭapayādi* notation a few times, e.g., *dāsadhirā* 2978 (p. 32), *nidhīdam* 890 (p. 33), *karagaṅgā* 3321 (p. 33) etc. It must remain a matter of speculation why he used the *Kaṭapayādi* system in this commentary and not elsewhere, or why he did not employ the word numerals in this commentary as he did in other works; it is also possible that these cases may be later interpolations.

3. *KAṬAPAYĀDI* SYSTEM IN NORTHERN INDIA

Al-Bīrūnī (973–1048), who spent several years in north-western India and produced an encyclopaedic account of Indian scientific theories and scientific literature, discusses the numerical notation in India. In this connection, he describes only the system of the word numerals, but does not show any awareness of the *Kaṭapayādi* system. In fact, he states expressly

31 [Āryabhaṭa II 1910, 15.1]:
gaṇite vyavahāre no pāṭīj̃ñānād ṛte 'dhikārī syāt |
yasmāt tasmād vakṣye sugamāṃ pāṭīṃ prasiddhasaṃjñābhiḥ ||
"Because one cannot become an expert in mathematics or commerce without the knowledge of arithmetic, I shall teach simple arithmetic with the well known symbols (*prasiddha-saṃjñā*)."

that "[t]he Hindus do not use the letters of their alphabet for numerical notation, as we use the Arabic letters in the order of the Hebrew alphabet."[32] Probably the system did not reach north India in the eleventh century.

However, there is some evidence that the *Kaṭapayādi* system was known in northern India at least from the fifteenth century onwards. It was not used in scientific texts, but only in pseudo-scientific works as a kind of secret code. The textual evidence comes from the works of Rāmacandra Vājapeyin [Sarma 1992], [Sarma 2006], [Pingree 1994, p. 467–478] and his younger brother Harṣa [Devasthali 1943], who flourished in the first half of the fifteenth century at Pātrapuñjangara in Naimiṣāranya, close to modern Sitapur in Uttar Pradesh. Rāmacandra composed many works on exact sciences, such as *Yantraprakāśa* on astronomical instruments, *Śulbavārttika* and *Kuṇḍākṛti* on ritual geometry, but he does not use the *Kaṭapayādi* notation there. He employs it in the *Samarasāra*[33] ("Essence of Warfare"), a slender work of 85 verses which, to judge from the number of extant manuscripts, seems to have been more popular than his scientific works. This text teaches how to draw various diagrams (*cakra*) with different arrangements of the letters of the alphabet. These diagrams are supposed to ensure or forecast victory on the battlefield or in a dispute. Here the *Kaṭapayādi* system is employed not for its numerical values, but to generate entirely different sets of numerical values for the letters of the Sanskrit alphabet. With these new values the names of the two combatants

32 [Al-Bīrūnī 1964, ch. XVI, p. 174]. Subsequently (p. 177–79), Al-Bīrūnī explains how the word numerals are constituted and provides a detailed list of these. On this see also [Chatterjee 1975]. Al-Bīrūnī did not know of the alphabetical notation devised by Āryabhaṭa I. As he states repeatedly [Al-Bīrūnī 1964, i. 370; ii. 16, 33], he had no direct access to Āryabhaṭa's work, his knowledge of Āryabhaṭa I's astronomy being based on Brahmagupta's work which frequently cites the former.

33 There are two printed editions, [Rāmacandra 1982–1983] and [Rāmacandra 1986]. Besides these, I have also used the manuscript [Rāmacandra 1762], which contains the commentary by Bharata.

or disputants are evaluated to forecast who will be victorious.[34] More interesting for the present context is the commentary which is purported to be the joint work of Rāmacandra and his younger brother Bharata.[35] This commentary contains three definitions of the *Kaṭapayādi* system, which are as follows:

kādayo 'ṅkāṣ (9) *ṭādayo 'ṅkāḥ* (9) *pādayaḥ pañca* (5) *kīrtitāḥ* |
yādayo 'ṣṭau (8) *tathā prājñair gaṇakair buddhimattaraiḥ* ||[36]

"The nine [consonants] starting from *ka*, the nine [consonants] starting from *ṭa*, the five [consonants] starting from *pa*, and the eight [consonants] starting from *ya*, are mentioned by the highly intelligent astronomers [as the alphabetical symbols to represent the digits from 1 to 9]."

kaṭapayavargair navanavapañcāṣṭa nañajñāḥ śūnyabodhakāḥ ||

"[The digits 1 to 9 are represented by the] nine [consonants] in sequence starting with *ka*, the nine [consonants] starting with *ṭa*, the five [consonants] starting with *pa*, and the eight [consonants] starting with *ya*, *na*, *ña* and *jña*[37] indicate zero."

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34 For example, verse 6 teaches how to draw the *Jaya-parājaya-cakra* ("Diagram of Victory and Defeat") where each of the cells in the first row is filled with the individual syllables of the phrase *śaṃ-me-gaṅ-gā-ga-ti-ste-da-ha-da-dhi*. The cells in the following rows are filled with the letters of the Sanskrit alphabet in their proper sequence. Now the syllables in the top row *śaṃ-me-gaṅ-gā-ga-ti-ste-da-ha-da-dhi* impart their numerical values according to the *Kaṭapayādi* system to the letters of the alphabet that stand below them in the respective columns. Thus, in the present diagram, the consonants *k*, *ṭ*, *p* and *y* do not have any more the value of 1 as in the standard system, but assume the values of the letters at the top of the respective column. Thus now $k = 5$, $ṭ = 8$, $p = 8$ and $y = 5$.

35 The title of the Jammu edition [Rāmacandra 1982–1983] states that the text is accompanied by a commentary by Bharata (*Bharata-saṃskṛta-ṭīkopetam*), but on p. 5, the editor mentions that the commentary is by Rāmacandra and Bharata. This commentary is substantially different from the one in the manuscript used by me. In the Hindi translation, the editor-translator adds one more definition of the *Kaṭapayādi* system which runs thus:
kaṭapayavargabhavair iha piṇḍāntyair akṣarair aṅkāḥ |
niñi (sic!) *ca śūnyaṃ jñeyaṃ tathā svare kevale kathitam* ||
[Subbarayappa & Sarma 1985, p. 48] read the second line as *ne-ñe śūnyaṃ* .... Actually the reading ought to be *na-ñe* ("*na* and *ña*").

36 In the Varanasi edition [Rāmacandra 1986], Rāmajanma Mishra attributes this verse to Rāmacandra himself and adds that this notation was invented for the sake of convenience and secrecy; cf. p. 4–5: *bhaṇitaṃ tathaiva śrī-rāmacandra-ācāryaiḥ: kādayo 'ṅkā ... iti lāghavārthaṃ gopanārthaṃ ca kalpitam.*

37 *Jña* is redundant because in this conjunct consonant only the last member, namely *ña*, has a numerical value which is zero, and this has already been indicated by the previous *ña*.

kādir navāṅkāḥ nava ṭādir aṅkāḥ
pādiḥ śarā yādi bhavanti cāṣṭau |
nañau ca śūnye ca svarāś ca śūnyāḥ ||

"The nine [consonants] starting from *ka*, the nine [consonants] starting from *ṭa*, the five [consonants] starting from *pa*, and the eight [consonants] starting from *ya* [represent the digits 1 to 9]; *na* and *ña* [stand] for zero; and the vowels [represent] zero."

Harṣa, another younger brother of Rāmacandra, composed the *Aṅka-yantra-cintāmaṇi*[38] ("Wishing Gem of Magic Squares") to teach how to draw various kinds of magic squares for therapeutic and other purposes.[39] Here verse 5, employing the *Kaṭapayādi* notation, teaches an elegant method of drawing pan-diagonal magic squares of the order four for any desired sum. I reproduce the verse below, after silently correcting the obvious scribal errors:

jñānaṃ śivāpade jñānaṃ jñānaṃ goptṛsurānanam |
iṣṭārdham ūnitaṃ teṣu guptasāraśivāpadaiḥ ||[40]

"[In the 16 cells of the square, write sucessively] 0 (*jñā*), 0 (*naṃ*), 5 (*śi*), 4 (*vā*), 1 (*pa*), 8 (*de*), 0 (*jñā*), 0 (*naṃ*), 0 (*jñā*), 0 (*naṃ*), 3 (*go*), 6 (*ptṛ*), 7 (*su*), 2 (*rā*), 0 (*na*), 0 (*naṃ*). [Then fill those cells having zeros by] half the desired sum reduced [successively] by 3 (*gu*), 6 (*pta*), 7 (*sā*), 2 (*ra*), 5 (*śi*), 4 (*vā*), 1 (*pa*), 8 (*daiḥ*)."

Let the desired sum be $2x$ and its half x. Then the magic square is constructed in the manner shown in Fig. 1 below. In the preceding verse, Harṣa explains which sum would produce what results.[41] The most potent, according to him, is the square with the magic constant 64 which brings

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38 I have used the manuscript from the Bombay University Library [Harṣa]; for its description, see [Devasthali 1944, Part II, No. 1719].

39 On the use of magic squares in Sanskrit medical texts, see [Roşu 1987], [Roşu 1986] and [Roşu 1989, p. 127, n. 37] where several extant manuscripts of the *Aṅkayantracintāmaṇi* are listed.

40 See f 3r of the manuscript [Harṣa] mentioned above. In the top margin of this page, there occurs the following gloss: *atha mūlaparibhāṣā. kādi nava 9 ṭādi nava 9 pādi pañca 5 yādy aṣṭau 8.*

41 *nārījīvaśiśuḥ śatena bhuvanaṃ vaśyaṃ catuḥsaṣṭito*
dvāṣaṣṭer jitavādakaṃ sthitaśubhaṃ dhānyaṃ caturviṃśateḥ |
aṣṭāviṃśatir ītināśana catuḥpañcāśato suprasūḥ
ṣaḍviṃśena rajasvalā gatabhayā dvātriṃśatā yātriṇaḥ ||4||
"By 100 (i.e. by means of a magic square with a constant of 100) the woman's life and the infant [will be saved]; by 64 one has the world under one's control; by 62 the disputant is defeated; 24 will make the fresh crop secure and nourishing; 28 will destroy the seasonal calamities; 55 will help easy delivery; 26 removes the fear (pains)

the whole world under one's control. Substituting then x by 32, the magic square for the sum 64 is obtained as shown in Fig. 2.

$x-3$	$x-6$	5	4
1	8	$x-7$	$x-2$
$x-5$	$x-4$	3	6
7	2	$x-1$	$x-8$

Fig. 1

29	26	5	4
1	8	25	30
27	28	3	6
7	2	31	24

Fig. 2

In Fig. 2, the sum of the numbers in any vertical column is 64; so also the sum of any horizontal row and the sum of each diagonal. Furthermore, any shorter diagonal parallel to the main diagonal, together with its complement, will also yield the same sum 64. For example, $27+8+5+24 = 64$. Therefore, this magic square is called pan-diagonal. In fact, it is much more than a pan-diagonal magic square; here the magic constant is produced also by the sum of any 2×2 square, including those that wrap around the edges of the whole square (e.g. $30 + 6 + 1 + 27 = 64$), by the sum of the digits in the four corners, by the sum of the digits at the corners of any 3×3 square, and so on [Sarma 2004].

Harṣa's magic square was adopted by some writers in the subsequent centuries. Thus, Raghunandana Bhaṭṭācārya (fl. ca. 1520–1575) [Pingree 1994, p. 341–368] of Bengal, in his *Jyotistattva*, prescribes this magic square with a slight modification by moving the top row to the bottom, for various therapeutic and other purposes. However, he uses the word numerals there and not the alphabetical notation. While Harṣa gives a mechanical rule for creating the square, Raghunandana's rule shows the logic behind the arrangement of the digits [Grierson 1881]. Nīlakaṇṭha Caturdhara, in his commentary on the *Śivatāṇḍavatantra*, which he completed in 1680 at Benares, makes use of this square with an explicit reference to Harṣa. He uses the alphabetical notation, but is also familiar with Raghunandana's rule for filling the cells [Minkowski 2008] [Roşu 1989, p. 129]. There may be similar other uses of the *Kaṭapayādi* notation in the vast pseudo-scientific literature. [42]

Through the contact with Islamic astronomy in the medieval period, astronomers in India became familiar with the alpha-numeric system called *abjad*, which is analogous to the *Kaṭapayādi* system and which was employed in the Islamic world to represent numbers in scientific texts,

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of the menstruating woman and 32 [will remove the fears of] the travellers." Though metrically correct, the text of the verse is rather corrupt.

[42] See [Goonetilleke 1882]. [Weber 1863, p. 771 ff.] reports that the *Tilaka* commentary on the *Rāmāyaṇa* mentions the number of verses in each canto in the *Kaṭapayādi* system.

in astronomical tables called *zīj-s*, and in the scales engraved on astronomical instruments like astrolabes and celestial globes. As a result of this encounter, Indian astronomers produced several books of astronomical tables in Sanskrit.[43] They also caused the production of astrolabes and celestial globes with legends in Sanskrit. In the books of tables as well as on the astronomical instruments, the numbers are represented by Devanāgarī numerals. Indian astronomers saw clearly that the *Kaṭapayādi* system of notation, though useful in mnemonic verses, does not give any advantage over common numerals in astronomical tables or on astronomical instruments. But two instrument makers decided to imitate the Islamic instruments in all respects in their Sanskrit instruments and therefore labelled the scales with the *Kaṭapayādi* notation. An anonymous Sanskrit astrolabe, now preserved in the Sampurnanand Sanskrit University at Varanasi, displays the *Kaṭapayādi* notation in some of the scales. The astrolabe is not dated; but on technical and stylistic considerations, it can be attributed to the seventeenth century. It was produced somewhere near Delhi, possibly at a place called Vaṃśāvatī-nagara situated on the latitude 28° N. (Fig. 3) [Sarma 1999].

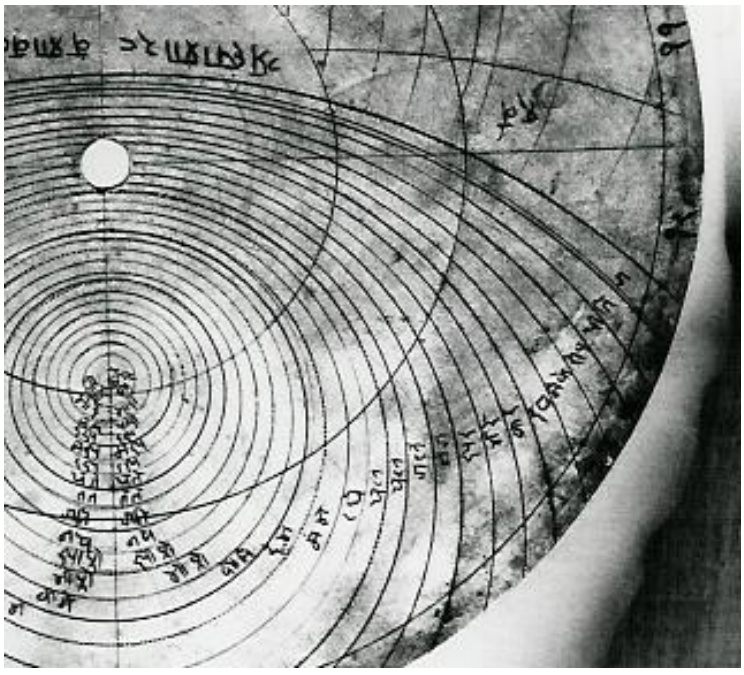

Fig. 3: Plate of the Sanskrit astrolabe

43 Cf. [Plofker 2009, p. 255–278, esp. 274–277].

Bhālūmal of Lahore [Sarma 1994, p. 522–523], who made astrolabes, celestial globes and other instruments in Arabic as well as in Sanskrit, produced in 1839 a Sanskrit celestial globe on which the scales were labelled in the *Kaṭapayādi* notation. This globe is now in a private collection in Milan.[44] In this globe, Bhālumal employs the *Kaṭapayādi* notation on the horizontal ring to represent groups of 6 degrees of arc thus: *ca* (6), *khaya* (12), *jaya* (18), *ghara* (24), *nala* (30), *cala* (36), *khava* (42), *java* (48), *vaśa* (54), *naṣa* (60), *caṣa* (66), *khasa* (72), *jasa* (78), *vaha* (84), *najha* (90). On the globe itself, both the ecliptic and the equator scales are likewise labelled in the *Kaṭapayādi* notation. Here groups of 6° are serially numbered. For economy of space, two consonants are conjoined here, both having numerical value. Thus *pla* (31), *phla* (32), *bla* (33), *bhla* (34), *mla* (35) .... *kva* (41), *khva* (42), *gva* (43), *ghva* (44), *ṇva* (45), *cva* (46), *chva* (47), *jva* (48) ... (See Figure 4).

Fig. 4: Sanskrit celestial globe made by Bhālūmal in 1839

44 The description is based on photographs kindly provided by Professor G. L'E. Turner, Oxford.

4. VARIANTS OF THE *KAṬAPAYĀDI* SYSTEM

In their *History of Hindu Mathematics*, Bibhutibhusan Datta and Avadhesh Narayan Singh declared in 1935 that there were four variants of the *Kaṭapayādi* system [Datta & Singh 1962, part I, p. 69–72], and this has been repeated by several writers since then. It is time that this assertion is re-examined. Datta and Singh mention the following four variants.

(i) The one described in the *Sadratnamālā*, which was quoted at the beginning of this paper. This is the standard system which originated in Kerala and is used there extensively.

(ii) The system followed by Āryabhaṭa II (see p. 49 above).

(iii) "A third variant of this system is found in some Pālī manuscripts from Burma. This is in all respects the same as the first variant except that $s = 5$, $h = 6$, $l' = 7$. The modification in the values of these letters are (sic!) due to the fact that the Pālī alphabet does not contain the Sanskrit *ś* and *ṣ*."[45]

(iv) "A fourth variant of the system was in use in South India, and is known as the Kerala System. This is the same as the first variant with the difference that the left-to-right arrangement of letters, just as in writing numerical figures, is employed."

What Datta and Singh call the third variant is found only in two passages in Pali texts composed in the nineteenth century in Burma. L. D. Barnett drew attention to the relevant expressions in 1907 and J. F. Fleet discussed these in 1911, citing the full passages [Fleet 1911, p. 792–793]. For the sake of completeness, we reproduce the passages from Fleet's article. The first passage is from the *Saddasāratthajālinī*, a grammatical work by Nāgita:

cakkē pattē guṇaggaraṃ sake pana alappāyaṃ ।
māghē māsē suniṭṭhitō taṃ sādhavō vicārentu ॥

The verse informs that the work was completed in the month of Māgha, when Jinacakka, the era counted from the Buddha's death, had reached the year 2353 (*gu-ṇa-gga-ra*) and Sakkarāj, the common Burmese era, had reached 1130 (*a-la-ppā-yam*). But since the difference between the two eras is about 1183 years, the equation Jinacakka 2353 = Sakkarāj 1130, is wrong. Therefore, Fleet emended *alappāyam* as *aḷappāyam* and invested the consonant with the value 7, as in Malayālam script (see p. 43 above). Then Jinacakka 2353 = Sakkarāj 1170 give AD 1807 as the date of the work. But one

45 [Datta & Singh 1962, part I, p. 72]. [Sarma 2003, p. 48] repeats this without attribution to Datta & Singh.

can also emend the other expression *gu-ṇa-gga-ra* as *gu-ya-gga-ra* (2313) and then the equation would be correct. Therefore, this passage does not provide any firm evidence for the value of $ḷ = 7$, because *ḷ* does not occur in the original but only in Fleet's emendation.

The second passage is from the *Saṃvegavatthudīpanī* by Jāgara and reads thus:

sabbha-khattiya-dhammēna dhammarājēna yācito ।
māpita-ratanapuṇṇēna katā saṃvegadīpanī ॥
niṭṭhitō ēsō saṃpattē sakkarājē raṭṭhakkhayaṃ ।
bhānuvakkhaṃ jinacakke pagguna-māsa-pañcamē ॥

"This tells us that the work was finished on the fifth day of Phālguna, when Sakkarāj had reached the year *ra*(2)-*ṭṭha*(2)-*kkha*(2)-*ya*(1), i.e. 1222, and the Jinacakka had reached the year *bhā*(4)-*nu*(0)-*va*(4)-*kkha*(2), i.e. 2404. In Sakkarāj 1222 the given day, the fifth of Phālguna, i.e. Tabaung waxing 5, was 13 February, A.D. 1861." [Fleet 1911, p. 792–793].

It is obvious that neither passage contains the values $s = 5$, $h = 6$ and $ḷ = 7$, as stated by Datta and Singh. Consequently, this so-called "third variant" is no different from the standard system as (i). It is, however, worth while to investigate how this standard *Kaṭapayādi* notation came to be employed in Pali texts in Burma in the nineteenth century.

Likewise, the fourth variant postulated by Datta and Singh is based on some erroneous assumption. From the writings of K. V Sarma and from the number of examples he provided therein, and also from the examples cited in this paper, it is amply clear that the system as practised in Kerala reads numbers from the right to the left. Therefore one can speak of only two variants. The first is the real *Kaṭapayādi* system as described in the *Sadratnamālā* with the right-to-left arrangement. This is *the* Kerala system. The reverse arrangement adapted by Āryabhaṭa II was confined only to his work *Mahāsiddhānta*, just as the alphabetical notation devised by the first Āryabhaṭa was limited to his *Āryabhaṭīya*. Neither was emulated by others.[46]

5. CONCLUSION

The *Kaṭapayādi* system of numeral notation originated in Kerala sometime before the seventh century AD, and has been employed continuously

46 In his *Vedic Mathematics*, Jagadguru Swāmī Śrī Bhāratī Kṛṣṇa Tīrthajī, the pontiff of a Hindu religious order, employs an alphabetical notation which he styles "The Vedic Numerical Code". It is just the standard *Kaṭapayādi* system but read from the left to the right; cf. [Tīrthajī 1991, p. 194–195].

from the time of the *Candra-vākyas* until the present times,[47] not only in astronomical and mathematical literature, but in other genres as well. Closely associated with this notation are the mnemonic tables which have contributed to the development of the Kerala school of mathematics, and also the *Kali-ahargaṇa* method of dating events in terms of the civil days elapsed since the commencement of the Kali era. The alphabetical notation was prevalent, to a limited extent, in the other regions of peninsular India, notably in Tamilnadu.

From the number of anonymous definitions current in northern India, it is evident that the *Kaṭapayādi* system was well known there. Yet it is surprising that this very convenient tool was not employed in any scientific work. The only instances noticed so far are in Tantric or magico-religious texts and in two cases on astronomical instruments. Why the true potential of this notation was not made use of in any astronomical or mathematical text in northern India will remain one of the enigmas in the history of transmission of scientific ideas within India [Sarma 2009a, p. 221–223].

6. APPENDIX
DEFINITIONS OF THE *KAṬAPAYĀDI* SYSTEM

For the sake of convenience, I put together all the definitions of the *Kaṭapayādi* system which I have discussed above. It will be noticed that most of these are anonymous and must be part of floating oral literature.

1. Śaṅkaravarman, *Sadratnamālā* (1819 AD), 3.3:

nañāv acaś ca śūnyāni saṃkhyāḥ kaṭapayādayaḥ |
miśre tūpāntahal saṃkhyā na ca cintyo hal asvaraḥ ||

"*na*, *ña* and the vowels [standing alone denote] zeros; [the consonants] beginning with *ka*, *ṭa*, *pa* and *ya*, [denote severally] the numbers [1 to 9]; in a conjunct [consonant, only] the consonant in the penultimate [place, which is followed by a vowel, denotes a] number; and a consonant not attached to a vowel should not be considered [as a number]."

2. Anonymous:

kādi nava, ṭādi nava, pādi pañca, yādy aṣṭau.

47 [Sarma 2003, p. 55, n. 15] mentions three astronomical works which were composed in the second half of the twentieth century in Sanskrit with auto-commentaries in Malayalam, employing the *Kaṭapayādi* system of numeral notation, viz. P. A. Poduval, *Gaṇitaprakāśikā*, Kannur, 1952; Pulioor Purushottaman Namputiri, 2nd edn, Quilon, 1952; V. P. Kunjikanna Poduval, *Śuddha-Dṛggaṇitam*, Kozhikode, 1956.

"The nine [consonants] starting with *ka*, the nine starting with *ṭa*, the five starting with *pa* and the eight starting with *ya* [respectively denote the numbers 1 to 9].

3. Anonymous:

kādayo 'ṅkās (9) *ṭādayo 'ṅkāḥ* (9) *pādayaḥ pañca* (5) *kīrtitāḥ* |
yādayo' ṣṭau (8) *tathā prājñair gaṇakair buddhimattaraiḥ* ||

"The nine [consonants] starting from *ka*, the nine [consonants] starting from *ṭa*, the five [consonants] starting from *pa*, and the eight [consonants] starting from *ya*, are mentioned by highly intelligent astronomers [as the alphabetical symbols to represent the digits 1 to 9]."

4. Anonymous:

kaṭapayavargair nava-nava-pañcāṣṭa na-ña-jñāḥ śūnya-bodhakāḥ ||

"[The digits 1 to 9 are represented by the] nine [consonants] in sequence starting with *ka*, the nine [consonants] starting with *ṭa*, the five [consonants] starting with *pa*, and the eight [consonants] starting with *ya*; *na*, *ña* and *jña* indicate zero."

5. Anonymous:

kādir navāṅkāḥ nava ṭādir aṅkāḥ
pādiḥ śarā yādi bhavanti cāṣṭau |
nañau ca śūnye ca svarāś ca śūnyāḥ ||

"The nine [consonant] starting from *ka*, the nine [consonants] starting from *ṭa*, the five [consonants] starting from *pa*, and the eight [consonants] starting from *ya* [represent the digits 1 to 9]; *na* and *ña* [stand] for zero; and the vowels [represent] zero."

6. Anonymous:

kaṭapayavargabhavair iha piṇḍāntyair akṣarair aṅkāḥ |
nañe śūnyaṃ jñeyaṃ tathā svare kevale kathite ||

" Here the [nine] digits [are denoted] by the letters of the alphabet that belong respectively to the groups [of consonants] beginning with *ka*, *ṭa*, *pa*, and *ya* and which are the last members of the conjunct consonants (*piṇḍa*); *na* and *ña* should be known as zero; so also when a vowel is pronounced separately."

7. Āryabhaṭa II, *Mahāsiddhānta* (ca. 950 AD), 1.2:

rūpāt kaṭapayapūrvā varṇā varṇakramād bhavanty aṅkāḥ |
ñnau śūnyaṃ prathamārthe ā chede ai ṭṛtīyārthe ||

"The consonants (*varṇa*) starting from *ka*, *ṭa*, *pa*, *ya* represent the numerals from one (*rūpa*) [in succession] in the order of the consonants, *ña* and *na* [denoting] the zero. [The numerical expressions, when] separated [from each other, have] *ā* and *ai* [at their end] in the nominative [plural] and in the instrumental [plural respectively]."

8. Ṭoḍarmal, in his commentary (1762 AD) on the *Gommaṭasāra* of Nemicandra Siddhānta-cakravartin [Nemicandra 1974a, p. 223] [Datta 1930–31, p. 78, n. 4]:

kaṭapaya-purastha-varṇair nava-nava-pañcāṣṭa-kalpitaiḥ kramaśaḥ |
svara-ña-na-śūnyaṃ saṃkhyā mātroparimākṣaraṃ tyājyam ||

"Numbers (*saṃkhyā*) [are expressed] by the consonants, starting [severally] from *ka*, *ṭa*, *pa*, *ya*, which are respectively nine, nine, five and eight in number; the vowels, *ña* and *na* denote zero. The vowel symbols and the 'upper' (*uparima-akṣara*) consonants are to be ignored."

Here *uparima-akṣara*, the "upper letter", means the first consonant in a conjunct consonant; in Devanāgarī script the first consonant is written half, i.e. without the vertical stroke. In Kannada (and Telugu), the first consonant is written above and the second below it in the form of a special symbol. Though cited by Ṭoḍarmal of Rajasthan in the eighteenth century, the definition is clearly from the Jain community of Karnataka.

It may be noted that none of the definitions refer to the direction of enumeration.

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Paper 52

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Nila School of Mathematics - A Note

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Notes

P. P. Divakaran, in his work on Indian mathematics, "*Mathematics of India : Concepts, Methods, Connections*" (Hindustan Book Agency/Springer, 2018), proposed a hypothesis about the identity or the roots of Saṇgamagrāma Mādhava the founder and originator of the Kerala School of Mathematics (Divakaran has christened the School as the Nila School). In this paper the author raises some doubts about the reliability of the arguments put forward by Divakaran to substantiate his hypothesis.

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Nila School of Mathematics – A Note

N. K. Sundareswaran

It is well known fact that there flourished in Kerala, to be specific, in some villages on the banks of Nila river, with Trikkandiyur – Alathiyur – Triprangode area as their centre, a community of mathematicians which carried out tremendous research on infinitesimal Calculus during 14^{th} to 17^{th} century. The members of this community were astronomer-mathematicians. Their work on Trigonometric power functions, π series, and the application of infinitesimal philosophy to approximations to these infinite series (though not explicitly pronounced) are now well-acclaimed. Such a concentration of mathematical talent, which extended for five-six generations, is unparalleled in history. Much has been written on the mathematical activities of this school.

The pioneering works of C. M. Whish, C. T. Rajagopal, Mukunda Marar and others in highlighting the importance of this mathematical tradition are ever to be remembered. It is K. V. Sarma, who brought out many of the seminal works of this school in print for the first time. His contributions in this respect are unique. Equally significant is the contribution of Rama Varma Maru Thampuran and Akhileswara Iyer who jointly brought out in print (though partially and with not of much perfection) in 1948, *Yuktibhāṣā* - a book written in Malayalam, the most significant literary product of this school. The contributions of T. A. Saraswathi Amma, who produced, for the first time, English renderings of original findings regarding trigonometric series and much of geometry (profusely applied by the school) embedded in the significant works like Nīlakaṇṭha's *Āryabhaṭīyabhāṣya*, *Yuktibhāṣā* and the *Kriyākramakarī* commentary on *Līlāvatī*, are not deservingly acknowledged by scholars[1].

The English translation of K. V. Sarma brought out (in 2008), with explanatory notes using modern notations, by Ramasubramanian, Srinivas and Sriram is a landmark in the study of this community of mathematics. This work, along with the English

[1] The passing (but significant) remarks of P. P. Divakaran (in [Div]) and references made by Vrinda P. M (in [Vr]) are not overlooked.

translation (with elaborate commentary) of *Tantrasaṅgraha* (by Ramasubramanian and Sriram), will certainly facilitate the further studies on the assessment of the mathematical output of this tradition.

Scholars used to designate this community of mathematical activity as Kerala School of mathematics. T. A. Saraswathi Amma refers to this as Āryabhaṭan school, since its members followed Āryabhaṭa. In her recent work on Indian mathematics, *Mathematics in India*, Kim Plofker calls this by the name 'School of Madhava'[2].

P. P. Divakaran, in his significant work on Indian mathematics, *Mathematics of India : Concepts, Methods, Connections*, names this community as Nila School. This work of Divakaran is beautifully-written. Divakaran discusses what is 'Indian' in Indian mathematics. He takes us along with him for a voyage through the mathematico-cultural history of India. Starting from the *Ṛgveda*, the journey ends with the culmination of activities of Nila School. 'The death of mathematics on the Nila was also its death in India', he says.

Divakaran has proposed a hypothesis about the identity or the roots of Saṅgamagrāma Mādhava the founder and originator of the Nila School of Mathematics. His story of Mādhava goes like this. Mādhava hailed from Tulu country. His forefathers migrated from Tulu country to Kerala. He belonged to Emprāntiri community (which fact has been suggested by earlier scholars like K. V. Sarma and K. Kunjunni Raja). This Mādhavan Emprān is the carrier of mathematics of Bhāskara II (of *Līlāvatī* fame) to Kerala. Earlier mathematicians and astronomers of Kerala like Śaṅkaranārāyaṇa and Sūryadevayajvan discuss much of Bhāskara I's mathematics and astronomy and not those of Bhāskara II. Nārāyaṇa the author of *Gaṇitakaumudī* might be the link between Bhāskara II and Mādhava. The village where Mādhava's forefathers settled, given as Saṅgamagrāma by earlier historians, may be Kudallur (Kūṭallūr) situated beside the confluence of the rivers Kunti and Niḷā.

[2] [Pl]

And through a post-script to the story he presumes that this family may be the same as the present day Kudallur family of Namputiris in the Nagalasseri village (This Kudallur family was, in the past, well known for the profound scholarship - of Sanskrit grammar - of its members). Divakaran cites in his support their family history. It goes like this: 'the main branch of the family got relocated from its original place near the confluence of rivers (Saṅgama) to the present location'. He further states that the members of this Kudallur family were, if not astronomers, rationalists who believed and taught the '*lokāyata*' system of materialistic philosophy – the epistemic currents of which are reflected in some statements of Nīlakaṇṭha (Keḷallūr Nīlakaṇṭha Somayājī).

It is proposed here to make some observations on this story about Saṅgamagrāma Mādhava.

1. Did Mādhava belong to Emprāntiri community? We do not have a clear evidence either for affirmation or negation. It is K. V. Sarma who stated for the first time that Mādhava belonged to Emprāntiri community. He quotes in his support a document - a colophonic statement given in a palm-leaf manuscript containing a Malayalam commentary on *Sūryasiddhānta*[3]. It contains a list of names and some other details of astronomer-cum-mathematicians of Kerala. It is from the passage, '*mādhavan ilaññippaḷḷi emprān/ vēṇvārōhaṃ muhūrttamādhavīyaṃ praśnamādhavīyaṃ mutalāya granthaṅṅaḷkku agaṇitapañcāṅgattinuṃ karttāvākunnu*/', appearing there, Mādhava is taken as an Emprāntiri. This passage further has it that Saṅgamagrāma Mādhava is the author of *Muhūrttamādhavīya* and *Praśnamādhavīya*. Sarma points out that the authorship of these works is wrongly attributed to Saṅgamagrāma Mādhava. Another Mādhava, Vidyāmādhava to be specific, is the real author. How can we fully rely upon this passage, which gives such wrong information? Further, at the end of the colophonic passage, the author states: '*iti śrutaṃ mayā*' [I have heard so]. This means that it is all hearsay. But some of the information given in the document is true and correct.

[3] The document is a manuscript (no. 9986) in the Oriental Institute, Baroda. Sarma wrote an article in Malayalam in the journal *Mathrubhumi* in the year 1957 [Sar MB]. He further wrote an article titled 'Some direct lines of astronomical tradition in Kerala' in 1972. This second article is cited in his book *A History of the Kerala School of Hindu Astronomy* [Sar HKSHA].

It may be noted that Ullur S. Parameswara Iyer gives Mādhava's name as Iriñāñāḍappaḷḷi Nampūtiri. Iyer further refutes the opinion that he belonged to Vāriyar community without giving any kind of explanation[4]. Vadakkumkur Rajarajavarmaraja has also referred to the conjecture that Mādhava belonged to Vāriyar community[5]. Sarma sweepingly rejects all these views relying exclusively upon the above colophonic passage.

2. Sūryadevayajvan is not a Keralite. He hailed from Gangaikondacholapuram in Tamil Nadu. See the Introduction to the edition of *Āryabhaṭīya* by K. S. Shukla and K. V. Sarma ([Shukla &Sarma]).

3. The present Kudallur family has been known by the name 'Kudallur Namputiri family'. Its members were also known as Namputiri-s and not as Emprāntiri-s.

4. Divakaran's statement that "They may not have been astronomers but the Kudallur Namputiris ("Embrantiri[s] . . . succeeded in later times in securing, or being thought fit to assume, the name of Namputiri" – Logan in *Malabar Maual*) were rationalists who believed in and taught the system of metaphysics known as *lokāyata*, the most extremely materialistic and this-worldly of Indian philosophical schools." is not supported by any corroborative evidence.

5. Divakaran finds this rationalistic and materialistic outlook and epistemic current finds in the works of Nīlakaṇṭha also. He writes: "This is the epistemic current that runs through Nilakantha's last books and it has been suggested that, for all his expertise in ritual and his standing in the Brahmin hierarchy, he can easily be taken for a *lokāyata* sympathizer in his scientific philosophy, at least in his later life. (It has also been said of Aryabhata)".

This view is based on a wrong notion – the notion that ritualistic Brahmanism did not accommodate rational and materialistic ideas. The *Pūrvamīmāṃsā*, taken as the most orthodox system of philosophy, and a declared ritualistic system, holds rational

[4] [Ullur], p.110
[5] [Vad], p.351

views. The early phases of *Pūrvamīmāṃsā* are even atheistic. They declare that Īśvara is not required either to bless or to punish, as the actions themselves will fetch the results. And as this world has ever remained as it is (there is no deluge or creation for them – they declare *na kadācidanīdṛśaṃ jagat*), a creator-god is also not essential. *Arthasaṅagraha*, a *Mīmāṃsā* work declares that going to temple or *vihāra* is not *Dharma* (*yāgādireva dharmaḥ/ na caityagamanādiḥ*). It may further be noticed that Nīlakaṇṭha (whose statements like '*devatāprasādo mativaimalyaheturēva/ na ca punaḥ brahmā ādityō vā svayamēva āgatya upadiśet/*[6]' prompt to take him as a sympathiser of *lokāyata*), cites passages from *Mīmāṃsā* works like *Tantravārttika* (also commentaries and sub-commentaries on it) to show that one has to have a rational approach towards disciplines like *Jyotiṣa*[7].

6. The Kudallur Namputiri family of Nagalasseri has another name 'Nāṟēri'. It is said that it may be the corrupt form of the Sanskrit word 'Nāgaśreṇi'. Also it is said that the family got this name as the members worship Nāgarāja (Ādiśeṣa, who is said to have incarnated as Patañjali, great grammarian and author of *Mahābhāṣya*). There is, still, an idol of Patañjali in the house, to which Pūjā is performed daily). The term Nāgarājaśrī got corrupted as Nagalasseri which became the name of the village.[8]

Regarding the two names of the family ('kūṭallūr' and 'nāṟeri'), it is said that it is the case with some Namputiri families. Another explanation is that one member from some other family might have entered into a family as '*dattaka*'. The name of his original family might have got appended to his descendants.

The name 'kūṭallūr' is mentioned in some old literary compositions like *Kamalinīkaḷahaṃsa* (*asti keraḷeṣu saṅgamagrāmaṃ nāma gṛham*)[9].

[6] *Jyotirmīmāṃsā* [Sar JM], p.2
[7] One may refer to [Sun 2009] and [Sun 2014] for further reading on this.
[8] See [Murali]
[9] Quoted in [Murali]

7. The family was well-known as a centre for higher learning in Sanskrit Grammar as has rightly been pointed out by Divakaran. It was known as a 'Gurukulam'. There is popular Sanskrit saying praising the excellence of this centre. It goes as

'*kaumudīpipaṭhiṣā yadi te syādabdapañcakamananyavicāraḥ/*
saṅgamālayamahīsuravaryasyālaye vasa kṛpānilayasya//'[10]

[If you intend to learn (*Siddhānta*)*kaumudī* (a text on Sanskrit grammar), go and stay for five years with a contemplative mind, in the family of Kudallur Namputiri, who is benevolent]. It may be noted that here also the family name is given as 'kūṭallūr'].

8. There were many great scholars, scholar poets and poets in the family. Kūṭallūr Nārāyaṇan Nampūtirippāḍ (who is said to be the author of the Śāstrakāvya *Subhadrāharaṇa* [11]); Nīlakaṇṭhan Nampūtirippāḍ, who is well known for his commentary on *Viṣṇusahasranāma*; Kuñňikkāvu Nampūtirippāḍ, who was well-known as a master of *Bhāgavata* exposition are some of the famous members of this family.

9. One of the last Sanskrit scholars of this family, Nārāyaṇan Nampūtirippāḍ by name, was a great promoter of 'kathakaḷi' the classical art form of Kerala. He is said to have formed a 'kaḷiyoga' a formal troupe of 'kathakaḷi' in which great maestros of the stage such as Pattikkanthodi Ravunni Menon, Kavalappara Narayanan Nair and Vazhenkada Kunju Nair were members[12].

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Paper 53

G. Raja Rajeswari and M. S. Sriram (2022)

Mahājyñayanaprakrāḥ: Infinite Series for the Sine and Cosine Functions in the Kerala Works

History and Development of Mathematics in India edited by Sita Sundar Ram and Ramakalyani V., and published in 2022 by National Mission for Manuscripts, New Delhi jointly with D. K. Printworld (P) Ltd., New Delhi. Chapter 20 *Mahājyñayanaprakrāḥ*, pp. 293–306

Notes

The short work *Mahj̄yñayanaprakrāḥ* describes the infinite series for the *jyā* ($R\sin\theta$) and the *śarā* ($R(1-\cos\theta)$) and provides a shorter derivation of them. This was discussed in a paper by David Gold and David Pingree in 1991 (see p. 462). However, that paper did not explain the derivation of the infinite series in the manuscript. In this paper we provide the derivation based on the *upapatti* provided by the author of the manuscript.

M. S. Sriram

For a biographical sketch of M. S. Sriram, see p. 589.

Source: National Manuscripts Mission, New Delhi

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Mahājyānayanaprakāraḥ

Infinite Series for the Sine and Cosine Functions in the Kerala Works

G. Raja Rajeswari
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Abstract: It is well known that the infinite series expansion for the sine and cosine functions were first discussed in the Kerala works on astronomy and mathematics and are invariably ascribed to Mādhava of Saṅgrāmagrāma (fourteenth century CE). The full proofs of these are to be found in the *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva (composed around 1530 CE). However, there is a Kerala work called the *Mahājyānayanaprakāraḥ*, which describes the infinite series for the *jyā* ($R \sin \theta$) and the *śarā* ($R(1 - \cos \theta)$) and provides a shorter derivation of them. This was discussed in a paper by David Gold and David Pingree in 1991. However, that paper did not explain the derivation of the infinite series in the manuscript. In this paper we provide the derivation based on the *upapatti* provided by the author of the manuscript.

Keywords: Infinite series, *jyā*, Kerala school, Mādhava, *śarā*, derivation.

Introduction

THE Kerala school of astronomy and mathematics (fourteenth–nineteenth centuries) is well known for its pioneering work on mathematical analysis, especially the discovery of the infinite

series for π and also sine and cosine functions. In modern notation (Sarma 1972), the infinte series for the latter are:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \ldots \quad (1)$$

$$1 - \cos\theta = \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} \ldots . \quad (2)$$

They do not appear in any of the discovered works of Mādhava, the founder of the school, but are invariably ascribed to him by the later astronomer-mathematicians of the school like Jyeṣṭhadeva and Śaṅkara Varier. The *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva (*c*.1530) is perhaps the first work to give the detailed derivation of all the infinte series (Sarma 2008).

K.V. Sarma was perhaps the first to notice a manuscript in Sanskrit named the *Mahājyānayānaprakāraḥ* which describes the infinite series for the sine and cosine functions and also gives the *upapatti* (derivation) for the same, though he did not discuss the manuscript in detail. This manuscript was available in the India Office Library in London.[1] A handwritten version of the manuscript was prepared by K.V. Sarma and it is available in the Prof. K.V. Sarma Research Foundation, Adyar , Chennai. K.V. Sarma ascribed the authorship of the manuscript to Mādhava himself.

In a paper published in 1991, David Gold and David Pingree gave a full edition of this manuscript, and also provided the translation. However they did not provide any explanation of the derivation of the infinte series, as described by the author. Gold and Pingree argued that the author could not have been Mādhava, but definitely from the "Mādhava school".

One of the authors of the present paper (G. Rajarajeswari) had worked on the manuscript for her MPhil thesis submitted to the University of Madras in August 2010. In that thesis, the manuscript had been translated into English afresh, and detailed explanatory notes had been provided. The present paper is essentially a summary of the thesis.

[1] Ff. 12-16 of a manuscript, Burnell 17e, India Office Library, London.

In this paper, we have explained the derivation of the series for $R \sin\theta$ and $R(1 - \cos\theta)$, completely based on the description of the derivation in the work. This derivation is very similar to the one in the *Gaṇitayuktibhāṣā*, but differs from it in some respects. Both the derivations are based on the iterative solution of the discrete version of the equ ations:

$$\sin\theta = \int_0^\theta \cos\theta' d\theta' = \theta - \int_0^\theta (1 - \cos\theta') d\theta',$$

$$1 - \cos\theta = \int_0^\theta \sin\theta' d\theta'.$$

Description of the Series for the R sine and the Numerical R sine Values

The manuscript has three sections, viz. the explanation of the series, the method to derive the numerical sine values and the derivation for both the sine and cosine series.

The author begins with the description of the series for R sine θ:

निहत्य चापवर्गेण चापं तत्तत्फलानि च।
हरेत् समूलयुग्वर्गैस्त्रिज्यावर्गहतैः क्रमात्।।

चापं फलानि चाधोऽधो न्यस्योपर्युपरि त्यजेत्।
जीवाप्त्यै संग्रहोऽस्यैव विद्वानित्यादिना कृतः।।

Mulitiply the arc ($r\theta$) and the [successive] results by the square of the arc, divide (each of the above numerators) by the squares of the successive even numbers increased by that number and multiplied by the square of the radius in order. Place the arc and the successive results so obtained one below the other and subtract each from the one above. These together give the *jīvā*, as collected together in the verse beginning with *vidvān*, etc.

Hence,

$$R\sin\theta = (R\theta) - \left[\frac{(R\theta)(R\theta)^2}{(2^2+2)R^2}\right. - \left[\frac{(R\theta)(R\theta)^2(R\theta)^2}{(2^2+2)(4^4+4)R^2R^2}\right. - \left[\frac{(R\theta)(R\theta)^2(R\theta)^2(R\theta)^2}{(2^2+2)(4^4+4)(6^6+6)R^2R^2R^2}\right. - \ldots$$

Here R is the *trijyā* (radius) of a circle whose circumference is 21600. In fact,

$$R \approx 3437 + \frac{44}{60} + \frac{48}{3600}.$$

In modern notation,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

He also explains how we can get numerical values of $R \sin\theta$ for any θ, using the method given by Śaṅkara Varier and others.

Derivation of the Series

OBTAINING THE COUPLED EQUATIONS
FOR THE JĪVĀ, R sin θ AND THE ŚARĀ, R(1 – COS θ)

We provide the essential steps in the author's derivation of the series. First, he obtains the *jīvā*, $R \sin\theta$ as the sum of the intermediate *koṭi*'s and the Śarā, $R(1 - \cos\theta)$ as a sum of the intermediate *jīvā*s.

> निहत्य चापवर्गेण इत्यादिना इष्टचापस्य ज्यानयने कीदृश्युपपत्तिः। तत्प्रदर्शनाय वृत्तमालिख्य मातृपितृरेखां च कुर्यात्।
>
> What is the proof for finding the *jīvā* for the desired arc, as given by the *śloka*, *nihatya cāpavargeṇa . . .*. For demonstrating that result, let a circle be drawn and let the "east–west and the north–south" (Y and X axis) (*mātṛ-pitṛ-rekhā*) be marked.

The author begins thus:

ES is a quadrant of a circle of radius R. *OE* and *OS* are the east–west and north–south lines.

Consider the

$$\text{arc } EC = R\theta.$$

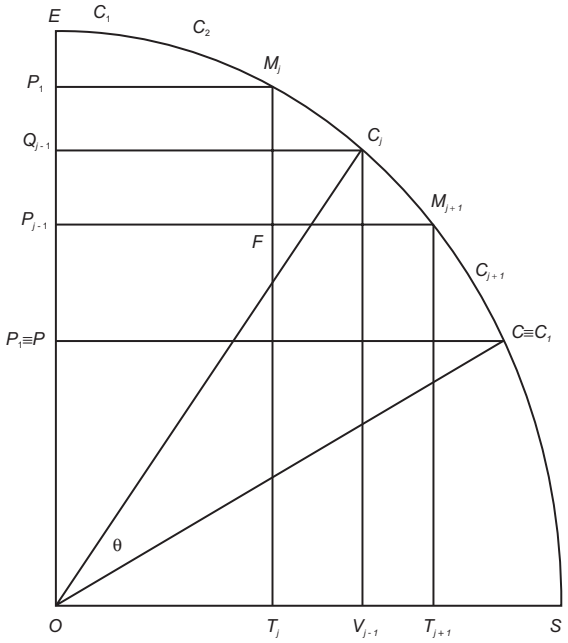
This is divided into n equal arc bits (where n is a large number):

$$EC_1 = C_1C_2 = \ldots C_jC_{j+1} \ldots = C_{n-1}C_n \equiv R\theta/n = \alpha$$

and

$$EC_j = R\theta_j = R\,\frac{j\theta}{n} = Rj\alpha.$$

Draw C_jP_j parallel to *OS* and C_jT_j parallel to *OE*:



$C_jP_j = R \sin \theta_j = R \sin (\alpha j)$.

C_n is the same as C and P_n is the same as P.

$C_nP_n = CP = R \sin \theta$.

$B_j = C_jP_j = R \sin (\alpha \cdot j)$ is the *bhujā* corresponding to the arc EC_j and $K_j = C_jT_j = OP_j\ R \cos (\alpha \cdot j)$ is the *koṭi* corresponding to the arc EC_j. Let M_{j+1} be the midpoint of the arc C_jC_{j+1}. Then,

$B_{j+1/2} = M_{j+1}Q_{j+1} = R \sin (\alpha \cdot (j + 1/2))$ is the *bhujā* corresponding to the arc EM_{j+1}, and $K_{j+1/2} = M_{j+1}V_{j+1} = OQ_{j+1} = R \cos (\alpha \cdot (j + ½))$ is the *koṭi* corresponding to the arc EM_{j+1}.

Now the *bhujā khaṇḍa* is the R sine difference or

bhujā khaṇḍa $= B_{j+1} - B_j = R \sin (\alpha \cdot (j + 1)) - R \sin (\alpha \cdot j) = FC_{j+1}$.

The *samasta jyā* of the arc-bit C_jC_{j+1} is the full-chord, C_jC_{j+1}. *Koṭi khaṇḍa* $= K_j - K_{j+1} = C_jT_j - C_{j+1}\ T_{j+1} = R \cos (\alpha j) - R \cos (\alpha(j + 1)) = FC_j$.

The author considers the two right triangles, $C_{j+1}FC_j$ and $OQ_{j+1}M_{j+1}$. He says:

तत्रतत्क्षेत्रद्वस्य तुल्याकारान्निर्णयः

The two geometrical figures are similar.

He explains that they are similar as OM_{j+1} is perpendicular to C_jC_{j+1} and C_{j+1}. F is perpendicular to T_jF and hence to OQ_{j+1}.

Then the author says:

तदानीं तस्य चापखण्डस्य समस्तज्यां भुजकोटिभ्यां पृथङ्निहत्य त्रिज्यया द्वे अपि विभज्य लब्धे कोटिखण्डं भुजाखण्डं च भवत:

Then when the full-chord of the arc-bit (*samasta jyā*) is multiplied by the R sine and R cosine (of the desired arc) and divided by the Radius (R) separately, R cosine – difference (*koṭi khaṇḍa*) and R sine – difference (*bhujā khaṇḍa*) of the arc-bit are the results.

So, according to him:

$$Bhujā\ khaṇḍa = \frac{Koṭi \times Samasta\ jyā}{R},$$

and

$$Koṭi\ khaṇḍa = \frac{Bhujā \times Samasta\ jyā}{R}.$$

This can be understood as follows:

Because of the C similarity of the triangles, we have

$$\frac{C_jF}{C_jC_{j+1}} = \frac{OQ_{j+1}}{OM_{j+1}}.$$

Here,

$C_jF = B_{j+1} - B_j = Bhujā\ khaṇḍa$.

$C_j\,C_{j+1}$ = Chord = *Samasta jyā* ≈ arc $C_jC_{j+1} = \alpha$.

OQ_{j+1} is the *koṭi* at the mid-point M_{j+1}. $OM_{j+1} = R$. Hence,

$$B_{j+1} - B_j = \frac{\alpha \cdot K_{j+½}}{R}.$$

Similarly,

$$K_j - K_{j+1} = \frac{\alpha B_{j+½}}{R}.$$

Then the author says:

एवं खण्डमध्यप्रसृतानां कोटिज्यानां योगेन खण्डं निहत्य त्रिज्यया विभज्य लब्दमिष्टे जीवा भवति।

When the arc-bit is multiplied by the sum of *koṭi jyā*s proceeding

from the mid-point of all the arc-bits and divided by the radius R, what results is the *jīvā*.

We explain this below:

Now, the R sine of the arc EC is $CP = R \sin\theta$:

$CP = R \sin\theta = P_nC_n - 0 = B_n - B_0$.

We write this as,

$$R \sin\theta = B_n - B_0 = (B_n - B_{n-1}) + (B_{n-1} - B_{n-2}) + ... + (B_1 - B_0).$$

Using the relation between the *bhujā khaṇḍa* and the mid-point (*koṭi*)

$$jīvā = R \sin\theta = \frac{\alpha}{R}\sum_{j=0}^{n-1} K_{j+1/2}.$$

Now,

$S = Śarā = R - R\cos\theta = R(1 - \cos\theta)$.

$S_{j+1/2} = R - K_{j+1/2} = R - V_{j+1}M_{j+1} = OE - OQ_{j+1} = EQ_{j+1}$.

Now,

$$K_{j+1/2} = R - S_{j+1/2}$$

Hence,

$$R\sin\theta = \frac{\alpha}{R}\sum_{j=0}^{n-1}(R - S_{j+1/2}), \text{or}$$

$$R\sin\theta = \frac{\alpha}{R}\cdot R\cdot n - \frac{\alpha}{R}\sum S_{j+1/2}$$

As $R\theta = \alpha n$.

$$\therefore R\sin\theta = R\theta - \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} S_{j+1/2}.$$

This is the discret *R*e version of

$$R\sin\theta = R\theta - \int_0^{\theta}(1 - \cos\theta')\,d\theta',$$

where $\frac{\alpha}{R}$ corresponds to $d\theta'$.

Similarly, we find

$$R(1 - \cos\theta) = S = \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} B_{j+1/2}.\cdot$$

This is the discrete version of

$$R(1 - \cos\theta) = \int_0^{\theta}\sin\theta'\,d\theta'.$$

Iterative technique for solving the coupled equations for R sin θ and R(1 – cos θ)

We have the equations

$$R\sin\theta = R\theta - \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} S_{j+1/2}.$$

and

$$R(1-\cos\theta) = S = \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} B_{j+1/2}.$$

The author uses an ingenious iterative technique to solve these equations and obtain the infinite series.

Zeroth Approximation

Here all *śarā*s are taken to be zero: $S_j = S_{j+1} = 0$.

Then,

$$R \sin\theta = R\,\theta.$$

and

$$S = R(1 - \cos\theta) = 0.$$

First Approximation

In this approximation, in the expression for S, which is a sum of the *bhujā*s, the *bhujā*s are taken to be the arcs themselves.

So,

$$S = R(1-\cos\theta) = \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} B_{j+1/2} \approx \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} \mathrm{Arc}\left(EC_{j+1/2}\right)$$
$$\approx \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} \mathrm{Arc}\left(EC_j\right).$$

Now, Arc $EC_j = j\alpha$.

$$\therefore S = \frac{\alpha}{R}\cdot\alpha\sum_{j=0}^{n-1} j$$
$$= \frac{\alpha}{R}\cdot\alpha\cdot n\frac{n-1}{2}$$
$$\approx \frac{\alpha^2 n^2}{2R} = \frac{(R\theta)^2}{2R}.$$
$$\therefore S = R(1-\cos\theta) = \frac{(R\theta)^2}{2R}.$$

In the words of the author:

खण्डमध्यादि वा खण्डादि वा खण्डोत्तरं इष्टचापान्तरं यत् संकलितं तस्मिन् खण्डगुणिते सति इष्टचापवर्गार्धं सम्पद्यते। अत्र यत्किञ्चिन्न्यूनातिरिको दृश्यते सवर्णस्याल्पत्वे कृते मण्डूकशिशूनां लाङ्गूलवत् लुप्यमाना तस्मिन्नेव निमज्जति।

When the arcs from the beginning of the desired arc to the middle or beginning or ending points of the arc-bits are summed over, and multiplied by the arc-bit then, half the square of the desired arc is obtained. Whatever appears as the deficiency or the excess will get eliminated just as the "tail of a frog's new offspring disappears in itself".

Here, we have used $\sum_{j=0}^{n-1} j \approx \frac{n^2}{2}$.

This is the discrete equivalent of

$$\int x dx = \frac{x^2}{2}.$$

In this approximation, summation is done by taking n to be very large, and the limit is taken properly. So, actually, "calculus" type of ideas are used.

Consider now the first approximation for $R \sin\theta$.

The author says:

जीवानयने तु खण्डमध्यप्रसृताभिः कोटिभिः खण्डस्य गुणने कर्तव्येऽपि कोटिज्यान्तरभूतैः शरखण्डं हत्वा त्रिज्यया विभज्य लब्धमिष्टचापाद्विशोध्यम् शिष्टं इष्टचापज्या भवति।

अथवा शरैक्येन खण्डं गुणन्यन् संकलितांशभूतानां ऐक्यं तत्संकलितसंकलितमेव। अतः संकलितज्ञानाय त्रिज्यया विभक्तमिष्टचापार्धमिष्टचापेन निहत्य त्रिसंख्येन विभज्य पुनः त्रिज्यया च हरेत्। लब्धं ज्यानयनाय चापाच्छोध्यं च भवति।

For finding the *jīvā*, multiply the *cāpa khaṇḍa* with the *koṭi*, corresponding to the middle of the arc-bit. Further, multiply by the *śarakhaṇḍam* which is the difference between the *koṭi jyā*s, and divide it by the radius R. Subtract the obtained result by the arbitrary arc. What results is the *jīvā.*

In other words, the product of the sum of the *śara*s multiplied by the arc-bit is the sum of the sums only. Therefore, to know the summation, half of the square of arc is divided by the radius R, multiplied by the desired arc, divided by the number 3 and

divided again by radius R. This result is to be subtracted from the desired arc to obtain the desired R sine (*jyā*).

We explain this below.

Now,

$$R\sin\theta = R\theta - \frac{\alpha}{R}\sum_{j=0}^{n-1} S_{j+1/2}$$

$$\approx R\theta - \frac{\alpha}{R}\sum_{j=0}^{n-1} S_j.$$

Now, $$S = \frac{\alpha}{R}\sum_{j=0}^{n-1} B_{j+1/2}.$$

This corresponds to the arc $R\theta = \alpha n$.

S_j corresponds to the arc $R\theta_j = \alpha j$. Hence,

$$S_j = \frac{\alpha}{R}\sum_{r=0}^{j-1} B_{r+1/2}.$$

Hence, the expression for $R \sin \theta$ is a sum of sums.

Now, $$B_{r+1/2} \approx \left(r + \frac{1}{2}\right)\alpha.$$

Hence,

$$R\sin\theta \approx R\theta - \frac{\alpha^3}{R^2}\sum_{j=0}^{n-1}\sum_{r=0}^{j-1}(r+½)$$

$$\approx R\theta - \frac{\alpha^3}{R^2}\sum_{j=0}^{n-1}\sum_{r=0}^{j}(r).$$

Now,

$$\sum_{j=0}^{n-1}\sum_{r=0}^{j}(r) = \sum_{j=0}^{n-1}\frac{j(j+1)}{2} = \frac{(n+1)n(n+1)}{1\cdot 2\cdot 3}.$$

This is the correct result for the double summation. In the large n limit, $(n-1)n(n+1)$ can be replaced by n^3. Then, using

$$\alpha n = R\theta$$

we have,

$$R\sin\theta \approx R\theta - \frac{\alpha^3 n^3}{R^2 1\cdot 2\cdot 3}$$

$$\approx R\theta - \frac{(R\theta)^3}{R^2 1\cdot 2\cdot 3}.$$

Now, consider the *śara* expression:

$$S = \frac{\alpha}{R} \sum B_{j+1/2}.$$

The author notes the following:

> पूर्वं शरानयनाय खण्डमध्यप्रसृतानां जीवानां योगेन खण्डेन गुणनीयेपि तासामज्ञातत्त्वाच्चापानामेव योगः कृतः।
>
> Earlier, for obtaining the *śara*, though the sum of *jīvā*s that proceeded from the middle of each part was to be considered; because the *jīvā*s were not known, the summation of the arcs itself was considered (that is, the *jīvā*s were taken to be the arcs themselves in the first approximation).
>
> इदानीं ज्यानयनाय चापादियच्छोध्यमिति जातम्। अतस्तेषामैक्यमाऽनेयम्। ज्ञातानि ज्याचापान्तरानि तु संकलितैक्यांशीभूतानि । तेषां योगस्यानयनाय ज्ञातज्याचापान्तं इष्टचापेने निहत्य चतुस्संख्येन हर्तव्यम्। तत्पुनः त्रिज्यया विभज्य पूर्वानीताच्छरात् शोध्यत्वं च।
>
> Now, in order to obtain the *jyā*, we know that "this" is the measure that needs to be subtracted from *cāpa*. Hence we need to find the sum of them. The differences of the *jīvā* and arc [at each *khaṇḍa madhya*] known, have become part of the sum of the sums.
>
> To find their sum, multiply the difference of *jyā* and *cāpa* (which is known) by the arbitrary arc, and divide it by number 4. Again divide it by Radius *R* and subtract it from the *śara* obtained before.

We explain thSis below:

$$S = \frac{\alpha}{R} \sum B_{j+½}$$

$$= \frac{\alpha}{R} \left(\sum_{j=0}^{n-1} (j+½)\alpha - \sum_{j=0}^{n-1} \frac{\{(j+½)\alpha\}^3}{R^2 \cdot 2 \cdot 3} \right).$$

$$\sum_{j=0}^{n-1} (j+½) \approx \frac{n^2}{2}.$$

This is the discrete equivalent of $\int x dx = \frac{x^2}{2}$.

$$\sum_{j=0}^{n-1}(j+\tfrac{1}{2})^3 \approx \sum_{j=0}^{n-1} j^3 \approx \frac{n^4}{4}.$$

This is the discrete equivalent of $\int x^3 dx = \frac{x^4}{4}$.

As $\alpha n = R\theta$,

$$S = \frac{\alpha^2 \cdot n^2}{2R} - \frac{\alpha^4 n^4}{R^3 4!}$$
$$= \frac{(R\theta)^2}{2R} - \frac{(R\theta)^4}{R^3 4!}.$$

Now, this expression for the *śara* has to be used to find the next approximation for $R \sin \theta$.

In the words of the author:

पुनः ज्याशोधनाय शराच्छोधितस्य संकलितं कार्यम्। तदर्थमानेयं तृतीयफलम्। इष्टचापेन हत्वा पञ्चसंख्येन विभज्य त्रिज्यया हर्तव्यम्।

Again to find the correction to the *jyā*, we have to find the sum of the corrections to the *śara* (at each *khaṇḍa madhya*). For that purpose, find the third result and multiply it by the arbitrary arc, and divide it by number 5, multiplied by the radius *R*.

We explain this below:

$$R\sin = R\theta - \frac{\alpha}{R}\sum_{j=0}^{n-1} S_j$$
$$= R\theta - \frac{\alpha}{R}\sum\left\{\frac{\alpha^2 j^2}{2R} - \frac{\alpha^4 j^4}{R^3 4!}\right\}$$
$$\approx R\theta - \frac{\alpha^3 n^3}{R^2 \cdot 2 \cdot 3} + \frac{\alpha^5 j^5}{R^4 5!},$$

where we have used

$\sum_{j=0}^{n-1} j^4 \approx \frac{n^4}{5}$ for large n and $\alpha n = R\,\theta$.

$$\therefore R\sin\theta = R\theta - \frac{(R\theta)^3}{R^2 \cdot 3!} + \frac{(R\theta)^5}{R^4 \cdot 5!}.$$

The author says:

एवमुत्तरोत्तरफलानि नेयानि।

Bring out the remaining results in the same manner.

So, this procedure should be continued to obtain the successive corrections to the *jīvā*, $R \sin \theta$ and the *śara*, $R(1 - \cos \theta)$.

The infinite series for the *jīvā*, $R \sin \theta$ and *śara*, $R(1 - \cos \theta)$ are stated again by the author.

He notes that $n^2 + n = n(n + 1)$. So he states the series for $R \sin \theta$ and $R(1 - \cos \theta)$ in the modern form (apart from the appearance of R):

$$R \sin \theta = R\theta - \frac{R\theta^3}{2 \cdot 3 \cdot R^2} + \frac{R\theta^5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot R^4} - \ldots$$

$$R(1 - \cos \theta) = \frac{R\theta^2}{2 \cdot R^2} - \frac{R\theta^4}{2 \cdot 3 \cdot 4 \cdot R^4} + \ldots.$$

Concluding Remarks

It appears that the infinte series for π and the sine and cosine functions had become common knowledge among the astronomer-mathematicians of Kerala by the sixteenth century. The *Yuktibhāṣā* gives a detailed derivation for the same. In a short Kerala manuscript *Mahājyānayanaprakāraḥ*, the author states the infinite series and discusses the method due to Śaṅkara Varier and others to compute them. More importantly, he gives a simple and elegant derivation of the series for $R \sin \theta$ and $R(1 - \cos \theta)$. It is a compact version of the derivation in the *Yuktibhāṣā*. In this paper we have explained this derivation using the modern notation.

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K. Ramasubramanian (2022)

The history and historiography of the discovery of calculus in India

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A world-class Indic scholar of great repute, a historian of Indian Sciences in Mathematics and Astronomy, Sanskrit Guru, Vedic Teacher and Indian Philosopher. His immensely broad erudition, his insights and his popular teachings have been inspiring a new generation of thousands of students, other scholars and teachers to learn and undertake research on the riches of India's scientific heritage and enabling its preservation. His creative and exciting teachings at IIT Bombay and his unique, unparalleled multi-disciplinary scholarship continues to make an impact with a powerful combination of Science, Sanskrit and Spirituality. His life's mission has been to inspire society to undertake research and give India its rightful place in the world of history of science and ignite interest in Sanskrit as a language of science.

(For a narrative biography, click here.)

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THE HISTORY AND HISTORIOGRAPHY OF THE DISCOVERY OF CALCULUS IN INDIA

K. RAMASUBRAMANIAN

ABSTRACT

Weaving through the emergence and convergence of various mathematical ideas that led towards the discovery of calculus in India provides an enthralling experience for aficionados of mathematics and its diverse history. This article attempts to briefly capture some of the milestones in the journey made by Indian mathematicians through two eras that paved the way for the discovery of infinite series for π and some of the trigonometric functions in India around the middle of the 14th century. In the first part we shall discuss the developments during what may be called the classical period, starting with the work of Āryabhaṭa (c. 499 CE) and extending up to the work Nārāyaṇa Paṇḍita (c. 1350). The work of the Kerala School starting with Mādhava of Saṅgamagrāma (c. 1340), which has a more direct bearing on calculus, will be dealt with in the second part. The third part recounts the story of the 19th century European discovery of infinite series in India which seems to have struck a wrong note among the targeted audience in Europe with a serious cascading effect.

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1. INTRODUCTION

Couched in sublime poetry in a variety of rhythmic meters and codified in the classical Sanskrit language, a journey through the history of mathematics in India could be extremely fascinating and at the same time quite challenging too. The journey would indeed be highly enriching to those who have mastered the language and understood the subtlety of expressions and figures of speech employed in it. However, for those untrained in the nuances of such a knowledge system, it would be difficult to appreciate the beautiful blend of mathematics and poetry—usually characterized with brevity without sacrificing the perspicuity—that we find in most of the texts composed over the last two millennia. The distinct style adopted by the Indian mathematicians for practising (thinking, codifying, transmitting, etc.) mathematics, by directly plunging into results without much mathematical elaborations, has been succinctly and beautifully brought out by A. A. K. Ayyangar in his article [17, P. 4.101]:[1]

> *The Hindu mind has always shown peculiar aptitude for fundamental thinking, digging down into the depths of thought with the minimum of external equipment, while other minds are after heavy superstructures with complicated scaffolding, tools and machinery. One extra-ordinary illustration of this trait of the Hindu mind we have in Ramanujan.*

Perhaps being fascinated by this peculiar way of doing mathematics by Hindus, using poetic verses, and aphoristic expressions, some of the of European scholars who were serving the British establishment in various capacities—starting from the final decades of the 18th century—embarked on their journey to study the civilizational basis of India, and the route adopted by Indians to excel in mathematics and astronomy, besides arts, architecture, aesthetics, philosophy and other disciplines.[2]

One such European scholar who got deeply attracted towards the mathematics and astronomy of the Kerala School was the then civil servant of the East India Company, Charles M. Whish (1792–1833). Having been posted at the Malabar region of Kerala for more than a decade, Whish started interacting with the local pundits and gained proficiency in both the local language Malayalam and Sanskrit. He also began to communicate some of his fascinating findings concerning the breakthroughs made by the native astronomers of Kerala, by way of both authoring papers and sharing them with the Madras Literary Society. A remarkable paper of his carrying the details of signal contributions made by the Kerala School of mathematicians, which flourished during the medieval period (14–16 centuries CE), got published in the Transactions of Royal Asiatic Society of Great Britain and Ireland in 1834—unfortunately, only posthumously—due to his premature death in 1833.

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1 Ayyangar, who came out in flying colors, with his Master's degree at the age of 18 years, has done remarkable research particularly with respect to second-order indeterminate equations.

2 See, for instance, [12].

It is this paper, which for the first time brings to the notice of European scholars the discovery of the infinite series for π, and some of the trigonometric functions by the Kerala mathematicians, almost three centuries before their advent in Europe. Strangely, this paper of Whish, instead of generating curiosity, discussion, and excitement among the European scholars, remained largely disregarded for almost a century. This deafening silence—along with the discount of its contents, among the historians of mathematics in the West—got broken only in the decades to follow from the 1940s, when some of the Indian mathematicians such as C. T. Rajagopal, Mukunda Marar, and others brought to fore the sophisticated mathematics produced by this school in the form of a series of articles [22, 23, 25, 26]. During the same period, Ramavarma Thampuran and Akhileswara Ayyar also brought out an edition of the first part (dealing with mathematics) of seminal text of Kerala astronomy and mathematics, *Yuktibhāṣā* (c. 1530), along with detailed explanations in Malayalam [28].

The Kerala School that we refer to in this article commences with Mādhava of Saṅgamagrāma (c. 1340–1420), the originator of this *guru-paraṃparā* or "lineage of teachers." His followers include Dāmodara, Parameśvara, Nīlakaṇṭha Somayājī, Jyesṭhadeva, Śaṅkara Vāriyar, and others. Though Mādhava's works containing the infinite series are not available to us, the later mathematicians in this tradition unanimously ascribe the series to Mādhava. In some of the recent studies, it has been convincingly argued by modern scholars that these series expansions for π and other trigonometric functions, and the evaluation of derivatives of various functions (while computing instantaneous velocities) rely indispensably on the central ideas of infinitesimal calculus, which include local approximation by linear function (see Section 3.4 of the present article).[3]

It is, however, important to understand that these breakthroughs achieved in the Kerala School of Mathematics cannot be narrowed to only the scope of work made in a span of two centuries. It is the continuum of mathematical ideas evolved by various Indian mathematicians spanning over nine centuries before—starting at least from the time of Āryabhaṭa (5th century)—till the dawn of the Kerala School that has led to the convergence point which has led Mādhava (14th century) to invent infinitesimal methods, thereby marking the advent of the discipline of calculus, though largely restricted to the consideration of the circular functions.[4]

This paper attempts to string the pearl of ideas and breakthroughs through the history of mathematics in India that led to this advent. The evolution of poignant ideas is traced in two parts. The first part, covered in Section 2, deals with precalculus breakthroughs and the germinating ideas for calculus that were intuitively apprehended in India well before Mādhava came on the scene. The second part, dealt with in Section 3, captures the discovery of calculus in the Kerala School. Section 4 of this paper recounts the story of how the revelations of the work of the Kerala School brought out by Whish seems to have struck a wrong note and alarmed some of the leading figures in the British academic establishment which led to the denigration and suppression of this work for almost a century.

3 The reader is also referred to the articles [13, 14, 24].

4 For a detailed discussion on this evolution readers may refer to [15, 27].

2. DEVELOPMENTS IN THE CLASSICAL ERA OF INDIAN MATHEMATICS

In this section, we shall consider some of the ideas and methods developed in Indian mathematics, during the period 450–1350 CE, which have a bearing on the later work of the Kerala School. In particular, we shall focus on the following topics: the notion and mathematics of zero and infinity; iterative approximations for irrational numbers; summation of powers of natural numbers; the discrete form of the harmonic equation for the sine function given by Āryabhaṭa; and the emergence of the notion of instantaneous velocity of a planet in astronomy.

2.1. Notion of zero and infinity

2.1.1. Philosophical and cultural context of zero and infinity

Select passages in *Upaniṣads*, as well as contemporary Buddhist and Jaina philosophy, point to the philosophical and cultural context that has possibly led to the development of the fundamental and intriguing concepts such as void and the infinite which later got incorporated in mathematics as zero and infinity. In this section, we present quotes from different ancient literature in this regard.

The *śānti-mantra* of the *Īśāvāsyopaniṣad* refers to the ultimate absolute reality, the *Brahman*, as *pūrṇa*, the perfect, complete or full. Talking of how the universe emanates from the *Brahman*, it states:

> पूर्णमदः पूर्णमिदं पूर्णात्पूर्णमुदच्यते।
> पूर्णस्य पूर्णमादाय पूर्णमेवावशिष्यते॥
> *pūrṇamadaḥ pūrṇamidaṃ pūrṇātpūrṇamudacyate।*
> *pūrṇasya pūrṇamādāya pūrṇamevāvaśiṣyate॥*
>
> That (*Brahman*) is *pūrṇa*; this (the universe) is *pūrṇa*; [this] *pūrṇa* emanates from [that] *pūrṇa*; even when *pūrṇa* is drawn out of *pūrṇa*, what remains is also *pūrṇa.*

In the *Kṛṣṇa-Yajurveda Taittirīya-Brāhmaṇa* (*Kāṭhaka* 3.49), we have the word *śūnya* (generally employed to mean zero in mathematics) appearing in the form of a compound word with a negative particle (*nañ*) tagged to it. This is in the context of describing the glory of the sun:

> वेदैरशून्यस्त्रिभिरेति सूर्यः।
> *vedairaśūnyastribhireti sūryaḥ।*

Pāṇini's *Aṣṭādhyāyī* (c. 500 BCE) has the notion of *lopa* which functions as a null-morpheme. *Lopa* appears in several *sūtras*, starting with

> अदर्शनं लोपः। (1.1.60).
> *adarśanaṃ lopaḥ।*
>
> That which gets voided is [termed] *lopaḥ*.

The word *śūnya* also appears twice as a symbol in Piṅgala's *Chandaḥ-śūtra* (c. 300 BCE). In Chapter VIII, while enunciating an algorithm for evaluating any positive integral power of 2 in terms of an optional number of squaring and multiplication (duplication) operations, *śūnya* is used as a marker:

> रूपे शून्यम्। द्विः शून्ये। (8.29-30).
> *rūpe śūnyam* | *dviḥ śūnye* |
>
> If you get one (*rūpe*) [as the remainder after doing modulo 2 arithmetic] place zero [as the marker]. If you get zero [as the remainder] place two.

Different schools of Indian philosophy have related notions such as the notion of absence (*abhāva*) in Nyāya School, and the *śūnyavāda* among the Bauddhas.

2.1.2. The mathematics of zero

The *Brāhmasphuṭa-siddhānta* (c. 628 CE) of Brahmagupta seems to be the first available text that thoroughly discusses the mathematics of zero. While describing arithmetic, the six operations with zero (*śūnya-parikarma*) are also discussed in Chapter XVIII on algebra (*kuṭṭakādhyāya*). While zero divided by zero is stated to be zero, any other quantity divided by zero is said to be *taccheda* (that with zero denominator). Of the six verses, two are presented below and the rest are paraphrased here [5, PP. 309–310]:

> धनयोर्धनमृणमृणयोः धनर्णयोरन्तरं समैक्यं खम्।
> ऋणमैक्यं च धनमृणधनशून्ययोः शून्यम्॥ ...
> खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा।
> ऋणधनयोर्वर्गः स्वं खं खस्य पदं कृतिर्यत् तत्॥
> *dhanayordhanamṛṇamṛṇayoḥ dhanarṇayorantaraṃ samaikyaṃ kham* |
> *ṛṇamaikyaṃ ca dhanamṛṇadhanaśūnyayoḥ śūnyam* || ...
> *khoddhṛtamṛṇaṃ dhanaṃ vā tacchedaṃ khamṛṇadhanavibhaktaṃ vā* |
> *ṛṇadhanayorvargaḥ svaṃ khaṃ khasya padaṃ kṛtiryat tat* ||
>
> … [The sum of] positive (*dhana*) and negative (*ṛṇa*), if they are equal, is zero (*kham*). The sum of a negative and zero is negative, of a positive and zero is positive and of two zeros, zero (*śūnya*). … Negative subtracted from zero is positive, and positive from zero is negative. Zero subtracted from negative is negative, from positive is positive, and from zero is zero (*ākāśa*).
>
> … The product of zero and a negative, of zero and a positive, or of two zeroes is zero. A zero divided by zero is zero. … A positive or a negative divided by zero is that with zero denominator (*taccheda*). The square (*kṛti*) of a positive or negative number is positive; the square and square-root (*padam*) of zero is zero.

Bhāskarācārya (c. 1150), while discussing the mathematics of zero in his work *Bījaganita*, explains that infinity (*ananta-rāśi*) which results when some number is divided by zero is called *khahara*. He also graphically describes [4, P. 6] the characteristic property of infinity that it is unaltered even if a huge quantity (*bahu*) is added to or taken away from it with a beautiful simile:[5]

> खहरो भवेत् खेन भक्तश्च राशिः॥ ...
> अस्मिन्विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु।
> बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत्॥
> *khaharo bhavet khena bhaktaśca rāśiḥ*॥ ...
> *asminvikāraḥ khahare na rāśāvapi praviṣṭeṣvapi niḥsṛteṣu*।
> *bahuṣvapi syāllayasṛṣṭikāle'nante'cyute bhūtagaṇeṣu yadvat*॥
>
> A quantity divided by zero will be (called) *khahara* (an entity with zero as divisor). … In this quantity, *khahara*, there is no alteration even if many are added or taken out, just as there is no alteration in the Infinite (*ananta*), Infallible (*acyuta*) [Brahman] even though many groups of beings enter in or emanate from [It] at times of dissolution and creation.

From the above illustrations it is discernible that Indian mathematicians began dabbling with the notions of zero and infinity in varied mathematical contexts.

2.2. Irrationals and iterative approximations

2.2.1. Approximation for surds in *Śulbasūtras*

Śulbasūtras (c. 800 BCE) that form a part of *Kalpasūtras* (one of the six *Vedāṅgas*) are essentially manuals that contain systematic procedures (algorithms) for the exact construction of altars that were laid out on leveled ground by manipulating cords of various lengths tied to a gnomon. The manuals also contain certain other mathematical details that are relevant to the construction, and are composed in the form of short, cryptic phrases—usually prose, although sometimes including verses—called *sūtras* (literally "string" or "rule, instruction"). The term for the measuring-cords called *śulba* got associated with the name to this set of texts as the *Śulbasūtras* or "Rules of the cord." Starting with simple shapes involving symmetrical figures such as squares and rectangles, triangles, trapezia, rhomboids, and circles, the texts move on to discuss the construction of complex shaped figures such that of falcon. Frequently, one also finds problems pertaining to transformation of one shape into another. Hence, the *Śulbasūtra* rules often involve what we would call area-preserving transformations of plane figures, and thus include the earliest known Indian versions of certain geometric formulas and constants. More interestingly, *Baudhāyana-śulvasūtra* gives the following approximation for $\sqrt{2}$ [33, (1.61–2), P. 19]:

---

5 This simile can be better appreciated by those who are reasonably familiar with the fundamental tenets of Hinduism and its philosophy.

प्रमाणं तृतीयेन वर्धयेत्तच्च चतुर्थेनात्मचतुस्त्रिंशोनेन। सविशेषः।
pramāṇaṃ tṛtīyena vardhayettacca caturthenātmacatustriṃśonena। saviśeṣaḥ।

The measure [of the side] is to be increased by its third and this [third] again by its own fourth less the thirty-fourth part [of the fourth]. That is the approximate diagonal (*saviśeṣa*).

$$\begin{aligned}\sqrt{2} &\approx 1 + \frac{1}{3} + \frac{1}{3\cdot 4} - \frac{1}{3\cdot 4\cdot 34}\\ &= \frac{577}{408}\\ &\approx 1.4142156.\end{aligned} \tag{1}$$

The above approximation is accurate to 5 decimal places. From certain other prescriptions [33, (1.58), P. 19] given in this text, one could discern the approximation for π to be given as $\pi \approx 3.0883$.

2.2.2. Approximation for π by Āryabhaṭa

Āryabhaṭa (c. 499) gives the following approximate value for π: [6]

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
अयुतद्वयविष्कम्भस्यासन्नो वृत्तपरिणाहः॥
caturadhikaṃ śatamaṣṭaguṇaṃ dvāṣaṣṭistathā sahasrāṇām।
ayutadvayaviṣkambhasyāsanno vṛttapariṇāhaḥ॥

One hundred plus four multiplied by eight and added to sixty-two thousand: This is the approximate measure of the circumference of a circle whose diameter is twenty-thousand.

Thus as per the above verse, $\pi \approx \frac{62832}{20000} = 3.1416$.

It appears that Indian mathematicians (at least in the Āryabhaṭan tradition) employed the method of successive doubling of the sides of a circumscribing polygon—starting from the circumscribing square leading to an octagon, etc.—to find successive approximations to the circumference of a circle. This method has been described in the later Kerala texts *Yuktibhāṣā* (c. 1530) of *Jyeṣṭhadeva* and the *Kriyākramakarī* commentary (c. 1535) of Śaṅkara Vāriyar on the *Līlāvatī*, of Bhāskarācārya.

2.3. Summation of geometric series

The result obtained by summing the geometric series $1 + 2 + 2^2 + \cdots + 2^n$ is stated in Chapter VIII of Piṅgala's *Chandaḥ-sūtra* (c. 300 BCE). It is quite remarkable that Piṅgala also gives a systematic algorithm for evaluating any positive integral power of a number (2 in this context) in terms of an optimal number of squaring and multiplication operations.

6 [2, P. 45], *Gaṇitapāda*, verse 10.

Mahāvīrācārya (c. 850), in his *Gaṇita-sāra-saṅgraha* gives the sum of a geometric series and also explains Piṅgala's algorithm for finding the required power of the common ratio between the terms of the series **[16, PP. 28–29]**:

> पदमितगुणहतिगुणितप्रभवः स्याद्गुणधनं तदाद्यूनम्।
> एकोनगुणविभक्तं गुणसङ्कलितं विजानीयात्॥
> *padamitaguṇahatiguṇitaprabhavaḥ syādguṇadhanaṃ tadādyūnam*।
> *ekonaguṇavibhaktaṃ guṇasaṅkalitaṃ vijānīyāt*॥
>
> The first term when multiplied by the product of the common ratio (*guṇa*) taken as many times as the number of terms (*pada*) [in the series], gives rise to the *guṇadhana*. This *guṇadhana*,[7] when diminished by the first term and divided by the common ratio less one, is to be understood as the sum of the geometrical series (*guṇa-saṅkalita*).

If a is the first term and r the common ratio, then what is stated in the verse above may be expressed as

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{(r - 1)}. \tag{2}$$

Vīrasena (c. 816), in his commentary *Dhavalā* on the *Ṣaṭkhaṇḍāgama*, has made use of the sum of the following infinite geometric series in his evaluation of the volume of the frustum of a right circular cone:[8]

$$1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n + \cdots = \frac{4}{3}. \tag{3}$$

The proof of the above result is outlined by Nīlakaṇṭha Somayājī in his *Āryabhaṭīya-bhāṣya*. Nīlakaṇṭha presents this discussion in the context of deriving an approximation for a small arc in terms of the corresponding chord in a circle. More details are presented in Section 3.1 of the article.

2.4. Āryabhaṭa's computation of Rsine-differences

In the mathematical section of *Āryabhaṭīya* (c. 499), Āryabhaṭa presents two different methods for the computation of tabular Rsine values. While the first is the usual geometric method, the second is an ingenious method which is based on computing the Rsine-differences employing the important property that the second-order differences of Rsines are proportional to the Rsines themselves:[9]

> प्रथमाद्यापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम्।
> तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि॥

---

7 This is a technical term employed to refer to ar^n in (2).

8 See, for instance, **[29, PP. 203–205]**.

9 **[2, P. 51]**, *Gaṇitapāda*, verse 12.

prathamāccāpajyārdhādyairūnaṃ khaṇḍitaṃ dvitīyārdham |
tatprathamajyārdhāṃśaistaistairūnāni śeṣāṇi ||

The first Rsine divided by itself and then diminished by the quotient will give the second Rsine-difference. The same first Rsine, diminished by the quotients obtained by dividing each of the preceding Rsines by the first Rsine, gives the remaining Rsine-differences.

Let the quadrant be divided into 24 equal parts, and let J_i denote $R\sin(i\alpha)$ where $\alpha = 225'$ for $i = 1, 2, \ldots, 24$. Now $J_1 = R\sin(225')$, $J_2 = R\sin(450')$, $\ldots$, $J_{24} = R\sin(90°)$, are the 24 Rsines. Let $\Delta_1 = J_1$, $\Delta_2 = J_2 - J_1$, $\ldots$, $\Delta_k = J_k - J_{k-1}$, be the first-order Rsine-differences. Then, the prescription given in the above verse may be expressed as

$$\Delta_2 = J_1 - \frac{J_1}{J_1} \tag{4}$$

$$= \Delta_1 - \frac{J_1}{J_1}. \tag{5}$$

In general,

$$\Delta_{k+1} = \Delta_k - \frac{J_k}{J_1} \quad (k = 1, 2, \ldots, 23). \tag{6}$$

Since Āryabhaṭa also takes $\Delta_1 = J_1 = R\sin(225') \approx 225'$, the above relations reduce to

$$\Delta_2 = 224', \tag{7}$$

$$\Delta_{k+1} - \Delta_k = \frac{-J_k}{225'} \quad (k = 1, 2, \ldots, 23). \tag{8}$$

The renowned mathematician David Mumford refers to the above equation as "the differential equation for the sine function in its finite difference form" [24].

2.5. Instantaneous velocity of a planet (*tātkālika-gati*)

In Indian astronomy, the motion of a planet is computed by making use of two corrections: the *manda-saṃskāra* which essentially corresponds to the equation of center and the *śīghra-saṃskāra* which corresponds to the conversion of the heliocentric longitudes to geocentric longitudes.

In Figure 1, C is the center of a circle on which the mean planet P_0 is located; CU is the direction of the *ucca* (aphelion or apogee as the case may be); P is the true planet which lies on the epicycle of (variable) radius r centered at P_0, such that P_0P is parallel to CU. If M is the mean longitude of a planet, α the longitude of the *ucca*, then the correction (*manda-phala*) $\Delta\mu$ is given by

$$R\sin(\Delta\mu) = \left(\frac{r}{K}\right) R\sin(M - \alpha). \tag{9}$$

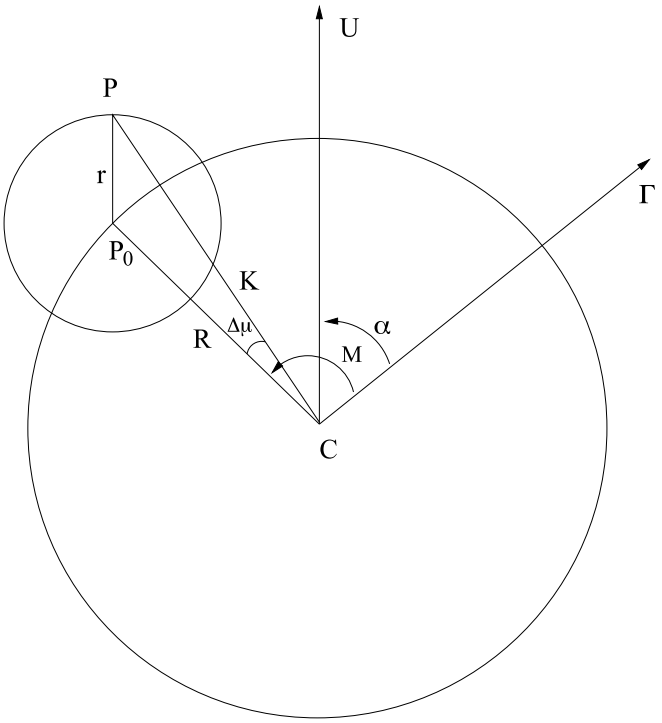

FIGURE 1
Manda correction.

Here K is the *karṇa* (hypotenuse) or the (variable) distance of the planet from the center of the concentric. The texts on Indian astronomy while giving the *manda-phala*, present the following formula:

$$R\sin(\Delta\mu) = \left(\frac{r_0}{R}\right) R\sin(M-\alpha), \tag{10}$$

where r_0 is the tabulated (or mean) radius of the epicycle in the measure of the concentric circle of radius R.

Thus there seems to have been an implicit understanding among the Indian astronomers in accepting this model that the true planet P moves on the variable epicycle of radius r in a way such that the following equation is satisfied:

$$\frac{r}{K} = \frac{r_0}{R}. \tag{11}$$

For small r, the left-hand side of (10) is usually approximated by the arc itself. Thus we have

$$\Delta\mu = \left(\frac{1}{R}\right)\left(\frac{r_0}{R}\right) R\sin(M-\alpha). \tag{12}$$

The *manda*-correction is to be applied to the mean longitude M, to obtain the true or *manda*-corrected longitude μ given by

$$\begin{aligned}\mu &= M - \Delta\mu \\ &= M - \left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right) R\sin(M-\alpha).\end{aligned} \tag{13}$$

If n_m and n_u are the mean daily motions of the planet and the *ucca*, then the true longitude of the planet on the next day may be expressed as

$$\mu + n = (M + n_m) - \left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right) R \sin(M + n_m - \alpha - n_u). \tag{14}$$

Thus the true daily motion (n), obtained by finding the difference of the two equations (13) and (14) is given by

$$n = n_m - \left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right)\left[R \sin\{(M - \alpha) + (n_m - n_u)\} - R \sin(M - \alpha)\right]. \tag{15}$$

The second term in the above is the correction to mean daily motion (*gati-phala*), which strictly involves evaluating the rate of change of the sine function. While an expression for this has been pursued by Bhāskara I (c. 629) in his *Mahābhāskarīya*, the correct formula for the true daily motion of a planet, employing the Rcosine as the "rate of change" of Rsine, seems to have been first given by Muñjāla (c. 932) in his short manual *Laghumānasa* [18, P. 125] and also by Āryabhaṭa II (c. 950) in his *Mahā-siddhānta* [20, P. 58]:

> कोटिफलघ्नी भुक्तिर्गज्याभक्ता कलादिफलम्॥
> *koṭiphalaghnī bhuktirgajyābhaktā kalādiphalam*॥
>
> The *koṭiphala* multiplied by the [mean] daily motion and divided by the radius gives the minutes of the correction [to the rate of the motion].

Essentially, the above verse gives the true daily motion in the form

$$n = n_m - (n_m - n_u)\left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right) R \cos(M - \alpha). \tag{16}$$

Bhāskarācārya (c. 1150) in his *Siddhānta-śiromaṇi* clearly distinguishes the true daily motion from the instantaneous rate of motion [32]. And he gives the Rcosine correction to the mean rate of motion as the instantaneous rate of motion. He further emphasizes the fact that the velocity is changing every instant and this is particularly important in the case of the moon because of its rapid motion [27, PP. 225–227].

3. KERALA SCHOOL OF MATHEMATICS AND ASTRONOMY

The banks of the river Nīlā in the south Malabar region of Kerala witnessed for over 300 years, beginning from about the mid-14th century, what may arguably be considered the golden age of Indian mathematics. The Kerala School of Mathematics and Astronomy pioneered by Mādhava (c. 1340–1420) of Saṅgamagrāma, extended well into the 19th century as exemplified in the work of Śaṅkaravarman (c. 1830), *Rājā* of Kaḍattanāḍu. Only a couple of astronomical works of Mādhava (*Veṇvāroha*, *Lagnaprakaraṇa* and *Sphuṭacandrāpti*) seem to be extant now. Most of his celebrated mathematical discoveries—such as the infinite series for π and the sine and cosine functions—are available only in the form of citations in later works.

Mādhava's disciple Parameśvara (c. 1380–1460) of Vaṭasseri is reputed to have carried out detailed observations for around 55 years. Though a large number of original works and commentaries written by him have been published, one of his important works on mathematics, the commentary *Vivaraṇa* on *Līlāvatī* of Bhāskarācārya, is yet to be published. Nīlakaṇṭha Somayājī (c. 1444–1550) of Kuṇḍagrāma, disciple of Parameśvara's son Dāmodara (c. 1410–1520), is the most celebrated member of Kerala School after Mādhava. Nīlakaṇṭha has cited several important results of Mādhava in his various works, the most prominent of them being *Tantrasaṅgraha* (c. 1500) and *Āryabhaṭīya-bhāṣya*. In the latter work, while commenting on the *Gaṇitapāda* of *Āryabhaṭīya*, Nīlakaṇṭha has also provided ingenious demonstrations or proofs for various mathematical formulae [21].

However, the most detailed exposition of the work of the Kerala School, starting from Mādhava, and including the seminal contributions of Parameśvara, Dāmodara, and Nīlakaṇṭha, is to be found in the famous Malayalam work *Gaṇita-yuktibhāṣā* (henceforth simply *Yuktibhāṣā*) (c. 1530) of Jyeṣṭhadeva (c. 1500–1610), who was a junior contemporary of Nīlakaṇṭha. The direct lineage from Mādhava continued at least till Acyuta Piśāraṭi (c. 1550–1621), a disciple of Jyeṣṭhadeva, who wrote many important independent works in Sanskrit, as well as a couple of commentaries in the local language Malayalam.

In the following sections we shall present an overview of the contribution of the Kerala School to the development of calculus (during the period 1350–1500), following essentially the exposition given in *Yuktibhāṣā*. In order to indicate some of the concepts and methods developed by the Kerala astronomers, we first take up the summation of infinite geometric series as discussed by Nīlakaṇṭha Somayājī in his *Āryabhaṭīya-bhāṣya*, that was alluded to just before. We then consider the derivation of binomial series expansion and the estimation of the sum of integral powers of integers, $1^k + 2^k + \cdots + n^k$ for large n, as presented in *Yuktibhāṣā*. These results constitute the basis for the derivation of the infinite series for $\frac{\pi}{4}$ and its various fast convergents given by Mādhava. Following this, we shall outline another interesting work of Mādhava on the estimation of the end-correction terms called the *antya-saṃskāra*,[10] that had enabled him to arrive at the transformation of the π-series to fast convergent ones—whose multifarious forms may be noted from a citation in Section 4.3.

3.1. Discussion of the sum of an infinite geometric series

In his *Āryabhaṭīya-bhāṣya*, while explaining the *upapatti* (rationale) behind an interesting approximation for the arc of a circle in terms of the *jyā* (Rsine) and the *śara* (Rversine), Nīlakaṇṭha presents a detailed demonstration of how to sum an infinite geometric series. The context of this discussion is Nīlakaṇṭha's pursuit to approximate the arc of a circle in terms of *jyā* (sine) and *śara* (versine). The verse that succinctly presents this approximation is the following:

10 Interestingly, this term in common parlance refers to the last rites to be performed.

सत्र्यंशादिषुवर्गात् ज्यावर्गाढ्यात् पदं धनुः प्रायः।
satryaṃśādiṣuvargāt jyāvargāḍhyāt padaṃ dhanuḥ prāyaḥ।

The arc is nearly (*prāyaḥ*) equal to the square root of the sum of the square of the *śara* added to one-third of it, and the square of the *jyā*.

In Figure 2, AB is the arc whose length (assumed to be small) is to be determined in terms of the chord lengths AD and BD. In the Indian mathematical literature, the arc AB, the semichord AD, and the segment BD are referred to as the *cāpa*, *jyārdha*, and *śara*, respectively. As can be easily seen from the figure, this terminology arises from the fact that these geometrical objects look like a bow, string, and arrow, respectively. Denoting them by c, j, and s, the expression for the arc given by Nīlakaṇṭha may be written as

$$c \approx \sqrt{\left(1 + \frac{1}{3}\right)s^2 + j^2}. \tag{17}$$

The proof of the above equation which has been discussed in detail by Sarasvati Amma **[29, PP. 179–182]** involves a summation of an infinite geometric series given by (19).

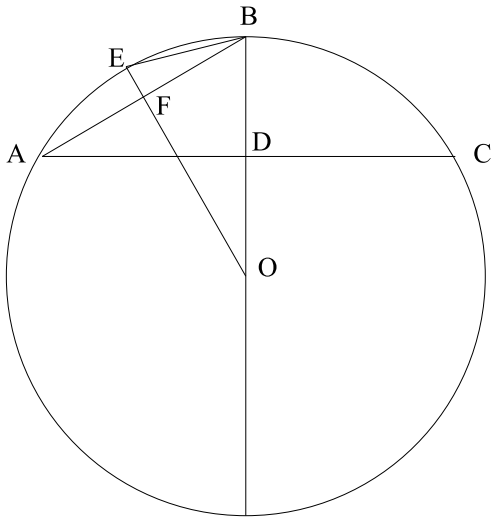

FIGURE 2
Arc-length in terms of *jyā* and *śara*.

The question that Nīlakaṇṭha poses as he commences his detailed discussion on the sum of geometric series is very important and pertinent to the current discussion. In fact, this is a general question that arises quite naturally whenever one encounters the sum of an infinite series **[1, P. 106]**:

कथं पुनः तावदेव वर्धते तावद्धर्धते च ?
kathaṃ punaḥ tāvadeva vardhate tāvadvardhate ca ?

How does one know that [the sum of the series] increases only up to that [limiting value] and that it certainly increases up to that [limiting value]?

Proceeding to answer the above question, Nīlakaṇṭha first states the general result

$$a\left[\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^3+\cdots\right]=\frac{a}{r-1}. \tag{18}$$

Here, the left-hand side is an infinite geometric series with the successive terms being obtained by dividing by a common divisor, r, known as *cheda*, whose value is assumed to be greater than 1. He further notes that this result is best demonstrated by considering a particular case, say $a = 1$ and $r = 4$. In his own words **[1, PP. 106–107]**:

> उच्यते — एवं यः तुल्यच्छेदपरभागपरम्परायाः अनन्तायाः अपि संयोगः, तस्य अनन्तानामपि कल्प्यमानस्य योगस्य आद्यावयविनः परम्परांशच्छेदात् एकोनच्छेदांशसाम्यं सर्वत्र समानमेव। तद्यथा — चतुरंशपरम्परायामेव तावत् प्रथमं प्रतिपाद्यते।
>
> *ucyate — evaṃ yaḥ tulyacchedaparabhāgaparamparāyāḥ anantāyāḥ api saṃyogaḥ, tasya anantānāmapi kalpyamānasya yogasya ādyāvayavinaḥ paramparāṃśacchedāt ekonacchedāṃśasāmyaṃ sarvatra samānameva। tadyathā — caturaṃśaparamparāyāmeva tāvat prathamaṃ pratipādyate।*
>
> It is being explained. Thus, in an infinite (*ananta*) geometrical series (*tulyaccheda-parabhāga-paramparā*)[11] the sum of all the infinite number of terms considered will always be equal to the value obtained by dividing by a factor which is one less than the common factor of the series. That this is so will be demonstrated by first considering the series obtained with one-fourth (*caturaṃśa-paramparā*).

What is intended to be demonstrated is

$$\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots\right]=\frac{1}{3}. \tag{19}$$

It is noted that one-fourth and one-third are the only terms appearing in the above equation. Nīlakaṇṭha first defines these numbers in terms of one-twelfth of the multiplier a referred to by the word *rāśi*. For the sake of simplicity, we take the *rāśi* to be unity:

$$3\times\frac{1}{12}=\frac{1}{4};\quad 4\times\frac{1}{12}=\frac{1}{3}. \tag{20}$$

Having defined them, Nīlakaṇṭha first obtains the sequence of results:

$$\begin{aligned}\frac{1}{3}&=\frac{1}{4}+\frac{1}{(4\cdot 3)},\\ \frac{1}{(4\cdot 3)}&=\frac{1}{(4\cdot 4)}+\frac{1}{(4\cdot 4\cdot 3)},\\ \frac{1}{(4\cdot 4\cdot 3)}&=\frac{1}{(4\cdot 4\cdot 4)}+\frac{1}{(4\cdot 4\cdot 4\cdot 3)},\end{aligned}$$

11 This compound word that has been coined in Sanskrit for the geometric series is very cute and merits attention. It literally means "A series of terms (*paramparā*) in which the successive ones (*parabhāga*) are obtained by the same divisor (*tulyaccheda*) [as the previous]."

and so on, which leads to the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right). \tag{21}$$

Nīlakaṇṭha then goes on to present the following crucial argument to derive the sum of the infinite geometric series: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by the right-hand side of the above equation) becomes extremely small, but never zero. Only when we take all the terms of the infinite series together, do we obtain the equality

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n + \cdots = \frac{1}{3}. \tag{22}$$

3.2. Derivation of binomial series expansion

The text *Yuktibhāṣā* presents a very interesting derivation of the binomial series for $(1 + x)^{-1}$ by making iterative substitutions in a simple algebraic identity. The method given here may be summarized as follows:

Consider the product $a(\frac{c}{b})$, where some quantity a is multiplied by the multiplier c, and divided by the divisor b. Here, a is called *guṇya*, c the *guṇaka* and b the *hāra*, which are all assumed to be positive integers, with $b > c$. Now the above product can be rewritten as

$$a\left(\frac{c}{b}\right) = a - a\frac{(b-c)}{b}. \tag{23}$$

In the expression $a\frac{(b-c)}{b}$ of the equation above, if we want to replace the division by b (the divisor) by division by c (the multiplier), then we have to make a subtractive correction (called *śodhya-phala*) which amounts to the following equation:

$$a\frac{(b-c)}{b} = a\frac{(b-c)}{c} - \left(a\frac{(b-c)}{c} \times \frac{(b-c)}{b}\right). \tag{24}$$

Now, in the second term (inside parentheses) if we again replace the division by the divisor b by the multiplier c, then we have to make a subtractive-correction once again. Proceeding thus we obtain an alternating series:

$$\begin{aligned} a\frac{c}{b} = a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \cdots + (-1)^{m-1} a\left[\frac{(b-c)}{c}\right]^{m-1} \\ + (-1)^m a\left[\frac{(b-c)}{c}\right]^m + \cdots . \end{aligned} \tag{25}$$

It may be noted that if we set $\frac{(b-c)}{c} = x$, then $\frac{c}{b} = \frac{1}{(1+x)}$. Hence, the series given by (25) is none other than the well-known binomial series

$$\frac{a}{1+x} = a - ax + ax^2 - \cdots + (-1)^m ax^m + \cdots ,$$

which is known to be convergent for $-1 < x < 1$.

Regarding the question of termination of the process, both texts, *Yuktibhāṣā* and *Kriyākramakarī*, clearly mention that logically there is no end to the process of generating *śodhya-phalas*.

It is also noted that the process may be terminated after having obtained the desired accuracy by neglecting the subsequent *phalas* as their magnitudes become smaller and smaller. In fact, *Kriyākramakarī* explicitly mentions that $(b - c)$ should be smaller than c, so that the successive *phalas* become smaller and smaller. In other words, the text, besides presenting a technique to turn a simple algebraic expression into an infinite series, also states the condition that would ensure the convergence of the series.

3.3. Estimation of sums of integral powers of natural numbers

The word employed in the Indian mathematical literature for summation is *saṅkalita*. *Yuktibhāṣā* gives a general method of estimating the sums of integral powers of natural numbers or *samaghāta-saṅkalita*.[12] The detailed procedure given in the text, which is tantamount to providing a proof by induction may be outlined as follows. Before proceeding further with the discussion, a brief note on the notation employed may be useful. We employ S to denote the sum with a subscript and superscript. The subscript denotes the number of terms that are being summed and the superscript denotes the nature of the numbers that are being summed. For the sum of natural numbers, we use (1) as the superscript. For squares of natural numbers, we use (2), and so on. Now, the sum of the first n natural numbers may be written as:

$$\begin{aligned} S_n^{(1)} &= n + (n-1) + \cdots + 1 \\ &= n + [n-1] + [n-2] + \cdots + \big[n - (n-2)\big] + \big[n - (n-1)\big] \\ &= n \cdot n - \big[1 + 2 + \cdots + (n-1)\big]. \end{aligned} \tag{26}$$

When n is very large, the quantity to be subtracted from n^2 is practically (*prāyeṇa*) the same as $S_n^{(1)}$, thus leading to the estimate

$$S_n^{(1)} \approx n^2 - S_n^{(1)}, \quad \text{or} \quad S_n^{(1)} \approx \frac{n^2}{2}. \tag{27}$$

The sum of the squares of the natural numbers up to n may be written as

$$S_n^{(2)} = n^2 + (n-1)^2 + \cdots + 1^2. \tag{28}$$

It can also easily be shown that

$$n S_n^{(1)} - S_n^{(2)} = S_{n-1}^{(1)} + S_{n-2}^{(1)} + S_{n-3}^{(1)} + \cdots . \tag{29}$$

For large n, we have already estimated that $S_n^{(1)} \approx \frac{n^2}{2}$. Thus, for large n, the right-hand side of (29) can be written as

$$n S_n^{(1)} - S_n^{(2)} \approx \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-3)^2}{2} + \cdots . \tag{30}$$

Thus, the excess of $n S_n^{(1)}$ over $S_n^{(2)}$ is essentially $\frac{S_n^{(2)}}{2}$ for large n, so that we obtain

$$n S_n^{(1)} - S_n^{(2)} \approx \frac{S_n^{(2)}}{2}. \tag{31}$$

12 The compound *sama-ghāta* in this context means the product of a number with itself.

Again, using the earlier estimate for $S_n^{(1)}$, we obtain the result

$$S_n^{(2)} \approx \frac{n^3}{3}. \tag{32}$$

Proceeding along these lines, *Yuktibhāṣā* presents an argument essentially based on mathematical induction that the summation of the kth powers of natural numbers for a large n may be written as

$$S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}. \tag{33}$$

3.4. Mādhava's infinite series for π

The infinite series for π attributed to Mādhava is cited by Śaṅkara Vāriyar in his commentaries *Kriyākramakarī* and *Yuktidīpikā*. Mādhava's quoted verse runs as follows [19, P. 379]:

> व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
> त्रिशरादिविषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात्॥
> *vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate।*
> *triśarādiviṣamasaṅkhyābhaktamṛṇaṃ svaṃ pṛthak kramāt kuryāt॥*
>
> The diameter multiplied by four and divided by unity [is found and saved]. Again the products of the diameter and four are divided by the odd numbers (*viṣamasaṅkhyā*) three, five, etc., and the results are subtracted and added sequentially [to the earlier result saved].

The words *paridhi* and *vyāsa* in the above verse refer to the circumference (C) and diameter (D), respectively. Hence the content of the verse above, expressed in the form of an equation, becomes

$$C = \frac{4D}{1} - \frac{4D}{3} + \frac{4D}{5} - \frac{4D}{7} + \cdots. \tag{34}$$

Rearranging the terms and using the notation π, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \tag{35}$$

We shall now present the derivation of the above result as outlined in *Yuktibhāṣā* of Jyeṣṭhadeva and *Kriyākramakarī* of Śaṅkara Vāriyar. For this purpose, let us consider the quadrant OP_0P_nA of the square circumscribing the given circle (see Figure 3). Let r be the radius of the circle. Divide the side $P_0P_n(=r)$ into n equal parts (n large). Then P_0P_i ($i = 1, 2, \ldots, n$) are the *bhujās* (sides) and $k_i = OP_i$ are the *karṇas* (hypotenuses) of the triangle to be conceived of. The points of intersection of these *karṇas* and the circle are marked as A_is.

It is straightforward to see that the *bhujās* P_0P_i, the *karṇas* k_i, and the East–West line OP_0 form right-angled triangles. Hence we have the relation

$$k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2. \tag{36}$$

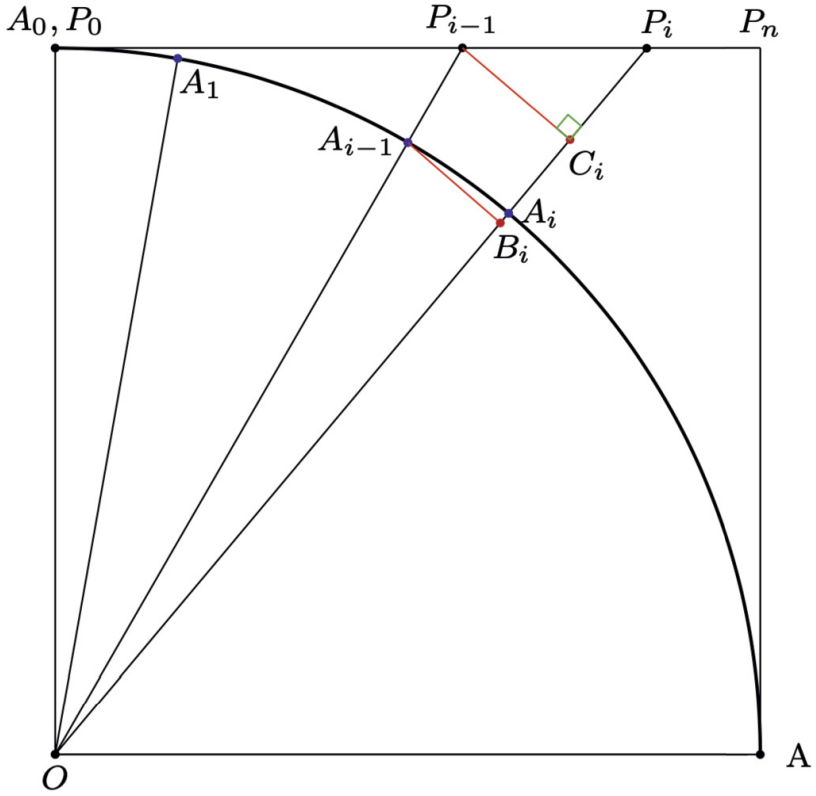

FIGURE 3
Geometrical construction used in the proof of the infinite series for π.

Considering two successive *karṇas*, and the pairs of similar triangles $OP_{i-1}C_i$ and $OA_{i-1}B_i$, and $P_{i-1}C_iP_i$ and OP_0P_i, it can be shown that the length of the segment $A_{i-1}B_i$ is given by

$$A_{i-1}B_i = \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \tag{37}$$

Now the text presents the crucial argument that, when n is large, the Rsines $A_{i-1}B_i$ can be taken as the arc-bits $A_{i-1}A_i$ themselves.

> परिधिखण्डस्य अर्धज्या परिध्यंश एव।
> *paridhikhaṇḍasya ardhajyā paridhyaṃśa eva*।
>
> The Rsines (*ardhajyā*) corresponding to the arc-bits (*paridhikhaṇḍa*) are essentially the arc-bits themselves.

Recalling that A_0 will merge with P_0, we can easily see that

$$\sum_{i=1}^{n} A_{i-1}A_i = \frac{C}{8}. \tag{38}$$

Thus, one-eighth of the circumference of the circle can be written as the sum of the contributions made by the individual segment $A_{i-1}B_i$ given by (37). That is,

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0k_1}\right) + \left(\frac{r^2}{k_1k_2}\right) + \left(\frac{r^2}{k_2k_3}\right) + \cdots + \left(\frac{r^2}{k_{n-1}k_n}\right)\right]. \tag{39}$$

It is further argued that the denominators may be replaced by the square of either of the *karṇas* since the difference is negligible. Hence we obtain:

$$\begin{aligned}\frac{C}{8} &= \sum_{i=1}^{n} \frac{r}{n}\left(\frac{r^2}{k_i^2}\right) \\ &= \sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+(\frac{ir}{n})^2}\right) \\ &= \sum_{i=1}^{n}\left[\frac{r}{n} - \frac{r}{n}\left(\frac{(\frac{ir}{n})^2}{r^2}\right) + \frac{r}{n}\left(\frac{(\frac{ir}{n})^2}{r^2}\right)^2 - \cdots\right]. \end{aligned} \tag{40}$$

In the series expression for the circumference given above, factoring out powers of $\frac{r}{n}$, the sums involved are the even powers of the natural numbers. Now, recalling the estimates that were obtained earlier (33) for these sums when n is large, we arrive at the result (35), which was rediscovered by Gregory and Leibniz almost three centuries later.

3.5. Derivation of end-correction terms (*antya-saṃskāra*)

It is well known that the series given by (35) for $\frac{\pi}{4}$ is an extremely slowly converging series. Mādhava seems to have found an ingenious way to circumvent this problem with a technique known as *antya-saṃskāra*. The nomenclature stems from the fact that a correction (*saṃskāra*) is applied towards the end (*anta*) of the series after we terminate it, by considering only a certain number of terms from the beginning. We can, of course, terminate the series at any term we desire, provided we find a correction $\frac{1}{a_p}$ to be applied, that happens to be a good approximation for the rest of the truncated terms in the series. This seems to have been the thought process that has gone in in discovering this *antya-saṃskāra* technique.

Suppose we terminate the series after the term $\frac{1}{p}$ and consider applying the correction term $\frac{1}{a_p}$, then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{\frac{p-3}{2}}\frac{1}{p-2} + (-1)^{\frac{p-1}{2}}\frac{1}{p} + (-1)^{\frac{p+1}{2}}\frac{1}{a_p}. \tag{41}$$

Three successive approximations to the correction divisor a_p given by Mādhava may be expressed as:

$$\begin{aligned} a_p(1) &= 2(p+2), \\ a_p(2) &= (2p+2) + \frac{4}{(2p+2)}, \\ a_p(3) &= (2p+2) + \frac{4}{2p+2+\frac{16}{2p+2}}. \end{aligned} \tag{42}$$

Yuktibhāṣā contains a detailed discussion on how these correction terms of successive orders are arrived at. While the discussion in the text goes only up to the three terms as above, presumably because the expressions become increasingly cumbersome, the idea that

the partial quotients of the continued fraction

$$(2p+2)+\cfrac{2^2}{(2p+2)+\cfrac{4^2}{(2p+2)+\cfrac{6^2}{(2p+2)+\cdots}}}$$

serve as correction factors to higher and higher orders is seen to be inherently present in the reasoning. A graph depicting the variation of error in the estimate of π using the three successive end-corrections by truncating the series at different values of p is shown in Figure 4. It may be noted that, when we use the third-order end-correction, by just considering about 25 terms in the series, we are able to obtain the π value correct to 10 decimal places.

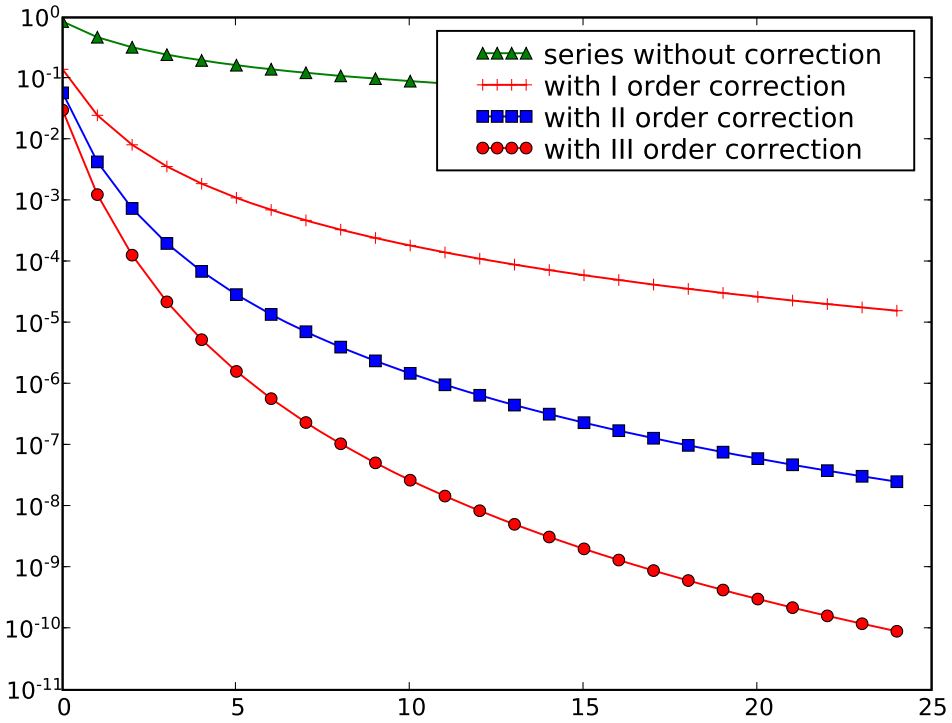

FIGURE 4
Graph depicting the accuracy that is obtained in estimating the value of π by truncating the series at different values of p and employing the three corrections given by (42).

The following accurate value of π (correct to 11 decimal places), given by Mādhava, has been cited by Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya* and by Śaṅkara Vāriyar in his *Kriyākramakarī*:[13]

13 **[1, P. 42]**, comm. on *Gaṇitapāda* verse 10; **[19, P. 377]**.

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः॥
vibudhanetragajāhihutāśanatriguṇavedabhavāraṇabāhavaḥ।
navanikharvamite vṛtivistare paridhimānamidaṃ jagadurbudhāḥ॥

The π value given above is

$$\pi \approx \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592\ldots \tag{43}$$

The 13-digit number appearing in the numerator has been specified using object-numeral (*bhūta-saṅkhyā*) system, whereas the denominator is specified by word numerals.[14]

4. HISTORIOGRAPHY OF THE INCEPTION OF CALCULUS IN INDIA

4.1. Brief note on Charles Whish and his collections

Charles Matthew Whish (1794–1833), as noted earlier, was instrumental in first bringing to the notice of modern mathematical scholarship the achievements of the Kerala School through his historic paper that got posthumously published in TRAS (1934) [36]. The fact that Whish had discovered them more than a decade before the paper got published is evident from the correspondence between John Warren and George Hyne that has been noted down by the former in his *Kālasaṅkalita* [35]. It may also be mentioned here that the collection of manuscripts that Whish had made—which the author of this paper had an occasion to look at—amply demonstrates the fact that he was interested not only in astronomy and mathematics, but also in a wide variety of topics that includes vedic literature, *itihāsas* and *purāṇas*. Fortunately, these manuscripts were deposited in the Royal Asiatic Society of Great Britain and Ireland in July 1836 by his brother, and are still well preserved in the Royal Asiatic Society, London.

The personal notes (see Figure 5) found in various manuscripts in Whish's collection also reveal that during his stay in South Malabar, he had got in touch with several scholars, and read some of the Sanskrit and Malayalam texts with them. Given his abiding interest to acquire scholarship in a variety of fields by familiarizing with the culture, language, and knowledge systems of India—and also share it back with his counterparts in Europe—it is highly unfortunate that Charles Whish suffered a premature death in 1833 at the age of 38 years.[15]

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14 In the *bhūta-saṅkhyā* system, *vibudha* = 33, *netra* = 2, *gaja* = 8, *ahi* = 8, *hutāśana* = 3, *tri* = 3, *guṇa* = 3, *veda* = 4, *bha* = 27, *vāraṇa* = 8, *bāhu* = 2. In word numerals, *nikharva* represents 10^{11}. Hence, *nava-nikharva* = 9×10^{11}.

15 The list of European tombs in the district of Cuddapah prepared by C. H. Mounsey in 1893 mentions: "Sacred to the memory of C. M. Whish, Esquire of the Civil Service, who departed this life on the 14th April 1833, aged 38 years."

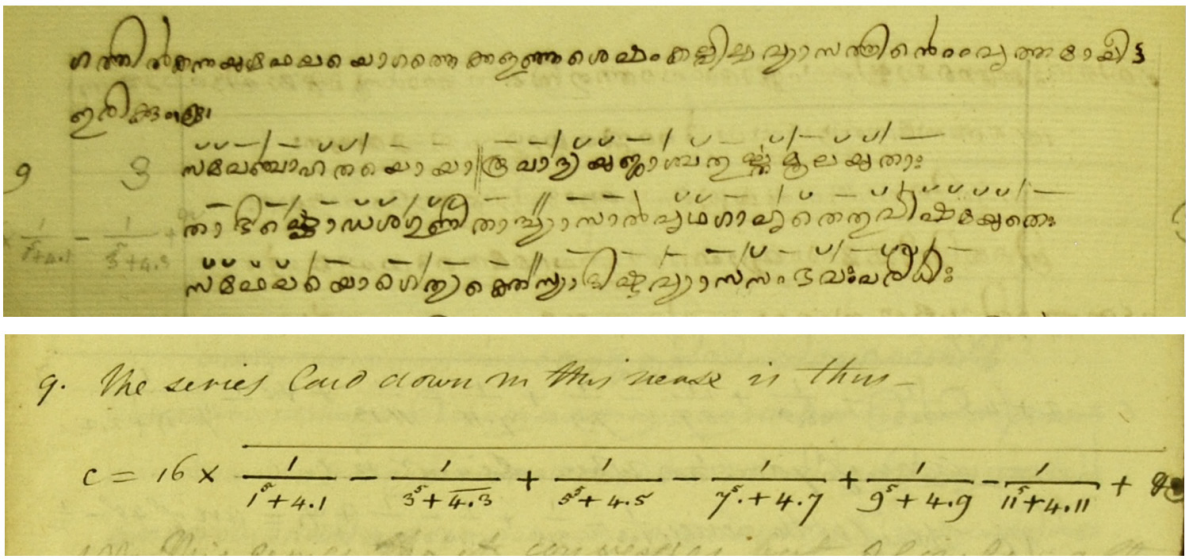

FIGURE 5
Excerpts from Whish's manuscript showing the verses in Malayalam along with his metrical markings and a portion from his mathematical notes in English (Courtesy: RAS, London).

4.2. About *Kālasaṅkalita*

Kālasaṅkalita, published in 1825 by John Warren who was the director of the Madras observatory for sometime, is a compendium of the different methods employed by the *pañcāṅga*-makers for reckoning time. The main purpose of preparing this text was to facilitate a comparison of the European and Indian chronologies, as is mentioned in the preface: "… their chief object being merely to explain the various modes according to which the Natives of India divide time, in these southern provinces, and to render their Kalendars intelligible. These may therefore be properly considered rather as instruments contrived for Chronological purposes, than as Astronomical Tracts."

It turns out that the text is useful in several other respects as well, especially from a historical perspective. Among other things, the one which is of particular interest to us in this paper is the exchange of ideas that took place among the three civil servants of the East India Company, namely, Warren, Whish, and Hyne, particularly with regard to the invention of the infinite series expansion by the "Natives."

4.3. Extracts from the exchanges between Whish, Hyne, and Warren

In the Second Memoir of *Kālasaṅkalita* on the Hindu Lunisolar year, before commencing his discussion on *śaṇku*[16] and the diurnal problems associated with it, John Warren notes:

16 The term *śaṇku* refers to a very simple contrivance, yet a powerful tool that has been extensively employed by Indian astronomers – right from the period of the *Śulvasūtras* (c. 800 BCE) – to carry out a variety of experiments related to shadow measurements.

Before entering into the resolution of the Problems which depend on the length of the Meridian shadow, it is proper to enquire …
Of their manner of resolving geometrically the ratio of the diameter to the circumference of a circle, I never saw any Indian demonstration: the common opinion, however, is that they approximate it in the manner of the ancients, by exhaustion; that is, by means of inscribed and circumscribed Polygons. However, a Native Astronomer who was a perfect stranger to European Geometry, gave me the well-known series $1 - \frac{1}{3} + \frac{1}{5} + \cdots$. *This person reduced the five first terms of the series before me, which he called Bagah Anoobanda, or Bagah Apovacha; to shew that he understood its use. This proves at least that the Hindus are not ignorant of the doctrine of series …*

This passage clearly indicates that John Warren is confronting a dilemma: on the one hand, he has met "a Native Astronomer who was a perfect stranger to European Geometry" giving the well-known series $1 - \frac{1}{3} + \frac{1}{5} + \cdots$ and, on the other hand, "he never saw any Indian demonstration" of the series. To the above passage, Warren appends a note where he mentions:

I owe the following note to Mr. Hyne's favor: "The Hindus never invented the series; it was communicated with many others, by Europeans, to some learned Natives in modern times. Mr. Whish sent a list of the various methods of demonstrating the ratio of the diameter and circumference of a Circle employed by the Hindus to the literary society, being impressed with the notion that they were the inventors. I requested him to make further inquiries, and his reply was that he had reasons to believe them entirely modern and derived from Europeans, observing that not one of those who used the Rules could demonstrate them. Indeed, the pretensions of the Hindus to such a knowledge of geometry, is too ridiculous to deserve refutation." I join in substance Mr. Hyne's opinion, but do not admit that the circumstance that none of the Sastras mentioned by Mr. Whish, who used the series could demonstrate them, would alone be conclusive.

John Warren returns to this issue in "Fragments II" attached at the end of his treatise *Kālasaṅkalita*, entitled *"On certain infinite Series collected in different parts of India, by various Gentlemen, from Native Astronomers."— Communicated by George Hyne, Esq. of the H. C.'s Medical Service*, which we reproduce below:

"My Dear Sir,
I have great pleasure in communicating the Series, to which I alluded …

$$C = 4D\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right), \tag{44}$$

$$C = \sqrt{12D^2} - \frac{\sqrt{12D^2}}{3 \cdot 3} + \frac{\sqrt{12D^2}}{3^2 \cdot 5} - \frac{\sqrt{12D^2}}{3^3 \cdot 7} + \cdots, \tag{45}$$

$$C = 2D + \frac{4D}{(2^2-1)} - \frac{4D}{(4^2-1)} + \frac{4D}{(6^2-1)} - \cdots, \tag{46}$$

$$C = 8D\left[\frac{1}{(2^2-1)} + \frac{1}{(6^2-1)} + \frac{1}{(10^2-1)} + \cdots\right], \tag{47}$$

$$C = 8D\left[\frac{1}{2} - \frac{1}{(4^2-1)} - \frac{1}{(8^2-1)} - \frac{1}{(12^2-1)} - \cdots\right], \tag{48}$$

$$C = 3D + \frac{4D}{(3^3-3)} - \frac{4D}{(5^3-5)} + \frac{4D}{(7^3-7)} - \cdots, \tag{49}$$

$$C = 16D\left(\frac{1}{1^5+4.1} - \frac{1}{3^5+4.3} + \frac{1}{5^5+4.5} - \cdots\right). \tag{50}$$

I am, my dear Sir, most sincerely, your's,

MADRAS, *17th August 1825.* *G. HYNE."*

Based on the nature of exchanges recorded by Warren in 1825, it is quite clear that:

1. Whish was convinced that the infinite series were discovered by the "Natives."
2. Hyne was convinced that the infinite series were NOT discovered by the "Natives" but was only borrowed, and that the Hindus were merely pretending as originators of the series.
3. Warren decides to go with the opinion of Hyne, though initially he felt that the latter's argument is not "conclusive."

Under such circumstances, with a lot of communication back and forth, one could only imagine how challenging it would have been for Whish[17] to swim against the current, and place on record his own understanding regarding the knowledge of the infinite series, or of their demonstration in the Indian astronomical tradition. The mere fact the paper authored by him in 1820s got accepted for publication in the 1830s posthumously, stands testimony to his courage, perseverance, assiduity, and tenacity with which he would wear down his opponents.

One of the remarkable statements in the paper of Whish that is of particular interest to us in the present context is: "A further account of the Yucti-Bhāshā, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper." Unfortunately, Whish did not survive to publish this paper with demonstrations from *Yuktibhāṣā*, which could have silenced all those who doubted whether these series listed by them were discovered by Indians.

17 It may also be recalled that Whish was hardly 30-year old in 1825, whereas George Hyne and Warren were seniors. Warren was the director of Madras Observatory around 1805 and Hyne was a senior member of Madras Literary Society who was appointed as the first Secretary of the Committee of Public Instruction by the Madras Government.

More striking and intriguing development connected with Whish's paper than what is narrated above, is the kind of consensus that seems to have emerged among the European indologists and historians of mathematics and astronomy to undermine and suppress it for almost a hundred years since its publication in 1834. Either the work itself was not referenced in their writings, or even if it were, some of the well-established mathematicians, such as Augustus De Morgan and scholar administrators such as Charles P. Brown dismissed it—far more strongly than was done by Hyne—by castigating it as "hoax" and "forgery" [11], [6, PP. 48–49].

Not providing reference to this paper of Whish on the contributions of the Kerala School, or discussing its contents, is certainly not out of ignorance, which is perfectly understandable. But strangely it seems to be a volitional act! See, for instance, the scholarly monograph of Geroge Thibaut (in German) on Indian Astronomy, Astrology, and Mathematics [34, P. 2] which makes note of 1827 article of Whish, on the Greek origin of the Hindu Zodiac. However, it mysteriously fails to mention this 1934 paper of Whish, though the paper is germane to the subject of his discussion. We present below a clip (Figure 6) of the relevant section from Thibaut's volume, along with a concise translation (done with the help Google).

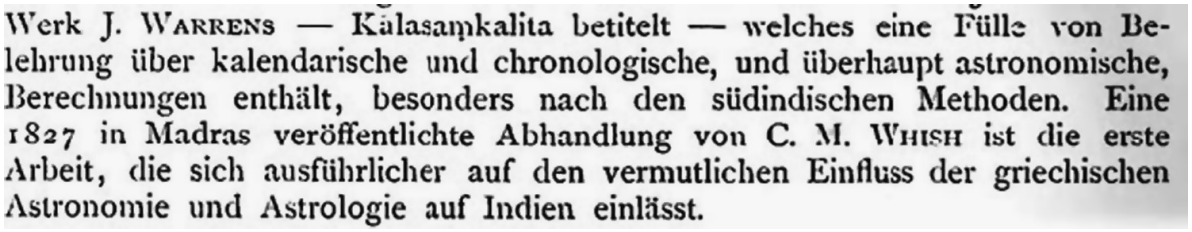
Werk J. Warrens — Kālasaṃkalita betitelt — welches eine Fülle von Belehrung über kalendarische und chronologische, und überhaupt astronomische, Berechnungen enthält, besonders nach den südindischen Methoden. Eine 1827 in Madras veröffentlichte Abhandlung von C. M. Whish ist die erste Arbeit, die sich ausführlicher auf den vermutlichen Einfluss der griechischen Astronomie und Astrologie auf Indien einlässt.

FIGURE 6
A clip of the relevant section from Thibaut's volume

> *J. WARREN'S work entitled Kālasaṅkalita, which contains a wealth of instruction on calendar and chronological, and generally astronomical, calculations, especially according to the South Indian methods. A treatise by C. M. WHISH, published in Madras in 1827, is the first to delve into the probable influence of Greek astronomy and astrology on India.*

Similarly, the popular translation of *Sūryasiddhānta* by Ebenezer Burgess [7, P. 174], and the review article by John Burgess of the European studies of Indian astronomy in the 18th and 19th centuries [8, PP. 746–750] do not refer to the 1934 paper of Whish while they take note of his other contributions.

Furthermore, David Eugene Smith (1860–1944), in his seminal two-volume history of mathematics completed in 1925, simply refers to the article of Whish, but does not touch upon its content except for noting that it deals with Indian values for π. Thus we find an interesting period of almost a century in European historiography where either both the title and the content, or at least the content of Whish's article remained an untouchable!

Fortunately, the references given by David Smith [31, P. 309] caught the attention of the renowned historian of Indian mathematics, Bibhutibhusan Datta, who drew attention to

the various infinite series mentioned in Whish's article in an article published in 1926 [10]. This was followed a decade later by Datta's colleague, Avadesh Narayan Singh, who referred to the various manuscripts of the Kerala texts which discuss these infinite series [30]. And the next decade finally saw the publication of a series of articles by C. T. Rajagopal and his collaborators and the edition of the mathematics part of *Yuktibhāṣā* by Ramavarma Thampuran and Akhileswara Ayyar (for details, please, see [22, 23, 25, 26, 28]).

5. CONCLUDING REMARKS

It is quite evident from the above mathematical and historical discussions that the mathematicians of the Kerala School, around the 14th century, had clearly mastered the technique of handling the infinitesimal, the infinite and the notion of limit—the three pillars on which the edifice of calculus rests upon. The context and purpose for which the Kerala mathematicians developed these techniques are different from those in which they got developed in Europe a couple of centuries later. It must also be mentioned here that the Kerala mathematicians had restricted their discussions to the quadrature of a circle and certain trigonometric functions.[18] However, their mathematical formulation of the problem involving the "infinitesimally" small and summing up the "infinite" number of the resulting infinitesimal contributions, along with a clear understanding of the mathematical subtleties involved in it, are not in any way fundamentally different from the way it would be formulated or understood today.

While there were a number of European mathematicians and indologists who expressed their appreciations for the contributions made by Indians, the historiography captured in Section 4, in no uncertain terms reveals that there were many others who promulgated their views and tried to suppress the discovery of Kerala mathematicians, by brazenly discounting their work.[19] The cascading effect of it has resulted in some well-known authors producing books even in 1930s—almost a century after the publication of the Whish's historic paper—containing descriptions such as "… the Hindus may have inherited some of the bare facts of Greek science, but not the Greek critical acumen. Fools rush in where angels fear to tread [9][20] …" that are quite misleading, derailing, and damaging. It is perhaps a fitting tribute to Whish that today at least most historians of mathematics are aware of this "neglected chapter" in the history of mathematics. For this reason, the following statement by David Mumford is quite relevant [24]:

> *It is high time that the full story of Indian mathematics from Vedic times through 1600 became generally known. I am not minimizing the genius of the Greeks and*

18 The mathematicians of Europe, however, took a different approach to the subject, by considering an arbitrary curve for analysis, and by providing formal definitions and generalized treatment to the topic.

19 The episode essentially reminds us of the important lesson: if we look through a malicious goggle, then even the genuine narratives may sound to be an elaborate hoax!

20 Quoted by A. A. K. Ayyangar in his article [3].

their wonderful invention of pure mathematics, but other peoples have been doing math in different ways, and they have often attained the same goals independently. Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge.

… the muse of mathematics can be wooed in many different ways and her secrets teased out of her. And so they were in India …

Apart from the topics discussed in the present article, several other ideas of calculus seem to have been employed by Indian astronomers in their studies related to planetary motion. For instance, one of the verses in the second chapter of *Tantrasaṅgraha* deals with the derivative of the inverse sine function.[21] We would also like to refer the reader to the literature for the very interesting proof of the sine and cosine series given in the *Yuktibhāṣā*. As has been remarked recently by Divakaran **[15, P. 335]** that, unlike the derivation that was given by Newton, which involved "guessing" successive terms "from their form," the *Yuktibhāṣā* approach of "integrating the difference/differential equation for sine and cosine is entirely different and very modern", which has also been briefly touched upon by Mumford in his article cited above.

For most of us who have got trained completely in the modern scheme of education, it may be hard to imagine doing mathematics without the "luxury" of expressing things "neatly" in symbolic forms. It is equally hard to think of expressing power series for trigonometric functions, derivatives of functions, and the like, purely in metrical forms. But that is how knowledge seems to have been preserved and handed down from generation to generation in India for millennia starting from Vedic age till the recent past. It only proves the point: equations may be handy but not essential; notations may be useful, but not indispensable. Formal definitions and structures are certainly valuable and helpful, but the absence of them does not inhibit or stagnate the birth and development of mathematical ideas. After all, mathematics is mathematics irrespective of how, where, and why it is practiced!

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21 In fact, Nīlakaṇṭha ascribes this verse—dealing with the instantaneous velocity (*tātkālikagati*) of the moon—to his teacher Dāmodara in his *Jyotirmīmāṃsā*. While the details of how Dāmodara, or someone else before him, arrived at this result is not evident to us, one thing is quite clear—the astronomers were adept at dealing with the derivatives of basic trigonometric functions.

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Appendix

A. *Yuktibhāṣā* derivation of Leibniz series

B. *Yuktibhāṣā* derivation of power series for sine and cosine functions

C. Missing papers

Appendix A

Yuktibhāṣā derivation of Leibniz series (2008)

Extracts from
Gaṇita-Yukti-Bhāṣā (Rationales in Mathematical Astronomy) of Jyeṣṭhadeva Volume I : Mathematics, Malayalam Text Critically Edited with English Translation by K. V. Sarma, with explanatory notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, Hindstan Book Agency/Springer, 2008.

The first part of the extracts (pp. 49–70 in the text form which the excerpts are taken) is the English translation of the of the relevant text in the original Malayalam *Yktibhāṣā* and the second part (pp. 183–200 in the text form which the excerpts are taken) is the explanatory notes on the English translation using modern mathematical notations.

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6.3 Circumference of a circle without calculating square-roots

6.3.1 Dividing the circumference into arc-bits: Approximating the arc-bits by *Jyārdha*-s (Rsines)

Now is described the procedure for arriving at the circumference of a circle of desired diameter without involving calculation of square-roots. Construct a square with the four sides equal to the diameter of the proposed circle. Inscribe the circle inside the square in such a manner that the circumference of that circle touches the centres of the four sides of the square. Then, through the centre of the circle, draw the east-west line and the north-south line with their tips being located at the points of contact of the circumference and the sides. Then, the interstice between the east-point and the south-east corner of the square will be equal to the radius of the circle. Divide this line into a number of equal parts by marking a large number of points closely at equal distances. The more the number of divisions, the more accurate (*sūkṣma*) would be the (calculated) circumference.

Then draw hypotenuse-lines from the centre of the circle to all these points. In (the triangles) so formed, *koṭi* will be the *pūrva-sūtra* (line from the centre towards east) and the *bhujā*-s will be the segments, along the eastern side of the square, between the tip of the east-line and the tips of the hypotenuse-lines. There, the (interstice) from the *pūrva-sūtra* towards the next hypotenuse to the south of it, will be the *bhujā*. For the second hypotenuse, the first two segments will be the *bhujā*. For the further hypotenuse-lines, the *bhujā*-s will increase by one segment each. The *bhujā* for the hypotenuse at the (south-east) corner of the square will be the largest. In all these cases,

the radius equal to the *pūrva-sūtra* will be the *koṭi*. Hence, the respective hypotenuse-lines will be the root of the sum of the squares of the radius and the respective *bhujā*.

Now, take the first segment, being the distance from the tip of the *pūrva-sūtra* to the tip of the first hypotenuse. Multiply it by the *pūrva-sūtra*, which is equal to the radius, and divide it by the first hypotenuse. The result will be the (perpendicular) distance from the east point to the first hypotenuse. (For the triangle formed), this perpendicular line will be a *koṭi*. The *bhujā* will be the (distance) from the meeting point of this *koṭi* and the first hypotenuse, and the tip of the hypotenuse. The corresponding *karṇa* will be the bit (of the eastern side of the square) joining the east point and the tip of the first hypotenuse. Take this as the *icchā-kṣetra*. We will now state a similar triangle which will be the *pramāṇa-kṣetra*. For this, the *koṭi* is the *pūrva-sūtra* from the centre of the circle to the mid-point of the eastern side of the (larger) square; the first hypotenuse will be the *karṇa*; the *bhujā* is the distance between the tips of the hypotenuse and the *koṭi*. The *icchā-kṣetra* is *tulyākāra* (similar) to this *pramāṇa-kṣetra*.

The reasoning for this is as follows: The *karṇa* of the *icchā-kṣetra* is parallel to the *bhujā* of the *pramāṇa-kṣetra* and the *karṇa* of the *pramāṇa-kṣetra* is parallel to the *bhujā* of the *icchā-kṣetra*. The *karṇa* of the *icchā-kṣetra*, which is a segment of the (eastern) side of the square, is perpendicular to the *pūrva-sūtra* which is the *koṭi* of the *pramāṇa-kṣetra*. Also, the *koṭi* of the *icchā-kṣetra*, which is being calculated as the *icchā-phala*, is perpendicular to the *karṇa* of the *pramāṇa-kṣetra*. For the above reason, the two figures are similar. Here, for these two figures, parallelness of their *bhujā* and *karṇa* (hypotenuse), and perpendicularity of their *koṭi* and *karṇa* (hypotenuse), render them into similar figures. On the other hand, if there is perpendicularity or parallelness of all the three sides, then also they wi11 be similar figures.

To cite an instance: On (the roof of a) square *maṇḍapa* (hall), for the slanting beam (*kazhukkol*), which might be taken as the *karṇa* of the *pramāṇa-kṣetra*, the joining tie (*vāmaṭa*, brim-plank) will be the *bhujā*. And the hole for

the tie (*vaḷattuḷa*) which corresponds to the *karṇa* of the *icchā-kṣetra* will be parallel to this (*bhujā*). The *bhujā* for this *karṇa* will be the slant of the hole for the tie on the side of the beam. Since the *bhujā* and the *karṇa* (of the two) are parallel, the slant in the hole of the tie is caused by the slant of the beam. These things have to be thought out in this manner. Hence the *koṭi* of the *icchā-kṣetra* can be derived by means of *trairāśika*.

Then there is a third triangle. For this the east-west line is the *karṇa*. The distance of the first hypotenuse from the tip of the east-west line, which is the *koṭi* of the above-said *icchā-kṣetra*, is the *bhujā* here. The segment of the first hypotenuse extending from the meeting point of the *bhujā* (stated above) up to the centre of the circle is the *koṭi*. This is thus.

Then there is a second *pramāṇa-kṣetra*. For that the *koṭi* is the east-west line. The *bhujā* is two segments of the side of the (original) square taken from the tip of this *koṭi*. The *karṇa* is the second hypotenuse drawn from the centre of the circle. This is the second *pramāṇa-kṣetra*. Now, its *icchākṣetra* is this: Its *koṭi* is the line starting from the tip of the first hypotenuse, perpendicular to the second hypotenuse and meeting the second hypotenuse. The *bhujā* is (the line) from the meeting point of this *koṭi* and the second hypotenuse, to the tip of the second hypotenuse. The *karṇa* is the second segment of the (eastern side of the) original square. This is the second *icchā-kṣetra*.

To cite a parallel: When the second beam (*kazhukkol*, making the roof of a square *maṇḍapa*, referred to earlier) from the centre is taken as the *karṇa* of the *pramāṇa-kṣetra*, the *bhujā* of the *pramāṇa-kṣetra* would be two beam segments. Hence, the second beam will be longer than the first one. In proportion, the holes for the joining rod in that will also be longer. Now, this will be the *karṇa* of the *icchā-kṣetra* and will be parallel to the brim-plank (*vāmaṭa*) which is the *bhujā* of the *pramāṇa-kṣetra*. Thus, as the slant in the (several) beams and the orientation of the holes in them are similar, so too will the *pramāṇa-kṣetra* and the *icchā-kṣetra* considered here will be similar.

Take the second segment of (the eastern) side of the square starting from the tip of the east-west line. Multiply that by the radius which is the *koṭi* of the *pramāṇa-kṣetra* and divide by the second hypotenuse which is the *pramāṇa*. The result will be the *koṭi* of the second *icchā-kṣetra*. Now, take this *koṭi* as the *bhujā* and the segment in the second hypotenuse, being the distance from its meeting point to the centre of the circle, as the *koṭi*. Take the first hypotenuse as the *karṇa*. Thus a third triangle is to be noticed here.

Thus, there are three triangles related to each of the segments of the side of the square commencing from the east-point and going up to the (south-east) corner. There, multiply each of these segments by *dik-sūtra*, the line from the centre towards the east (viz., radius). Divide each product by the longer of the (two) hypotenuse-lines which touch the tips of corresponding segment. The result, in each case, will be the perpendicular distance from the tip of the previous hypotenuse to the longer hypotenuse. These are the *koṭi*-s of the *icchā-kṣetra*-s. Each of these will then be considered as *bhujā*. The corresponding *koṭi* would be the segment of the longer hypotenuse, from the meeting point of this *bhujā* to the centre of the circle. And the *karṇa* would be the lesser of the two hypotenuses from the centre touching the two ends of the *bhujā*. These are the triangles. Some of these will be *pramāṇa-kṣetra*-s in what follows. The corresponding *icchā-kṣetra*-s will be portions of the *pramāṇa-kṣetra*-s which are within the circle. The *icchā* (*rāśi*) here would be the radius which is a portion of the *pramāṇa-karṇa*. The *icchā-phala* would be the perpendicular distance from the tip of this radius to the bigger hypotenuse.

These *icchā-phala*-s are nothing but the *jyārdha*-s, Rsines, of the bits of circumference between the successive hypotenuse-lines. Now, multiply each segment of the (eastern) side of the square commencing from the east-point, twice by the radius and divide by the product of the two hypotenuse-lines meeting the ends of the segment. The result, in each case, will be the *jyārdha*, Rsine of the bit of the circumference between the two hypotenuse-lines. When the segments of the square are extremely small, these *jyārdha*-s, Rsines, will be almost (*prayeṇa*) same as the corresponding arcs.

6.3.2 Circumference in terms of the *Karṇa*-s (hypotenuses)

Now, since the (eastern) side of the square has been divided into equal segments, the multiplicand is the same in all the above cases. The multiplier, in each case, is the square of the radius. Since the divisors are the products of adjacent hypotenuse-lines, they are all different. Now, in this situation, it is possible to take the product of the two hypotenuse-lines as half the sum of their squares, since they (hypotenuses) are practically the same. This being the case, divide the dividend separately by the squares of the two hypotenuse-lines, add the results obtained and halve them. The result of dividing by half the sum of the squares would be equal to this.

We shall consider each of the segments of the (eastern) side of the square as being divided by the square of the hypotenuse drawn to the northern tip of each segment. There, the first is the east-west line. When division is made by the square of this, the multiplier also being the same, viz., square of the radius, the result will be just the segment itself. The last hypotenuse is the line to the (south-east) corner (from the centre). When division is made by its square, half the segment will be the result, since the square of the last hypotenuse is double the square of the radius. When the divisor is double the multiplier, half the multiplicand is the result.

Now, there are two hypotenuse-lines touching the two tips of each segment. With reference to this, find the sum of the results obtained by dividing by the squares of the first of the (two) hypotenuse-lines. Find also the sum of the results obtained by dividing by the second of the (two) hypotenuse-lines. The difference between these two (sums) will be the difference between the first term of the first sum and the last term of the second sum. That will be, half the length of the segment. For the terms in between, since the divisors are the same the results will also be the same. There is no difference (between the two sums) as regards the terms starting from the second to the penultimate. (The difference between the two sums is) thus half the length of the segment. There, the result on dividing by the first divisor (the square of the first hypotenuse) is the segment itself. The result on dividing by the last

divisor (the square of the last hypotenuse) is half the segment. In the case of dividing by half the sum of the squares of the hypotenuse-lines, the difference is one-fourth of the segment. When the segment becomes extremely small, this one-fourth becomes negligible. Hence, it follows that we should take as the divisor the square of just one hypotenuse (in each term).

6.3.3 *Śodhya-phala*-s: Iterative corrections

Now, for each of the segments, we shall consider that the square of the larger hypotenuse is taken as the divisor. Then, multiply the segments by the square of the radius and divide by the square of the larger hypotenuse associated with them. The results, will be the *jyārdha*-s of the corresponding circumference-bits (arcs-bits) contained in the interstices of the hypotenuse-lines.

Now, multiply the respective segments by the difference between the multiplier (the square of the radius) and the respective divisor (the square of the hypotenuse) and divide by the square of the respective hypotenuse. Subtract the results from the respective segments. The results will be the *jyā* (Rsine) of the respective circumference-bit of arc contained in the interstices of the hypotenuse-lines. There, the square of the sum of those segments in the (eastern) side of the square, lying between the east-point and the respective tips of the hypotenuse, is the difference between the multiplier and the divisor. The multiplier is the square of the radius.

There, if the multiplication is done by the difference between the multiplier and the divisor, and division is done by the multiplier itself, the result will be larger (than the one referred to above), since the multiplier is smaller than the divisor. Now, place this result at two places, multiply the value at one place by the difference between the multiplier and the divisor and divide by the divisor; the result obtained is subtracted from the earlier result (kept at the other place). That will be the actual result.

The above would mean that in the case of calculating the *śodhya-phala* (the subtractive term obtained above), if the multiplication is done by the dif-

ference between the multiplier and the divisor and division is done by the multiplier (instead of the divisor), then a similar subtraction should be made from that term also as was done before. There, in the second *śodhya-phala* (subtractive term obtained by iteration) also, the result of dividing the difference between the multiplier and divisor by the divisor would give rise to another subtractive term from the *śodhya-phala*, which will be the third result. Here also, if the division is made by the multiplier, a fourth *śodhya-phala* would be obtained (following the same procedure as before). If in all (the *śodhya-phala*-s), the division is made by the multiplier, then there will be no end to the series of subtractive terms (*śodhya-phala-paramparā*), as always (at the end of any number of steps) the last term has a division by the divisor. If there is to be no division by the divisor, then there will be no end to the number of subtractive terms. When the (subtractive term) becomes very small, it can be discarded.

6.3.4 *Phala-yoga*-s, and their series: *Phala-paramparā*

When we proceed in the above manner, the first (*phala-yoga*) is the sum of the multiplicands, which is the sum of the segments of the (eastern) side of the square, being the radius. The second is subtractive from the first. The third is subtractive from the second. This being the case, add together all the odd terms; add also all the even terms amongst themselves. From the sum of the odd-s subtract the sum of the even-s. The result will be one-eighth of the circumference. The above is the case when the multiplier is small. When, however, the multiplier is large, then all the *phala*-s (results) have to be added to the multiplicand.

Since, here, the multipliers and divisors are the squares of the *koṭi* (which is the radius) and *karṇa*-s (hypotenuses), the differences between the multipliers and divisors would be the square of the *bhujā*-s. The first *bhujā* would be the first of the equal segments into which the (eastern) side of the square has been divided. Two segments together make up the second *bhujā*. Three segments together make up the third *bhujā*. In this manner, the *bhujā*-s will be

successively made up of a number of segments, the number increasing by one at each step (*ekādyekottara*). Further, they have to be conceived as being of infinitesimal length (*aṇu-parimāṇa*), for the sake of accuracy (*sūkṣmatā*) of the result. Then, consider them also as whole numbers. Starting from unity, consider the sum of the squares of successive numbers. Multiply by that sum, the multiplicand which is the segment (of the side of the square) which has been conceived both as an infinitesimal and as a unit (as *aṇu-parimita* and *rūpa*) and divide by the square of the radius. The quotient got will be the first *phala-yoga*, sum of results.

For the second *phala-yoga*, the first *phala* will be the multiplicand. The multiplicands are of different (measures); the difference between the multiplier and divisor which is the square of the *iṣṭa-bhujā* (portion of the eastern side of the square) are also several. Hence, there is no easy way to multiply by the sum of the difference between the multipliers and divisors. Hence, multiply the *bhujā-varga-saṅkalita*, summation of the squares of the *bhujā*-s (successive portions of the eastern side of the square), which is the sum of the difference between the multipliers and divisors, twice by the first multiplier which is unity, and divide the product twice by the square of the radius. The result will be the second *phala-yoga*. Here, the multiplier will be the summation of the squares of the squares (*varga-varga*) of numbers from unity increased consecutively by one (*ekādyekottara*). And the divisor will be the square of the square of radius. Here the radius (measured by the segment as a unit) will be the *pada*, number of terms for the summation. Then, the third *phala-yoga* is also to be derived from the first multiplicand. There the multiplier is the *samaṣaḍghāta-saṅkalita*, summation of the sixth power of numbers from one onwards (*ekādyekottara*), and the divisor is the sixth power of the radius.

Thus, (for the further *phala-yoga*-s) the divisor would be two powers raised further, (of the radius), and summation of the same powers (of numbers) is the multiplier. There, from the cubes are obtained, the summation of squares; from the fifth powers, the summation of the fourth powers; from the seventh powers, the summation of the sixth powers.

There, multiply the cube which forms the multiplier with the square of the radius, which is the divisor. The result will only be the radius. Thus, everywhere when the multipliers are divided by their respective divisors, the result will be the radius.

Now, since the cubes have to be divided by 3, divide the radius by 3. Then it would be that the summation of the squares of the (segments of the) radius has been divided by the square of the radius. In the same manner, division of the radius by 5 would be the same as division of the summation of the fourth powers by the fourth power. Thus, the result of dividing the radius successively by the odd numbers, 3, 5 etc. would be successive outcomes in the above-stated *phala-paramparā* (sequence of results). Hence, has it been said:

> *triśarādiviṣamasaṃkhyābhaktam ṛṇaṃ svaṃ pṛthak kramāt kuryāt* |
>
> Take the results of the division by the odd numbers 3, 5 etc. as additive and subtractive in order (cited also in *Yuktidīpikā* com. on *Tantrasaṅgraha*, II. 271[1]).

There, in this series of results (*phala-paramparā*), when subtractions have to be done at each step, it amounts to subtracting the sum of the odds from the sum of the multiplicands, and adding the sum of the evens. Hence has it also been said: *ṛṇaṃ svam pṛthak kramāt kuryāt*, 'do subtraction, or addition, in order'.

Next, *sama-ghāta-saṅkalitānayana*, the principle of 'summation of equal powers (of natural numbers)' is to be explained as it is useful in the above context. In that context, the principle of computing the summation of (natural) numbers and their squares are also explained here. Incidentally

[1]This and several other verses of Mādhava which have been cited in *Yuktibhāṣā*, have also been cited by *Śaṅkara* in his commentary *Yuktidīpikā* on *Tantrasaṅgraha*. It may however be noted that *Yuktidīpikā* declares, at the end of each chapter, that it is only presenting the subject as expounded by *Jyeṣṭhadeva* (in *Yuktibhāṣā*).

(*uttarottara-saṅkalitaikyānayana*) the principle of repeated summations of (natural numbers) is also explained.

6.3.5 *Śodhya-phala*-s and *Phala-yoga*

Now, the square of the east-west line (the radius) is the multiplier and the square of the corresponding hypotenuse is the divisor. Therefore, the respective difference between the multiplier and the divisor is the square of the segment in the (eastern) side of the square occurring in the interstice between the tip of the hypotenuse and the east-west line. The multiplicand is the segment of the (eastern) side of the square between the tip of the desired hypotenuse and the tip of the adjacent (smaller) hypotenuse. The *icchā-phala* is the (*ardha-jyā*) Rsine of the circumference bit (arc) which occurs in between the said two hypotenuses. Thus are obtained all the results. Here, the multiplicands are all the same, since the segments (on the eastern side of the square) between two hypotenuses are all equal. When all these results are calculated and added together, the result is the circumference of that portion of the circle which lies between the east-west line and the hypotenuse joining the (south-east) corner (of the square).

Here, the *bhujā* corresponding to the hypotenuse next to the east-west line is one segment of the side of the square. For the second hypotenuse, the *bhujā* is given by two segments. In this manner, the *bhujā* for the succeeding hypotenuses will include one segment more (than the previous). Thus the successive *bhujā*-s, starting from the first, are obtained by adding one segment each to the previous one. Hence, the sum of their squares is equal to the sum of the difference between the multipliers and the divisors. Since the multiplicand for all is the same, if that is multiplied by the sum of the difference between the multiplier and divisor, the result will be the *phala-yoga*, if the divisor by which the product is to be divided is just 1.

Suppose the divisor to be 1. It also taken to be the square of the radius for doing the calculations. Now, find the product of the result arrived at as above and the difference between the multiplier and the divisor. When this is lesser than the dividend, then the result got by dividing with the divisor

will be equal to the result got by dividing by the multiplier. When it is not less, but in fact larger than the dividend, then the result there-from and the difference between the multiplier and the divisor should be multiplied, and quotient obtained by dividing that by the divisor should be subtracted from the quotient got by dividing by the multiplier. Then also the result will be the same. When the subtractive term is obtained, if we instead divide by the multiplier, the result will be slightly larger. In that case a subtractive term (*śodhya-phala*) has to be derived from that also. If the process is repeated, further terms (*śodhya*-s) shall have to be subtracted. Then, starting from the end, when all these (*śodhya*-s) have been subtracted, the result will be obtained.

6.3.6 *Śodhya-phala*-s: An example

Here, let the dividend be 100. Let the divisor be 10 and the multiplier 8. Suppose also that by multiplying this, 100 is got. Here, if the division is done by the divisor (10), the result is 10. When the divisor 10 is to be subtracted once from the dividend (to get 90), if instead (the multiplier) 8 is subtracted, (to get 92), the difference between the divisor and the multiplier, viz., 2, will remain back in the dividend. Again, as many times as (the divisor and the multiplier are) subtracted, so many times of the difference between the divisor and the multiplier will remain back in the dividend. Hence, if the product of the result (100/10) and the difference between the multiplier and the divisor (10-8=2) is subtracted from the dividend (100) and the remainder (100-20=80) is divided by the multiplier (8), the result (80/8 = 10) will be equal to the result got by dividing (100) by the divisor (10). Here, when the dividend (100) is divided by the multiplier (8), the result is 12.5. When this result is multiplied by the difference between the multiplier and divisor, (viz., 2) the result got is 25. When this is divided by the divisor, 10, the result is $2\frac{1}{2}$. When this is subtracted from the earlier result, $12\frac{1}{2}$, the remainder is 10.

Here, if 25 is divided by 8, the result is 3 and one-eighth. This is *śodhya*, to be reduced, the result got is larger. Now, when this result, (3 and one-eighth), is multiplied by the difference between the multiplier and the divisor (2), and the product divided by the divisor (10), the result is half plus one-

eighth. When this is subtracted from the second result (namely 3 and one-eighth) we got $2\frac{1}{2}$. Then, this is the amount to be subtracted from the first result (namely $12\frac{1}{2}$). In this manner, the respective results are multiplied by the difference between the multiplier and the divisor and divided by the divisor, and the result when subtracted from the previous result will be more accurate. Then it shall be subtracted from the result earlier; then (the still corrected result) from the one earlier than that, and so on. In this manner, the result obtained at the first step can be made to correspond to the actual.

Now, the respective squares of the *bhujā*-s (portions of the eastern side of the square) are the difference between the multiplier and divisor of (the multiplicand, viz.,) the segment of the square. The results got at each stage shall have to be multiplied by the square of the *bhujā*-s. Since there is no other result (from which to start), calculate the second result from the *bhujā-khaṇḍa*, the segment (of the side of the square) which is the multiplicand for the first result. (The method) for this: Since the multiplier for both the first and the second results are the square of the *bhujā*, to get the next result multiply twice with the square of the *bhujā*, the multiplicand which is the *bhujā-khaṇḍa*. The divisor of the first result is the square of the radius; so, divide (the product arrived at as above) by the square of that (i.e., the fourth power of the radius) for the second result. Here, since the multiplicands are the same, the multiplication can be done with the sum of the multipliers. Here, the multipliers are the squares of the squares (fourth power) of the *bhujā-khaṇḍa* multiplied by successive numbers starting with unity. Their sum is the sum of the difference of the multipliers and divisors. This sum is termed *ekādyekottara-varga-varga-saṅkalita*, summation of the squares of squares of numbers 1, 2, etc. Here the divisor is the square of square of the radius. Thus, for getting the first result, the multiplier is two equal *bhujā*-s which have been multiplied amongst themselves, and the divisor is two radii which are multiplied amongst themselves.

Then, for the second result: The multiplier is the equal bhuja segments multiplied four-fold and the divisor is radius multiplied four-fold. Then, for the third result: The multipliers and divisors are, (respectively), the same ones multiplied amongst themselves six-fold. Thus, for the fourth (result) the multipliers and divisors are (respectively) the same ones multiplied eight-

fold amongst themselves. The multiplicand, at all places, is the same, the *bhujā-khaṇḍa*. Hence, for obtaining the *phala-yoga*, the final result, the final multiplier will be the sum of the multipliers. Now, when the first *phala-yoga* is calculated, the *bhujā*-s associated with the hypotenuses commencing from the east-west line up to the line joining the (centre to the south-east) corner of the square, will be multiples of the *bhujā-khaṇḍa*, commencing from one segment of the side of the square and increasing at the rate of one segment, the last of them being the side of the square equal to the radius. The sum of their squares is the sum of the multipliers. This is termed *ekādyekottara-varga-saṅkalita*. Thus, in the calculation of the second *phala-yoga*, the sum of the multipliers is the sum of the squares of the squares of the successive *bhujā*-s commencing from one *bhujā-khaṇḍa*, and growing by the addition of one *bhujā-khaṇḍa* to the previous one, the last one being the radius. Thus the sum of the multipliers in terms that follow, will be the summation of (the multipliers) the same numbers of *bhujā-khaṇḍa* raised to powers 6, 8 etc.

6.4 *Saṅkalita*: Summation of series

Now is described the methods of making the summations (referred to in the earlier sections). At first, the simple arithmetical progression (*kevala-saṅkalita*) is described. This is followed by the summation of the products of equal numbers (squares). Though not useful in the (calculations dealt with in the present work), the summation of the products of 3 and 5 identical numbers are also described here, since they occur among matters useful herein.

6.4.1 *Mūla-saṅkalita*: Sum of natural numbers

Here, in this *mūla-saṅkalita* (basic arithmetical progression), the final *bhujā* is equal to the radius. The term before that will be one segment (*khaṇḍa*) less. The next one will be two segments less. Here, if all the terms (*bhujā*-s) had been equal to the radius, the result of the summation would be obtained by multiplying the radius by the number of *bhujā*-s. However, here, only one *bhujā* is equal to the radius. And, from that *bhujā*, those associated with the smaller hypotenuses are less by one segment each, in order. Now, suppose

the radius to be of the same number of units as the number of segments to which it has been divided, in order to facilitate remembering (their number). Then, the number associated with the penultimate *bhujā* will be less by one (from the number of units in the radius); the number of the next one, will be less by two from the number of units in the radius. This reduction (in the number of segments) will increase by one (at each step). The last reduction will practically be equal to the measure of the radius, for it will be less only by one segment. In other words, when the reductions are all added, the sum thereof will practically (*prāyeṇa*) be equal to the summation of the series from 1 to the number of units in the radius; it will be less only by one radius length. Hence, the summation will be equal to the product of the number of units in the radius with the number of segments plus one, and divided by 2. The summation of all the *bhujā*-s of the different hypotenuses is called *bhujā-saṅkalita*.

Now, the smaller the segments, the more accurate (*sūkṣma*) will be the result. Hence, do the summation also by taking each segment as small as an atom (*aṇu*). Here, if it (namely, the *bhujā* or the radius) is divided into *parārdha* (a very large number) parts, to the *bhujā* obtained by multiplying by *parārdha* add one part in *parārdha* and multiply by the radius and divide by 2, and then divide by *parārdha*. For, the result will practically be the square of the radius divided by two. In order that the number might be full, it is divided by *parārdha*. Thus, if the segments are small, only one small segment shall have to be added to get the summation. Hence, not adding anything to (the units in) the *bhujā*, if it is multiplied by the radius and divided by 2 it will be *bhujā-saṅkalita* when it has been divided into extremely small segments. Thus, the square of the radius divided by 2 will be the *saṅkalita* when the segment (*bhujā-khaṇḍa* into which the *bhujā* or the side of the square is divided) is very small.

6.4.2 *Varga-saṅkalita*: Summation of squares

Now is explained the summation of squares (*varga-saṅkalita*). Obviously, the squares of the *bhujā*-s, which are summed up above, are the *bhujā*-s each multiplied by itself. Here, if the *bhujā*-s which are all multipliers, had all

been equal to the radius, their sum, (*saṅkalita* derived above), multiplied by the radius would have been the summation of their squares. Here, however, only one multiplier happens to be equal to the radius, and that is the last one. The one before that will have the number of segments one less than in the radius. (Hence) if that, (i.e., the second one), is multiplied by the radius, it would mean that one multiplied by the penultimate *bhujā* would have been the increase in the summation of the squares. Then (the segment) next below is the third. That will be less than the radius by two segments. If that is multiplied by the radius, it will mean that, the summation of the squares will increase by the product of the *bhujā* by two (segments). In this manner, the summation in which the multiplication is done by the radius (instead of the *bhujā*-s) would be larger than the summation of squares by terms which involve the successively smaller *bhujā*-s multiplied by successively higher numbers. If (all these additions) are duly subtracted from the summation where the radius is used as the multiplier, the summation of squares (*varga-saṅkalita*) will result.

Now, the *bhujā* next to the east-west line is less than the radius by one (segment). So if all the excesses are summed up and added, it would be the summation of the basic summation (*mūla-saṅkalita-saṅkalita*). Because, the sums of the summations is verily the 'summation of summations' (*saṅkalita-saṅkalita*). There, the last sum has (the summation of) all the *bhujā*-s. The penultimate sum is next lower summation to the last. This penultimate sum is the summation of all the *bhujā*-s except the last *bhujā*. Next to it is the third sum which is the sum of all the *bhujā*-s except the last two. Thus, each sum of the *bhujā*-s commencing from any *bhujā* which is taken to be the last one in the series, will be less by one *bhujā* from the sum (of the *bhujā*-s) before that.

Thus, the longest *bhujā* is included only in one sum. But the *bhujā* next lower than the last (*bhujā*) is included both in the last sum and also in the next lower sum. The *bhujā*-s below that are included in the three, four etc. sums below it. Hence, it would result that the successively smaller *bhujā*-s commencing from the one next to the last, which have been multiplied by numbers commencing from 1 and added together, would be sum-

mation of summations (*saṅkalita-saṅkalita*). Now, it has been stated earlier that the summation (*saṅkalita*) of (the segments constituting) a *bhujā* which has been very minutely divided, will be equal to half the square of the last *bhujā*. Hence, it follows that, in order to obtain the summation (*saṅkalita*) of the *bhujā*-s ending in any particular *bhujā*, we will have to square each of the *bhujā*-s and halve it. Thus, the summation of summations (*saṅkalita-saṅkalita*) would be half the summation of the squares of all the *bhujā*-s. In other words, half the summation of the squares is the summation of the basic summation. So, when the summation is multiplied by the radius, it would be one and a half times the summation of the squares. This fact can be expressed by stating that this contains half more of the summation of squares. Therefore, when the square of the radius divided by two is multiplied by the radius and one-third of it subtracted from it, the remainder will be one-third of the whole. Thus it follows that one-third of the cube of the radius will be the summation of squares (*varga-saṅkalita*).

6.4.3 *Ghana-saṅkalita* and *Varga-varga-saṅkalita*: Summation of third and fourth powers

Now, to the method or deriving the summation of cubes: Summation of cubes, it is clear, is the summation where the square of each *bhujā* in the summation of squares is multiplied by the *bhujā*. Now, by how much will the sum of cubes increase if all the *bhujā*-squares were to be multiplied by the radius. By the principle enunciated earlier, the *bhujā*-square next to the last will increase by itself being multiplied by 1. The *bhujā*-squares below will increase by their multiples of two, three etc., in order. That sum will be equal to the summation of the summation of squares (*varga-saṅkalita-saṅkalita*). It has already been shown that the summation of squares is equal to one-third the cube of the radius. Hence one-third the cube of each *bhujā* will be equal to the summation of all the *bhujā*-squares ending with that *bhujā*. Hence, it follows that the summation of summation of *bhujā*-squares (*varga-saṅkalita-saṅkalita*) is equal to one-third the sum of the *bhujā*-cubes (*ghana-saṅkalita*). Therefore, the summation of squares multiplied by the radius will be equal to the summation of the cubes plus a third of itself. Hence, when one-fourth

of it is subtracted, what remains will be the summation of cubes. Hence, it also follows that the summation of cubes (*ghana-saṅkalita*) is equal to one-fourth the square of the square of the radius.

Then, when this summation of cubes is multiplied by the radius, by the principle enunciated earlier, it follows that the result will be the summation of the squares of squares (*varga-varga-saṅkalita*) together with the summation of the summation of cubes (*ghana-saṅkalita-saṅkalita*). It has just been stated that the summation of cubes is one-fourth the square of square (of the radius). Hence, by the principle enunciated earlier, it also follows that one-fourth the sum of squares of squares is the summation of the summation of cubes. Hence, this being in excess of the result by one-fourth of itself, if one-fifth of it is subtracted, it follows that the summation of squares of squares (*varga-varga-saṅkalita*) will be equal to one-fifth of the radius raised to the power of five.

6.4.4 *Samaghāta-saṅkalita*: General principle of summation

Now, the square of the square (of a number) is multiplied by itself, it is called *sama-pañca-ghāta* (number multiplied by itself five times). The successive higher order summations are called *sama-pañcādi-ghāta-saṅkalita* (and will be the summations of powers of five and above). Among them if the summation (*saṅkalita*) of powers of some order is multiplied by the radius, then the product is the summation of summations (*saṅkalita-saṅkalita*) of the (powers of the) multiplicand (of the given order), together with the summation of powers (*sama-ghāta-saṅkalita*) of the next order. Hence, to derive the summation of the successive higher powers: Multiply each summation by the radius. Divide it by the next higher number and subtract the result from the summation got before. The result will be the required summation to the higher order.

Thus, divide by two the square of the radius. If it is the cube of the radius, divide by three. If it is the radius raised to the power of four, divide by four. If it is (the radius) raised to the power of five, divide by five. In this

manner, for powers rising one by one, divide by numbers increasing one by one. The results will be, in order, the summations of powers of numbers (*sama-ghāta-saṅkalita*). Here, the basic summation is obtained from the square, the summation of squares from the cube, the summation of cubes from the square of the square. In this manner, if the numbers are multiplied by themselves a certain number of times (i.e., raised to a certain degree) and divided by the same number, that will be the summation of the order one below that. Thus (has been stated) the method of deriving the summations of (natural) numbers, (their) squares etc.

6.4.5 Repeated summations

Now, are explained the first, second and further summations: The first summation (*ādya-saṅkalita*) is the basic summation (*mūla-saṅkalita*) itself. It has already been stated (that this is) half the product of the square of the number of terms (*pada-vargārdha*). The second (*dvitīya-saṅkalita*) is the summation of the basic summation (*mūla-saṅkalitaikya*). It has been stated earlier that it is equal to half the summation of squares. And that will be one sixth of the cube of the number of terms.

Now, the third summation: For this, take the second summation as the last term (*antya*); subtract one from the number of terms, and calculate the summation of summations as before. Treat this as the penultimate. Then subtract two from the number of terms and calculate the summation of summations. That will be the next lower term. In order to calculate the summation of summations of numbers in the descending order, the sums of one-sixths of the cubes of numbers in descending order would have to be calculated. That will be the summation of one-sixth of the cubes. And that will be one-sixth of the summation of cubes. As has been enunciated earlier, the summation of cubes is one-fourth the square of the square. Hence, one-sixth of one-fourth the square of the square will be the summation of one-sixth of the cubes. Hence, one-twenty-fourth of the square of the square will be the summation of one-sixth of the cubes. Then, the fourth summation will be, according to the above principle, the summation of one-twenty-fourths of

the square of squares. This will also be equal to one-twenty-fourth of one-fifth of the fifth power. Hence, when the number of terms has been multiplied by itself a certain number of times, (i.e., raised to a certain degree), and divided by the product of one, two, three etc. up to that index number, the result will be the summation up to that index number amongst the first, second etc. summations (*ādya-dvitīyādi-saṅkalita*).

6.5 Conclusion: Calculation of the circumference

Here, it is necessary to calculate also the summation of squares, the summation of the squares of squares and the sum of numbers raised to the power of 6 etc. Hence it was directed to divide by numbers starting with 3, 5, etc. The divisors for these are the square of the radius, square of the square etc. But the result of dividing the cube of the radius by the square of the radius, the quotient is the radius itself. Hence, the radius divided by three is the first sum of the results (*phala-yoga*). And this is the sum of the differences of the respective multiplicands and the respective results (*phala*). Therefore, subtract it from the sum of multiplicands. The result will be half the side of the square from the tip of the east-west line to the (south-east) corner.

Likewise, when the fifth power (of the radius) is divided by the square of the square (of the radius), the result is only the radius. Thus, the radius divided by 5 is the second result. In this manner, when the radius is divided by the odd numbers 7, 9 etc., further results will be obtained. The results got are to be alternatively added to and subtracted from the radius. In this way, we obtain one-eighth of the circumference of the circle.

Here, as the multiplier is smaller than the divisor, the succeeding terms become less and less, and when the terms become very small, then they can be discarded and the calculation ended; then the result will be mostly accurate. The result got will be the portion of the circle lying between the tip of the east-west line and the line (from the centre) joining the (south-east) corner. When this is multiplied by 8, (the circumference) of the circle will be complete. First multiply the radius which is the dividend by 8; that will be four times the diameter. It is to this diameter that the necessary

procedures have to be adopted as explained with reference to the verse *vyāse vāridhinihate...* (cited also in *Yuktidīpikā*, com. on *Tantrasaṅgraha*, II. 271), and the prescribed corrections are to be applied to get (the circumference of) the circle.

6.6 *Cāpīkaraṇa*: Conversion of the Rsine to arc

Using the principles enunciated above, the *jyā*-s (Rsines) can be converted into arcs as given in the verses

iṣṭajyātrijyayorghātāt koṭyāptaṃ prathamaṃ phalam |
jyāvargaṃ guṇakaṃ kṛtvā koṭivargaṃ ca hārakam ||

prathamād phalebhyo'tha neyā phalatatirmuhuḥ |
ekatryādyojasaṃkhyābhirbhakteṣveteṣvanukramāt ||

ojānāṃ saṃyutestyaktvā yugmayogaṃ dhanurbhavet |
doḥkoṭyoralpameveṣṭaṃ kalpanīyamiha smṛtam ||

labdhīnāmavasānaṃ syānnānyathāpi muhuḥ kṛte |

The product of the desired *jyā* (Rsine) and the radius (Rsine 90), divided by the *koṭi* (Rcosine) is the first result. Making the square of the Rsine the multiplier and the square of the Rcosine the divisor, derive a series of successive results (by multiplying and dividing) the first result successively. Divide these results, in order, by the odd numbers 1, 3 etc., find the sum of the terms in the even places and subtract them from the sum of the terms in the odd places. The result will be the corresponding arc (*dhanus*). Here, the lesser of the *doḥ* and *koṭi* is to be taken as the desired *jyā*. Otherwise there will be no end to the quotients when the successive terms are calculated. (cited also in *Yuktidīpikā*, II. 206-209)

vyāsavargād ravihatāt padaṃ syāt prathamaṃ phalam |
tadāditastrisaṃkhyāptaṃ phalaṃ syāduttarottaram ||

rupādyayugmasaṃkhyābhirhṛteṣveṣu yathākramam |
viṣamānāṃ yutestyaktvā samaṃ hi paridhirbhavet ||

Multiply the square of the diameter by 12 and find the root. That shall be the first result. That result divided by 3 shall be the second result. Further and successive results are obtained (by dividing the further results repeatedly by 3, in order). All these results are divided, in order, by the odd numbers 1, 3 etc. Find the sum of the odd results (now got) and subtract from it the sum of the even results. The remainder is the circumference. (cited also in *Yuktidīpikā*, II. 212-214)[2]

Now is stated the method to find the arc of the *bhujā* or *koṭi* (Rsine or Rcosine), whichever is smaller. Here too, first it is supposed that the Rsine is smaller. Multiply this desired Rsine by the radius and divide by the Rcosine (*koṭi-jyā*). The result would be the first result. Then multiply this result itself by the square of the Rsine and divide by the square of the Rcosine. This would be the second result. Likewise calculate the third result, by multiplying the second result by the square of the Rsine and dividing by the square of the Rcosine. Calculate further results in the same way, using the selfsame multiplier and divisor. Divide the sequence of results (*phala-paramparā*), in order, by the odd numbers 1, 3, 5 etc. Of the results now got, find the sum of the first, third, fifth etc., and subtract therefrom the sum of the second, fourth etc. The remainder is the (desired) arc. To get the complementary-arc (*koṭi-cāpam*), subtract this from (the arc of the circle) equal in measure to three signs (90 degrees). When the complementary arc is smaller, that is what is to be calculated to start with.

The rationale (*upapatti*) for this (is as follows): As in the (earlier) case where (the circumference of) the circle was sought to be got from the radius, here a square is constructed (touching the sides of the circle). The sine-chord (*jyā*) is so chosen such that the versine-chord (*śara*), is stretched along the (portion of the) east-west line starting from the centre going up to the circle. Extend the hypotenuse drawn from the centre of the circle and passing through the

[2]This result is discussed in the next Section (6.7).

tip of the sine-chord up to the side of the square lying outside the circle. Here, this would be the longest hypotenuse. The part of the (eastern) side of the square that lies between the tip of this hypotenuse and the tip of the east-west line will now be the first result. Then, it is instructed to derive a series of results arrived at by taking the above-said part of the side of the square as the multiplicand, its square as multiplier and with the square of the east-west line as divisor. There, in all the results, if the entire (half) side of the square (*bhujā-bhāga*) is the multiplicand, the multipliers and the divisors will be the same. When such is the case, the result will be the multiplicand itself at all places. Then, the multiplicand itself is being divided by the odd numbers (from 1). Here, however, the multipliers and the divisors which are the Rsine (*bhujā*) and the Rcosine (*koṭi*), are not the same, and the results will be (different and) become successively smaller. (In this manner) all the results have to be calculated in order. Obviously, for this it would be preferable to have smaller multipliers and divisors. Therefore (in the instructions contained in the verses) it is not the portion of the square, which stretches from the east-west line to the last hypotenuse, that is taken as the multiplier, but the (corresponding) interstice inside the circle, which is the Rsine. Then, the divisor would be its Rcosine. The respective results would be the multiplicands. This is the difference (between the calculation of one-eighth of the circumference and that of the given arc). Here also, the division is made by the odd numbers derived from *varga-saṅkalita* etc. Thus is (explained) the conversion of chords into arcs.

6.3 Circumference of a circle without calculating square-roots

6.3.1 Dividing the circumference into arc-bits: Approximating the arc-bits by *Jyārdha*-s (Rsines)

In Figure 6.3 we show the quadrant $OEAS$ of the square which circumscribes a circle of radius $r = OE$. OE is the east line and OS the south line. Divide the tangent EA into n equal parts by marking the points $A_o \equiv E,\ A_1,\ A_2....A_n \equiv A$, at equal distances given by,

$$\frac{EA}{n} = \frac{r}{n}.$$

It is noted that more the number of points (larger the n), more accurate will be the circumference determined by this method.

[1]If we have already calculated l_n, half the side of the circumscribing regular polygon of 2^{n+1} sides (and also k_{n-1}, a_{n-1}, the previous *karṇa* and base-segment), then we can calculate l_{n+1} (and $k_n,\ a_n$) using the relations:

$$\begin{aligned} k_n^2 &= r^2 + l_n^2, \\ a_n &= \frac{\left[k_n - (\frac{r^2 - l_n^2}{k_n})\right]}{2}, \\ l_{n+1} &= l_n - (k_n - r)\left(\frac{l_n}{a_n}\right). \end{aligned}$$

These lead to the simple recursion relation $l_{n+1} = \frac{r}{l_n}[\sqrt{(r^2 + l_n^2)} - r]$, with the initial condition $l_1 = r$.

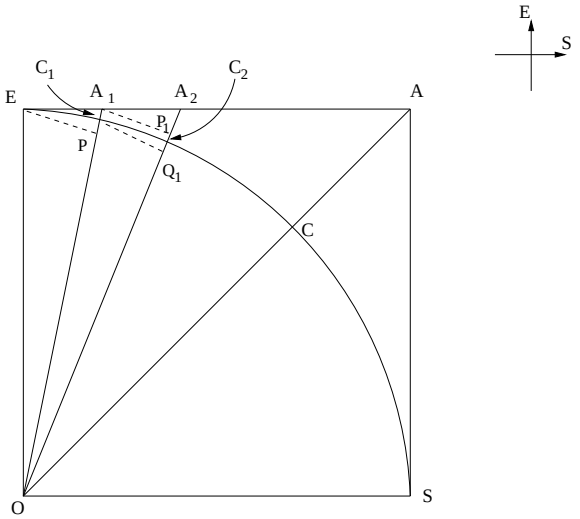

Figure 6.3: Dividing the circumference into arc-bits.

Join the hypotenuses *karṇa*-s $OA_1,\ OA_2, \ldots OA_n \equiv OA$, and let them meet the circle at $C_1,\ C_2, \ldots C_n \equiv C$. The quadrant of the circle is divided into n arc-bits, $EC_1,\ C_1C_2, \ldots C_{n-1}C_n$, which lie between successive hypotenuses. Each of the *karṇa*-s, $k_i = OA_i$, is given by

$$\begin{aligned} k_i^2 = OA_i^2 &= OE^2 + EA_i^2 \\ &= r^2 + \left(\frac{ir}{n}\right)^2. \end{aligned} \tag{6.15}$$

Draw the perpendicular EP from the east point E to the first hypotenuse OA_1. The triangles EPA_1 and OEA_1 are similar. Hence

$$\frac{EP}{EA_1} = \frac{OE}{OA_1}. \tag{6.16}$$

Therefore

$$EP = EA_1 \times \frac{OE}{OA_1} = \left(\frac{r}{n}\right)\left(\frac{r}{k_1}\right). \tag{6.17}$$

The similarity of the triangles EPA_1 and OEA_1 can be understood by noting that the *karṇa* of one triangle is parallel to the *bhujā* of the other and vice versa, and the *karṇa* of one is perpendicular to the *koṭi* of the other and vice versa. It is also illustrated by considering the junction of the slanting beam and the joining tie in the roof of a square hall *maṇḍapa* as in Figure 6.4.

Now, draw the perpendicular A_1P_1 from the tip A_1 of the first hypotenuse to the second hypotenuse OA_2. Again the triangles $A_1P_1A_2$ and OEA_2 are similar. Hence

$$\frac{A_1P_1}{A_1A_2} = \frac{OE}{OA_2}. \tag{6.18}$$

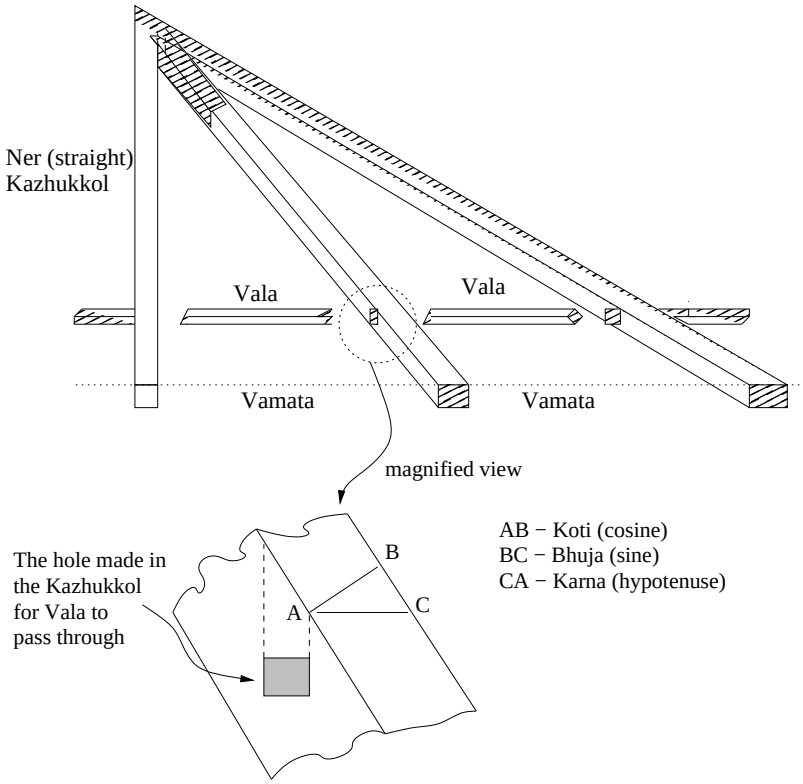

Figure 6.4: Slanting beam and joining tie in a *Maṇṭapa*.

Therefore

$$\begin{aligned} A_1P_1 &= A_1A_2 \times \frac{OE}{OA_2} \\ &= A_1A_2 \times \left(\frac{r}{k_2}\right) \\ &= \left(\frac{r}{n}\right)\left(\frac{r}{k_2}\right). \end{aligned} \tag{6.19}$$

Now, draw the perpendicular C_1Q_1 from the point C_1 where the first hypotenuse meets the circle to the second hypotenuse OA_2. The triangle OC_1Q_1 which lies inside the circle is similar to the triangle OA_1P_1. Hence $C_1Q_1 = b_1$, which is also the *jyārdha* (Rsine) of the arc-bit C_1C_2, is given by

$$\begin{aligned} b_1 = C_1Q_1 &= A_1P_1 \times \frac{OC_1}{OA_1} \\ &= A_1A_2\left(\frac{r}{k_2}\right) \times \left(\frac{r}{k_1}\right) \\ &= \left(\frac{r}{n}\right)\left(\frac{r^2}{k_1k_2}\right). \end{aligned} \tag{6.20}$$

Similarly, it can be shown that the *jyārdha* or Rsine, $b_i = C_iQ_i$, of the arc-bit C_iC_{i+1}, is given by the corresponding segment A_iA_{i+1} of the side of the square multiplied by the square of the radius and divided by the product of the two diagonals (OA_i and OA_{i+1}) meeting the ends of the segment. For this, first the perpendicular distance $d_i = A_iP_i$, from the tip of the hypotenuse OA_i to the hypotenuse OA_{i+1}, is calculated using the fact that the triangles $A_iP_iA_{i+1}$ and OEA_{i+1} are similar (see Figure 6.5):

$$\begin{aligned} d_i = A_iP_i &= A_iA_{i+1} \times \frac{OE}{OA_{i+1}} \\ &= A_iA_{i+1}\left(\frac{r}{k_{i+1}}\right) \\ &= \left(\frac{r}{n}\right)\left(\frac{r}{k_{i+1}}\right). \end{aligned} \tag{6.21}$$

Then the Rsine, $b_i = C_iQ_i$, is calculated using the fact that the triangles OC_iQ_i and OA_iP_i are similar

$$\begin{aligned} b_i = C_iQ_i &= A_iP_i \times \frac{OC_i}{OA_i} \\ &= A_iA_{i+1}\left(\frac{r}{k_{i+1}}\right) \times \left(\frac{r}{k_i}\right) \\ &= \left(\frac{r}{n}\right)\left(\frac{r^2}{k_ik_{i+1}}\right). \end{aligned} \tag{6.22}$$

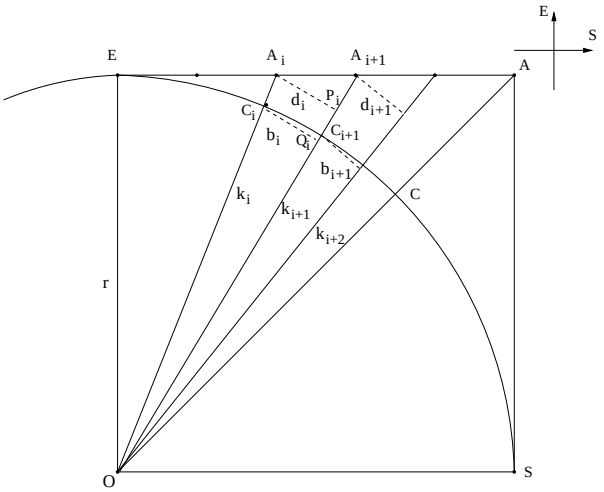

Figure 6.5: Circumference approximated by Rsines.

6.3.2 Circumference in terms of the *Karṇa*-s (hypotenuses)

It is said that when the size $(\frac{r}{n})$ of the equal segments into which the side of the square is divided is very small (n is very large), then these Rsines will be almost the same as the arc-bits. Then, one-eighth the circumference of the circle (the arc EC, between the first and last hypotenuses) can be approximated by the sum of the Rsines

$$\begin{aligned} \frac{C}{8} &\approx b_o + b_1 + \ldots + b_{n-1} \\ &= \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_o k_1}\right) + \left(\frac{r^2}{k_1 k_2}\right) + \ldots + \left(\frac{r^2}{k_{n-1} k_n}\right)\right], \end{aligned} \tag{6.23}$$

where $b_o = EP$ is the Rsine of arc-bit EC_1 and $k_o = r$. When n is very large,

$$k_i \approx k_{i+1}, \tag{6.24}$$

and hence

$$\frac{1}{(k_i k_{i+1})} \approx \frac{1}{2}\left(\frac{1}{k_i^2} + \frac{1}{k_{i+1}^2}\right). \tag{6.25}$$

Hence, the above expression for one-eighth the circumference of the circle becomes

$$\frac{C}{8} \approx \left(\frac{r}{n}\right) r^2 \left(\frac{1}{2}\right) \left[\left(\frac{1}{k_0^2}+\frac{1}{k_1^2}\right)+\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)+\ldots \right. \\ \left. \ldots+\left(\frac{1}{k_{n-1}^2}+\frac{1}{k_n^2}\right)\right]. \tag{6.26}$$

We have already noted that the first hypotenuse $k_0 = r$, and it can be easily seen that the last hypotenuse is given by $k_n^2 = 2r^2$. Now, when n is very large, the above expression (6.26) simplifies to

$$\frac{C}{8} \approx \left(\frac{r}{n}\right) \left[\left(\frac{r^2}{k_1^2}\right)+\left(\frac{r^2}{k_2^2}\right)+\left(\frac{r^2}{k_3^2}\right)+\ldots+\left(\frac{r^2}{k_n^2}\right)\right], \tag{6.27}$$

as the two expressions (6.26) and (6.27) differ by the amount

$$\begin{aligned} \left(\frac{r}{n}\right)\left(\frac{1}{2}\right)\left[\left(\frac{r^2}{k_o^2}\right)-\left(\frac{r^2}{k_n^2}\right)\right] &= \left(\frac{r}{n}\right)\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) \\ &= \left(\frac{r}{n}\right)\left(\frac{1}{4}\right), \end{aligned} \tag{6.28}$$

which becomes negligible when n is large.

6.3.3 *Śodhya-phala*-s: Iterative corrections

As we have seen above, each of the Rsines b_i of the arc-bits is approximated by

$$b_i = \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i+1}^2}\right). \tag{6.29}$$

Thus the segment $(\frac{r}{n})$ is multiplied by the "multiplier", namely the square of the radius, and divided by the "divisor", namely the square of the hypotenuse. Here the divisors, the *karṇa*-s k_i^2 given by (6.15), are larger than or equal to the multiplier r^2. This division can be expressed as a sequence of iterative corrections *śodhya-phala*-s as shown below.

Consider the product $a\,(\frac{c}{b})$, where some quantity a is multiplied by the multiplier c, and divided by the divisor b, which is larger than the multiplier

c. We first rewrite this product in the following form

$$a\ \left(\frac{c}{b}\right) = a - a\ \frac{(b-c)}{b}. \tag{6.30}$$

Now, if in the expression $\left(\frac{b-c}{b}\right)$ we want to replace the division by b (the divisor) by division by c (the multiplier), then we have to make a subtractive correction (*śodhya-phala*)

$$\frac{(b-c)}{b} = \frac{(b-c)}{c} - \frac{(b-c)}{c}\ \ \frac{(b-c)}{b}. \tag{6.31}$$

Thus our original relation (6.30) becomes

$$a\left(\frac{c}{b}\right) = a - \left[a\frac{(b-c)}{c} - \left(a\frac{(b-c)}{c}\right)\frac{(b-c)}{b}\right]. \tag{6.32}$$

Now in the second correction term in (6.32), if we again replace the division by the divisor b, by the multiplier c, then we have to employ the relation (6.31) to get another subtractive term

$$a\left(\frac{c}{b}\right) = a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - a\left[\frac{(b-c)}{c}\right]^2\frac{(b-c)}{b}. \tag{6.33}$$

Thus, after taking m *śodhya-phala*-s we get

$$\begin{aligned} a\left(\frac{c}{b}\right) \ &= \ a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \ldots + (-1)^{m-1}a\left[\frac{(b-c)}{c}\right]^{m-1} \\ &\quad +(-1)^m a\left[\frac{(b-c)}{c}\right]^{m-1}\frac{(b-c)}{b}. \end{aligned} \tag{6.34}$$

It is stated that if we keep on dividing only by the multiplier c, then there will be no end to these correction terms (*phala-parampar*ā). It is also said that when the correction term becomes very small they can be discarded.[2]

[2]The resulting infinite series expansion is convergent for $0 \leq b-c < c$, which condition is satisfied in our case where $b = k_i^2$ and $c = r^2$, as they satisfy $r^2 \leq k_i^2 < 2r^2$ for $0 \leq i < n$. For the same reason, the correction terms in (6.34) become very small when m becomes very large. If we set $\frac{(b-c)}{c} = x$, then $\frac{c}{b} = \frac{1}{(1+x)}$. Hence, what is given above is the binominal series for $(1+x)^{-1}$ which is convergent for $-1 < x < 1$.

6.3.4 *Phala-yoga*-s and their series: *Phala-paramparā*

Using the above method, each of the Rsines of the arc-bits can now be expressed as

$$\begin{aligned} b_{i-1} &= \left(\frac{r}{n}\right)\left(\frac{r^2}{k_i^2}\right) \\ &= \left(\frac{r}{n}\right)\left[\frac{r^2}{\left(r^2+\left(\frac{ir}{n}\right)^2\right)}\right] \\ &= \left(\frac{r}{n}\right) - \left(\frac{r}{n}\right)\frac{1}{r^2}\left(\frac{ir}{n}\right)^2 + \ldots + (-1)^m\left(\frac{r}{n}\right)\frac{1}{r^{2m}}\left(\frac{ir}{n}\right)^{2m} \\ &\quad + \ldots \quad . \end{aligned} \tag{6.35}$$

Now using this expansion for the Rsines in the expression for one-eighth of the circumference and keeping terms with the same powers of $(\frac{r}{n})$ together, we get

$$\begin{aligned} \frac{C}{8} &= b_o + b_1 + b_2 + \ldots + b_{n-1} \\ &= \left(\frac{r}{n}\right)[1+1+\ldots+1] \\ &\quad - \left(\frac{r}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{r}{n}\right)^2 + \left(\frac{2r}{n}\right)^2 + \ldots + \left(\frac{nr}{n}\right)^2\right] \\ &\quad + \left(\frac{r}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{r}{n}\right)^4 + \left(\frac{2r}{n}\right)^4 + \ldots + \left(\frac{nr}{n}\right)^4\right] \\ &\quad - \left(\frac{r}{n}\right)\left(\frac{1}{r^6}\right)\left[\left(\frac{r}{n}\right)^6 + \left(\frac{2r}{n}\right)^6 + \ldots + \left(\frac{nr}{n}\right)^6\right] \\ &\quad + \ldots \quad . \end{aligned} \tag{6.36}$$

Each of the terms in (6.36) is a sum of results (*phala-yoga*) which we need to calculate when n is very large, and we have a series of them (*phala-paramparā*) which are alternatively positive and negative. Clearly the first term is just the sum of the *bhujā-khaṇḍa*-s, the bits into which the tangent or the eastern side of the square, of side equal to radius r was divided. Thus the first term is equal to the radius r. The next summation will involve *bhujā-varga-saṅkalita*, the sum of the squares of the *bhujā*-s or portions of

the side of the square, increasing by one bit at a time. The next *phala-yoga* will involve *bhujā-varga-varga-saṅkalita* or sum of fourth powers of the *bhujā*-s and so on.

It is said that for the sake of accuracy (*sūkṣmatā*) of the result, the *bhujā-khaṇḍas* $\frac{r}{n}$, will have to be infinitesimal (*aṇu-parimāṇa*). It will be shown in Section 6.4 that when n is large, the successive sums (*phala-yoga*) are $\frac{r}{3}, \frac{r}{5}$, and so on. This leads to the expression for the circumference

$$\frac{C}{8} = r - \frac{r}{3} + \frac{r}{5} - \dots \ . \tag{6.37}$$

Or, equivalently if d is the diameter of the circle

$$C = 4d\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right). \tag{6.38}$$

which is the result enunciated in the verse of Mādhava, *vyāse vāridhinihate...* (cited also in *Yuktidīpikā* II. 271).[3]

6.3.6 *Śodhya-phala*-s : An example

Here, an example is given of how the iterative subtractive corrections can be obtained. The basic identity that is iterated is of the form

$$\frac{a}{b} = \left(\frac{a}{c}\right) - \left(\frac{a}{b}\right)\frac{(b-c)}{c}. \tag{6.39}$$

Here, if in the second term we divide by the "multiplier" c, instead of the "divisor" b, we have to subtract the next subtractive term

$$\frac{a}{b} = \frac{a}{c} - \left[\left(\frac{a}{c}\right)\left(\frac{(b-c)}{c}\right) - \left(\frac{a}{c}\right)\left(\frac{(b-c)}{c}\right)\left(\frac{(b-c)}{b}\right)\right], \tag{6.40}$$

and so on. In the example given, $a = 100$, the divisor $b = 10$ and the multiplier $c = 8$. First it is noted that

$$\frac{100}{10} = \frac{100}{8} - \left(\frac{100}{10}\right)\left(\frac{(10-8)}{8}\right). \tag{6.41}$$

[3]This and several verses of Mādhava, which have been cited in *Yuktibhāṣā*, have also been cited by Śaṅkara in his commentary *Yuktidīpikā* on *Tantrasaṅgraha*. It may however be noted that *Yuktidīpikā* declares, at the end of each Chapter, that it is only presenting the subject as expounded by Jyeṣṭhadeva (in *Yuktibhāṣā*).

Then it is noted that if in the second term we take $\frac{100}{8}$ instead of $\frac{100}{10}$, then the second term becomes $\frac{25}{8}$ instead of $\frac{25}{10}$. So the subtraction is in excess and is to be *śodhya*, further reduced. Then it is explained that we get back the correct correction $\frac{25}{10}$, by subtracting from $\frac{25}{8}$ the product of that with $\frac{(10-8)}{10}$. Thus

$$\frac{100}{10} = \frac{100}{8} - \frac{100}{8}\left(\frac{(10-8)}{8}\right)\left[1 - \left(\frac{(10-8)}{10}\right)\right], \tag{6.42}$$

and so on.

6.4 *Saṅkalita*: Summation of series

The *bhujā-khaṇḍa*-s, the bits into which the tangent or the eastern side of the square of side equal to radius r was divided, are all equal to $\frac{r}{n}$. The *bhujā*-s are given by its integral multiples $\frac{r}{n}, \frac{2r}{n}, \ldots \frac{nr}{n}$. In the series expression (*phala-paramparā*) derived earlier for the circumference, there occur *saṅkalita*-s or summations of even powers of the *bhujā*-s such as the *bhujā-varga-saṅkalita*, $(\frac{r}{n})^2 + \left(\frac{2r}{n}\right)^2 + ... + (\frac{nr}{n})^2$, *bhujā-varga-varga-saṅkalita*, $(\frac{r}{n})^4 + \left(\frac{2r}{n}\right)^4 + ... + (\frac{nr}{n})^4$, and so on. If we take out the powers of *bhujā-khaṇḍa* $\frac{r}{n}$, the summations involved are that of even powers of the natural numbers, namely *edādyekottara-varga-saṅkalita*, $1^2 + 2^2 + ... + n^2$, *edādyekottara-varga-varga-saṅkalita*, $1^4 + 2^4 + ... + n^4$, and so on. A method is now given to estimate for large n the *sama-ghāta-saṅkalita*, or the sum of a general series of integral powers of the natural numbers.

6.4.1 *Mūla-saṅkalita*: The sum of natural numbers

The first summation, the *bhujā-saṅkalita*, may be written in the order from the final *bhujā* to the first *bhujā* as

$$S_n^{(1)} = \left(\frac{nr}{n}\right) + \left(\frac{(n-1)r}{n}\right) + ... + \left(\frac{r}{n}\right).$$

Now, conceive of the *bhujā-khaṇḍa* $\frac{r}{n}$ as being infinitesimal (*aṇu*) and at the same time as of unit-measure (*rūpa*), so that the radius will be the measure

of n, the *pada*, or the number of terms. Then

$$S_n^{(1)} = n + (n-1) + ... + 1. \tag{6.43}$$

If each of the terms were of the measure of radius (n) then the sum would be nothing but n^2, the square of the radius. But only the first term is of the measure of radius, the next is deficient by one segment (*khaṇḍa*), the next by two segments and so on till the last term which is deficient by an amount equal to radius-minus-one segment.

$$\begin{aligned} S_n^{(1)} &= n + [n-1] + [n-2].... + [n-(n-2)] + [n-(n-1)] \\ &= n.n - [1 + 2 + ... + (n-1)]. \end{aligned} \tag{6.44}$$

When n is very large, the quantity to be subtracted from n^2 is practically (*prāyeṇa*) the same as $S_n^{(1)}$, thus leading to the estimate

$$S_n^{(1)} \approx n^2 - S_n^{(1)}, \tag{6.45}$$

or, equivalently

$$S_n^{(1)} \approx \frac{n^2}{2}. \tag{6.46}$$

It is stated that the result is more accurate, the smaller the segments, or larger the value of n. In fact, the well-known exact value of the sum of the first n natural numbers

$$S_n^{(1)} = \frac{n(n+1)}{2}, \tag{6.47}$$

is also quoted as another way of justifying the above estimate when n is very large. With the convention that the $\frac{r}{n}$ is of unit-measure, the above estimate (6.46) is stated in the form that the *bhujā-saṅkalita* is half the square of the radius.

6.4.2 *Varga-saṅkalita* : Summation of squares

With the same convention that $\frac{r}{n}$ is the measure of the unit, the *bhujā-varga-saṅkalita* will be

$$S_n^{(2)} = n^2 + (n-1)^2 + ... + 1^2. \tag{6.48}$$

In above expression, each *bhujā* is multiplied by itself. If instead, we consider that each *bhujā* is multiplied by the radius (n in our units) then that would give raise to the sum

$$n\,[n + (n-1) + ... + 1] = n\,S_n^{(1)}. \tag{6.49}$$

This sum is exceeds the *bhujā-varga-saṅkalita* by the amount

$$nS_n^{(1)} - S_n^{(2)} = 1.(n-1) + 2.(n-2) + 3.(n-3) + \ldots + (n-1).1.$$

This may be written as

$$\begin{aligned} nS_n^{(1)} - S_n^{(2)} = (n-1) + (n-2) + (n-3) \quad &+\ldots \quad +1 \\ +(n-2) + (n-3) \quad &+\ldots \quad +1 \\ +(n-3) \quad &+\ldots \quad +1 \\ &+\ldots \quad . \end{aligned} \tag{6.50}$$

Thus,

$$nS_n^{(1)} - S_n^{(2)} = S_{n-1}^{(1)} + S_{n-2}^{(1)} + S_{n-3}^{(1)} + \ldots. \tag{6.51}$$

The right hand side of (6.51) is called the *saṅkalita-saṅkalita* (or *saṅkalitai-kya*), the repeated sum of the sums $S_i^{(1)}$ (here taken in the order $i = n-1,\ n-2, \ldots 1$). For large n, we have already estimated in (6.46) that $S_n^{(1)} \approx \frac{n^2}{2}$. Thus, for large n

$$nS_n^{(1)} - S_n^{(2)} = \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-3)^2}{2} + \ldots. \tag{6.52}$$

Thus, the right hand side of (6.52) (the *saṅkalita-saṅkalita* or the excess of $nS_n^{(1)}$ over $S_n^{(2)}$) is essentially $\frac{S_n^{(2)}}{2}$ for large n, so that we obtain

$$nS_n^{(1)} - S_n^{(2)} \approx \frac{S_n^{(2)}}{2}. \tag{6.53}$$

Again, using the earlier estimate (6.46) for $S_n^{(1)}$, we obtain the result

$$S_n^{(2)} \approx \frac{n^3}{3}. \tag{6.54}$$

Thus *bhujā-varga-saṅkalita* is one-third the cube of the radius.

6.4.3 *Ghana-saṅkalita* and *Varga-varga-saṅkalita*: Summation of third and fourth powers

The *ghana-saṅkalita* is given by

$$S_n^{(3)} = n^3 + (n-1)^3 + \ldots + 1^3. \tag{6.55}$$

Again, when we consider by how much $n\,S_n^{(2)}$ is in excess over this, we get

$$\begin{aligned} nS_n^{(2)} - S_n^{(3)} &= 1.(n-1)^2 + 2.(n-2)^2 + 3.(n-3)^2 \ldots + (n-1).1^2 \\ &= S_{n-1}^{(2)} + S_{n-2}^{(2)} + S_{n-3}^{(2)} + \ldots \quad . \end{aligned} \tag{6.56}$$

Here again, the right hand side of (6.56) is *varga-saṅkalita-saṅkalita*, the repeated summation of the summation of squares. As the sum of squares has already been estimated in (6.54) for large n, we get

$$\begin{aligned} nS_n^{(2)} - S_n^{(3)} &\approx \frac{(n-1)^3}{3} + \frac{(n-2)^3}{3} + \frac{(n-3)^3}{3} + \ldots \\ &\approx \left(\frac{1}{3}\right) S_n^{(3)}. \end{aligned} \tag{6.57}$$

Thus we get the estimate

$$S_n^{(3)} \approx \frac{n^4}{4}. \tag{6.58}$$

Similar reasoning leads to the following estimate in the case of *varga-varga-saṅkalita*

$$S_n^{(4)} \approx \frac{n^5}{5}. \tag{6.59}$$

6.4.4 *Samaghāta-saṅkalita* : General principle of summation

Essentially the same procedure is to be followed in the case of a general *samaghāta-saṅkalita*, (summation of equal powers) given by

$$S_n^{(k)} = n^k + (n-1)^k + \ldots + 1^k. \tag{6.60}$$

We first compute the excess of $nS_n^{(k-1)}$ over $S_n^{(k)}$ to be a *saṅkalita-saṅkalita* or repeated sum of the lower order *saṅkalita*-s $S_r^{(k-1)}$

$$nS_n^{(k-1)} - S_n^{(k)} = S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + S_{n-3}^{(k-1)} + \ldots \quad . \tag{6.61}$$

If the lower order *saṅkalita* $S_n^{(k-1)}$ has already been estimated to be, say,

$$S_n^{(k-1)} \approx \frac{n^k}{k}, \tag{6.62}$$

then, the above relation (6.61) leads to

$$\begin{aligned} nS_n^{(k-1)} - S_n^{(k)} &\approx \frac{(n-1)^k}{k} + \frac{(n-2)^k}{k} + \frac{(n-3)^k}{k} + \dots \\ &\approx \left(\frac{1}{k}\right) S_n^{(k)}. \end{aligned} \tag{6.63}$$

Thus we obtain the estimate[4]

$$S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}. \tag{6.64}$$

6.4.5 Repeated summations

Now are discussed *saṅkalita-saṅkalita* or *saṅkalitaikya* (repeated summations). There the first summation (*ādya-saṅkalita*) $V_n^{(1)}$ is just the *mūla-saṅkalita* or the basic summation of natural numbers, which has already been estimated in (6.46)

$$\begin{aligned} V_n^{(1)} = S_n^{(1)} &= n + (n-1) + (n-2) + \dots + 1 \\ &\approx \frac{n^2}{2}. \end{aligned} \tag{6.65}$$

The second summation (*dvitīya-saṅkalita* or *saṅkalita-saṅkalita* or *saṅkalitaikya*) is given by

$$\begin{aligned} V_n^{(2)} &= V_n^{(1)} + V_{n-1}^{(1)} + V_{n-2}^{(1)} + \dots \\ &= S_n^{(1)} + S_{n-1}^{(1)} + S_{n-2}^{(1)} + \dots \ . \end{aligned} \tag{6.66}$$

As was done earlier, this second summation can be estimated using the estimate for $S_n^{(1)}$

$$V_n^{(2)} \approx \frac{n^2}{2} + \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \dots \ . \tag{6.67}$$

Therefore

$$V_n^{(2)} \approx \left(\frac{1}{2}\right) S_n^{(2)}. \tag{6.68}$$

[4]These are estimates for large n. Exact expressions for $S_n^{(1)}$, $S_n^{(2)}$ and $S_n^{(3)}$, have been known in Indian mathematics for a long time and may be found, for instance, in *Āryabhaṭīya, Gaṇita*, 21-22.

Using the earlier estimate (6.54) for $S_n^{(2)}$, we get an estimate for the *dvitīya-saṅkalita*

$$V_n^{(2)} \approx \frac{n^3}{6}. \tag{6.69}$$

Now the next repeated summation can be found in the same way

$$\begin{aligned} V_n^{(3)} &= V_n^{(2)} + V_{n-1}^{(2)} + V_{n-2}^{(2)} + \dots \\ &\approx \frac{n^3}{6} + \frac{(n-1)^3}{6} + \frac{(n-2)^3}{6} + \dots \\ &\approx \left(\frac{1}{6}\right) S_n^{(3)} \\ &\approx \frac{n^4}{24}. \end{aligned} \tag{6.70}$$

It is noted that proceeding this way we can estimate repeated summation $V_n^{(k)}$ of order k, for large n, to be[5]

$$\begin{aligned} V_n^{(k)} &= V_n^{(k-1)} + V_{n-1}^{(k-1)} + V_{n-2}^{(k-1)} + \dots \\ &\approx \frac{n^{k+1}}{1.2.3.\dots.(k+1)}. \end{aligned} \tag{6.71}$$

6.5 Conclusion: Calculation of the circumference

Using the estimates for the various *samaghāta-saṅkalita*, summation of powers of the *bhujā*-s, we can derive the series for the circumference (6.37) mentioned earlier. We had in (6.36)

$$\begin{aligned} \frac{C}{8} &= \left(\frac{r}{n}\right)[1+1+\dots+1] \\ &- \left(\frac{r}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{r}{n}\right)^2 + \left(\frac{2r}{n}\right)^2 + \dots + \left(\frac{nr}{n}\right)^2\right] \\ &+ \left(\frac{r}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{r}{n}\right)^4 + \left(\frac{2r}{n}\right)^4 + \dots + \left(\frac{nr}{n}\right)^4\right] \\ &- \dots\dots\ . \end{aligned} \tag{6.72}$$

[5]These are again estimates for large n. Exact expressions for the first two summations, $V_n^{(1)}$ and $V_n^{(2)}$, are given in *Āryabhaṭīya, Gaṇita*, 21. Exact expression for the k-th order repeated summation $V_n^{(k)}$ has been given (under the name *vāra-saṅkalita*), by Nārāyaṇa Paṇḍita (c.1350) in his *Gaṇitakaumudī*, 3.19, and they are also given later in Chapter 7 of *Yuktibhāṣā*.

For large n, using the estimate (6.64), $S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}$, we arrive at the result of Mādhava cited earlier

$$\frac{C}{8} = r - \frac{r}{3} + \frac{r}{5} - \dots \ , \tag{6.73}$$

$$C = 4d\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right). \tag{6.74}$$

It is remarked that the magnitude of the succeeding terms becomes smaller and smaller, and when the terms become very small, then they can be discarded and the calculation be terminated; then the result will be mostly accurate.

6.6 *Cāpīkaraṇa*: Conversion of the Rsine to arc

If $s = r\theta$ is the arc (*dhanus*) of a circle of radius r subtending an angle θ at the centre, $r\sin\theta$ and $r\cos\theta$ are its *jyā* (Rsine) and *koṭi* (Rcosine) respectively, and if the *jyā* is less than the *koṭi*, then the arc is obtained by the rule *iṣṭajyātrijyayorghātāt...* (cited also in *Yuktidīpikā*, II. 206):[6]

$$s = r\left[\frac{jy\bar{a}(s)}{ko\dot{t}i(s)}\right] - \left(\frac{r}{3}\right)\left[\frac{jy\bar{a}(s)}{ko\dot{t}i(s)}\right]^3 + \left(\frac{r}{5}\right)\left[\frac{jy\bar{a}(s)}{ko\dot{t}i(s)}\right]^5 - \dots . \tag{6.75}$$

This result is obtained in the same way as the series for one-eighth the circumference derived earlier. Consider the quadrant of the square $OEAS$ circumscribing the circle as in Figure 6.6.

Choose the (*bhujā*) *jyā* PQ of the given arc EP in such a way that its *koṭi* OQ and the *śara* (Rversine) QE are along the west-east line. Extend the radius OP to meet the square at B. The portion $EB = t$, of the eastern side of the square, is now divided into n equal parts and, following the same argument as before in Section 6.3.1, we arrive at the expression for the arc $EP = s$:

$$s \approx \left(\frac{t}{n}\right)\left[\left(\frac{r^2}{k_o^2}\right) + \left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \dots + \left(\frac{r^2}{k_n^2}\right)\right], \tag{6.76}$$

[6] This corresponds to the result

$$s = r\theta = r\left(\frac{r\sin\theta}{r\cos\theta}\right) - \left(\frac{r}{3}\right)\left(\frac{r\sin\theta}{r\cos\theta}\right)^3 + \left(\frac{r}{5}\right)\left(\frac{r\sin\theta}{r\cos\theta}\right)^5 - \dots \ .$$

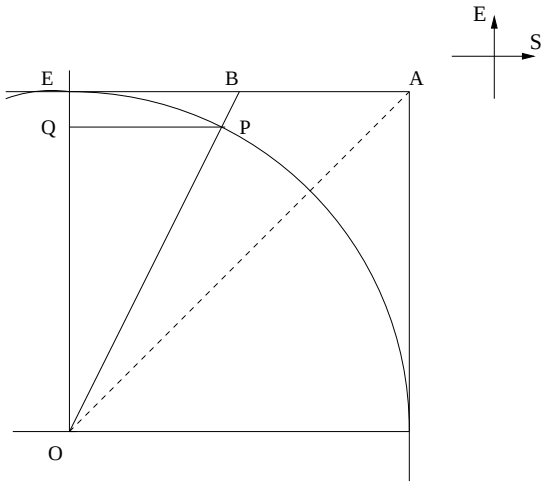

Figure 6.6: Conversion of the Rsine to arc.

where the hypotenuses k_i^2 are given by

$$k_i^2 = r^2 + \left(\frac{it}{n}\right)^2 . \tag{6.77}$$

Calculating the *śodhya-phala*-s as before, we get[7]

$$\begin{aligned} s &= \left(\frac{t}{n}\right)[1+1+\ldots+1] \\ &- \left(\frac{t}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{t}{n}\right)^2+\left(\frac{2t}{n}\right)^2+\ldots+\left(\frac{nt}{n}\right)^2\right] \\ &+ \left(\frac{t}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{t}{n}\right)^4+\left(\frac{2t}{n}\right)^4+\ldots+\left(\frac{nt}{n}\right)^4\right] \\ &- \ldots \ . \end{aligned} \tag{6.78}$$

Estimating the *saṅkalitas* $S_n^{(k)}$ as before, we get

$$s = r\left(\frac{t}{r}\right) - \left(\frac{r}{3}\right)\left(\frac{t}{r}\right)^3 + \left(\frac{r}{5}\right)\left(\frac{t}{r}\right)^5 - \ldots \ . \tag{6.79}$$

[7]Since $t < r$, we continue to have $r^2 < k_i^2 < 2r^2$, which ensures that the infinite series expansion converges.

If we note that

$$\frac{t}{r} = \frac{EB}{OE} = \frac{PQ}{OQ} = \frac{jy\bar{a}(s)}{ko\dot{t}i(s)}, \tag{6.80}$$

then we obtain the series (6.75) given above.

Appendix B

Yuktibhāṣā derivation of power series for sine and cosine functions (2008)

Excerpts from

Gaṇita-Yukti-Bhāṣā (Rationales in Mathematical Astronomy) of Jyeṣṭhadeva Volume I : Mathematics, Malayalam Text Critically Edited with English Translation by K. V. Sarma, with explanatory notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, Hindstan Book Agency/Springer, 2008.

The first part of the extracts (pp. 94–105 in the text form which the excerpts are taken) is the English translation of the of the relevant text in the original Malayalam *Yktibhāṣā* and the second part (pp. 221–234 in the text form which the excerpts are taken) is the explanatory notes on the English translation using modern mathematical notations.

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7.5 Computations of *Jyā* and *Śara* by *Saṅkalita*-s

7.5.1 First and second order differences of Rsines

Thus, when the Rsines (and Rcosines) which have their tips at the midpoint of the arc-bits are considered as *pramāṇa-phala*-s there will result the Rsine and Rcosine associated with the junction of the arc-bits which are the *bhujā* and *koṭi* of the hypotenuse, which is the full-chord with its tip at the junction of the arcs. There, those that are obtained from the midpoint of the first arc are those relating to the tip of the first arc-bit. There too, it is definite that if the *pramāṇa-phala* is east-west, the *icchā-phala* is north-south and if (the *pramāṇa-phala*) is north-south, the *icchā-phala* would be east-west. There is something more to be noted. If for the *pramāṇa-phala*-s the tip is at the centre of the arc-bit, for the *icchā-phala*-s, the tip is at the tip of the arc-bits. And, if for the *pramāṇa-phala*-s the tip is at the tip of the arc-bits, for the *icchā-phala*-s the tip is at the midpoint of the arc-bits. This is also definite. Here, in the derivation of all differences, the *icchā* and *pramāṇa* are the full-chord and *trijyā*. They are the same in all the cases. It is because there is difference in the *pramāṇa-phala*-s, that there is difference in *icchā-phala*-s.

Now, multiply the *icchā-rāśi* by the difference of the Rcosines which have their tips at the midpoint of the arc-bits. The results would be the differences between the Rsine differences which have their tips at the tips of the arc-bits. If, however, (the *icchā-rāśi*) is multiplied by the Rsine-difference at the midpoint of the arc-bits, one gets the difference of the Rcosine differences at the tips of the arc-bits. When the Rsine of the junction of the first arc-bit is multiplied by the full-chord of the arc-bit and divided by *trijyā*, the Rcosine difference which has its tip at the midpoint of the first arc-bit is got. Then, multiply that difference by the full-chord and divide by the radius. The result will be the amount by which the Rsine difference which has its tip at the tip of the second arc-bit is smaller than the Rsine difference which has its tip at the tip of the first arc-bit. Hence, when the first Rsine is multiplied by the square of the full-chord of the arc-bit and divided by the square of the *trijyā*, the result would be the difference between the first Rsine difference and the second Rsine difference.

Now, the tabular Rsines at the junction of the arc-bits are called also *piṇḍa-jyā*-s. Now, multiply the several *piṇḍa-jyā*-s by the square of the full-chord (of the arc-bit) and divide by the square of *trijyā*. The results are the differences of the Rsine differences (*khaṇḍa-jyāntara*). When we consider the *piṇḍa-jyā* at the juncture of any two arc-bits, the result would be the difference between the Rsine differences associated with the two arc-bits. Here, in the place of the multiplier, the *phala* can be taken and in the place of the divisor, the multiplicand can be taken. Then, multiply *piṇḍa-jyā* by the respective second order difference (*khaṇḍāntara*) and divide by the corresponding *piṇḍa-jyā*. Then also the second order difference will result. Thus (has been set out) the method of deriving the first and second differences.

Next is explained the derivation of the sums and repeated summations of the second differences and using them for the computation of desired Rsine and Rversines. It has been stated earlier that for the first arc-bit the Rsine difference (*khaṇḍa-jyā*) is the same as *piṇḍa-jyā*. Multiply this by the square of the full-chord (of the arc-bit) and divide by the square of *trijyā*. The result will be the difference between the first and second Rsine differences. When

this difference is subtracted from the first sine difference, the second Rsine difference is got. Then, when that is added to the first Rsine difference, the result will be second *piṇḍa-jyā*. If this is multiplied by the square of the full-chord and divided by the square of *trijyā*, it will be the difference between the second and third Rsine differences. When this is subtracted from the second Rsine difference, the third Rsine difference will result. If this is added to the second *piṇḍa-jyā*, we get the third *piṇḍa-jyā*. Thus, when the respective *piṇḍa-jyā*-s are multiplied and divided (by the above multiplier and divisor) the second difference (*khaṇḍāntara*) next to it is obtained. Then, commencing from the beginning, if the second differences up to the desired arc-bit are added together and subtracted from the first Rsine difference, the remainder will be the desired Rsine difference.

Now, if all the second differences are to be added together at one stretch, add together all the tabular Rsines up to the desired sine, multiply by the square of the full-chord and divide by the square of *trijyā*. The result will be the sum of the second differences. When this is subtracted from the first Rsine difference, the remainder will be the desired Rsine difference.

7.5.2 Desired Rsines and Rversines from *Jyā-saṅkalita*

Here, if the sum of the Rversine-differences (*śara-khaṇḍa*) centred at the midpoint of the arc-bits is multiplied by the full-chord and divided by *trijyā*, then also the sum of the second differences will be obtained. To obtain the sum of the Rversine differences about the midpoints (of the arc-bits), the sum of the *piṇḍa-jyā*-s at the tip of the arc-bits should be multiplied by the full-chord of the arc-bit and divided by *trijyā*. Thus can be obtained the summation of the Rversine differences at the middle of the arc-bits.

Now to the derivation of the sum of Rsine differences. In a quadrant, there are 24 Rsines. Suppose the 8th sine is required. Multiply the first *piṇḍa-jyā* by 7; multiply the second *piṇḍa-jyā* by 6, the third by 5, the fourth by 4, the fifth by 3, the sixth by 2 and the seventh by 1. Add all these together. The

result is called the 'repeated summation of Rsines' (*jyā-saṅkalita*). Repeated summations have been dealt with in detail earlier, in the context of computing the circumference and diameter of a circle. Here, multiply the repeated summation of the Rsines by the square of the fullchord of the arc-bit and divide by square of the radius. Subtract the quotient from the first Rsine difference multiplied by 8. The result will be the 8th Rsine.

In this manner we get the result that when the repeated summation of the Rsines (*jyā-saṅkalita*) up to the tip of a particular arc-bit is done, the result will be the difference between the next higher Rsine and the corresponding arc (*jyā-cāpāntara*). Here, the arc-bit has to be conceived as being as minute as possible. Then, the first Rsine difference will be the same as the first arc-bit. Hence, if it is multiplied by the desired number, the result will certainly be the desired arc. Therefore, when the result of the *jyā-saṅkalita* is subtracted from the desired arc, the result will be the desired Rsine.

The statement that in a quadrant there are 24 Rsines is only a convention, since some specified number of divisions has to be made (for definitive calculations). In this case, therefore, by multiplying the second differences, starting from the last second difference of the desired arc up to the first and second differences, respectively, by the numbers 1, 2, 3 etc., the result will be the repeated summation of the second differences (*khaṇḍāntara-saṅkalita*). And it has been obtained since this is the difference between the required arc and the required Rsine.

It is clear that the all the Rsines up to that of the desired arc are the means for deriving the difference between the corresponding Rsine and the arc. Since all these Rsines are not known, consider the arcs themselves as the Rsines and perform the *cāpa-saṅkalita*. Here, the desired arc itself is the last Rsine. One arc-bit less than this arc is the next previous Rsine. Similarly, consider that the arcs lower by successive arc-bits form the successive Rsines, in that order. Here again, consider that there are as many arc-bits as there are minutes in the desired arc. Then perform the repeated summation of the sum of numbers one, two, etc. (*ekādyekottara-saṅkalita*). That will result in

the sum of the Rsines. When that is multiplied by 1 minute which is the full-chord, there will be no difference in the number. Now, divide this by *trijyā*. The result will be the sum of the Rversine-differences (*śara-khaṇḍa-yoga*) at the middle of the arc-bits. Since the bit is small, the sum of the Rversine differences centred at the tip of the arc-bits is practically equal to this and, as such, can be taken as this itself. Now, the smaller the arc-bit, the more exact (*sūkṣma*) the Rsine would be. Here, taking the arc-bit to be one minute divided by *parārdha*, multiply it by the denominator which is *parārdha*, perform the *saṅkalita* and divide by the denominator. The result will be practically equal to the *saṅkalita* performed without having been multiplied by the denominator.

7.5.3 First, second and third repeated summations

Here, however, repeated summation (*saṅkalita*) is made up to the number, which is the number of infinitesimally small parts contained in the required arc. And that number is the number of terms in the series. It is easy to conceive this if it is visualized in a summation figure (*saṅkalita-kṣetra*), in which there are as many rows as there are terms with number 1 in the first row. It would be easier if that is conceived of as a square bit. In the second row there will be two bits, in the third line three and so on, increasing by one bit per line; the last line will have as many bits as there are terms (*pada-saṅkhyā*). Here, the number is the desired arc. Multiply the degrees herein by a denominator equal to an atom (*aṇu*) and the resulting number of atoms will be the number of terms. Now, the number of terms multiplied by itself increased by 1 and divided by 1 multiplied by 2, (i.e., $n(n+1)/(1\times 2)$) the result would be the summation (*saṅkalita*). This is the first summation (*saṅkalita*).

Now, the second (repeated) summation. This is the sum of the above (first) *saṅkalita*, and the *saṅkalita* with one row less, with two rows less, with three rows less etc. Thus the second *saṅkalita* is the sum of (first) *saṅkalita*-s with

one term less at each stage. The third *saṅkalita* would be got by taking the (second) *saṅkalita* as that of the last term and adding up the results by reducing the terms by one.

The method for deriving the above *saṅkalita*-s: The *pada*, (*pada* + 1), (*pada* +2), these three are to be multiplied together. This is to be divided by 6, which is the product of 1, 2 and 3. The result got would be the second *saṅkalita*. In this manner, take the numbers ascending by one each, which have been multiplied together. (Their result) should be divided by the product of the same number of the (natural numbers) 1, 2, 3 etc. The result will be the next following *saṅkalita*. Here, if the arc-bit is conceived as infinitesimal, the sine will be accurate. When units which are practically zero (*śūnya-prāya*) are added one each to the number of terms (*pada*) the resulting numbers practically do not change at all. Hence, the squares and cubes of the required arc should be divided by the successive products of one (two) etc. (*ekādi-ghāta*, i.e. $1,\ 1\times 2,\ 1\times 2\times 3,\ 1\times 2\times 3\times 4$ etc.). Then the results will be accurate.

Hence half the square of the arc is the first *saṅkalita*. Then the second *saṅkalita* is one-sixth of the cube of the desired arc. Since the first *saṅkalita* is half the square, it can be taken as the last term of the second *saṅkalita*. Then half the square of the number of terms reduced by one would be the penultimate term. When addition is made successively of all these terms the result will be the *saṅkalita* of half the square of the desired arc. And, that is half of the summation of the squares (*varga-saṅkalita*). It has been stated earlier that one-third the cube of the number of terms is the summation of squares. Hence, half the above is one-sixth of the cube. From this it results that the third *saṅkalita* is one-sixth of the summation of cubes (*ghana-saṅkalita*). But this will be one-twenty-fourth of the square of squares. Thus, has been derived, what was stated in detail earlier for a *saṅkalita*, that equal numbers (the number of terms) are multiplied amongst themselves a certain number of times and then the denominator for it is the product of the integers 1, 2, 3 etc. taken up to the same number.

7.5.4 Successive corrections to *Jyā* and *Śara*

Now, here the first *saṅkalita* is the sum of the Rsines beginning with the first Rsine up to the required Rsine. Taking the full-chord to be 1, there will be no change in the above sum when multiplied by it (i.e., by 1). Divide it by the radius. The result will be the Rversine (*śara*) which is the sum of the Rversine differences (*śarakhaṇḍa-yoga*). Multiply this Rversine by the sum of the arc-bits, divide by the radius and by three. The result will be the difference between the arc and the Rsine. Thus, the result of dividing one-sixth the cube of the desired arc by the square of the radius, will also be the difference between the arc and the sine.

Now, when the desired Rversine is divided by the radius, the result is the difference between the first and last differences (*ādyāntya-khaṇḍāntara*). Now, find by the sum of Rsines, the difference between the first and penultimate differences. From this can be got, the summation of the second order differences (*khaṇḍāntara-saṅkalita*) being the sum of all the second order differences starting from the first sine difference and summing all differences of differences, from the second *saṅkalita* which is one-sixth of the cube of the desired arc. This is the same as the difference between the Rsine and the arc.

But this is only approximate (*prāyika*), since in place of the *saṅkalita* of the Rsines the *saṅkalita* of the arcs had been taken for calculation. Hence, in this way the sum of the differences at each step between the Rsines and the arcs, would be the excess in the *saṅkalita* of the arcs over the *saṅkalita* of the Rsines. When from the sum of the arcs the Rversine is obtained, and then the differences between the Rsines and the arcs accounted for divided by the radius, and the result subtracted from the Rversine, the Rversine would become more accurate (*sūkṣma*).

Here, Rsine-arc difference at the last stage had been from the second *saṅkalita*. Therefore, the successive Rsine-arc differences (*jyā-cāpāntara*) should all be found starting from the second *saṅkalita*, with the number of terms (*pada*)

going down by one. Thus the sum of Rsine-arc differences will be the third *saṅkalita*. Then from the fourth *saṅkalita*, the *saṅkalita* of the Rsine-arc differences is to be found by the method indicated earlier. Then, this *saṅkalita* will be the excess that had resulted previously by using the *saṅkalita* of arcs in place of what was desired, namely, the *saṅkalita* of Rsines. When, this *saṅkalita* of the arcs is divided by the square of the radius and the result subtracted from the earlier calculated difference between the Rsine and the arc, the resulting difference between the desired Rsine and the desired arc will be more accurate.

Now, when the difference between the Rsine and the arc which was obtained first, is multiplied by the desired arc and divided by the radius, the result will be the correction for the Rversine (*śara-saṃskāra*). When this correction to the Rversine is again multiplied by the desired arc and divided by the radius, the result will be the correction for the Rsine-arc difference (*jyā-cāpāntara-saṃskāra*). Now, obtain this correction to the difference between the Rsine and the arc, from the sum of the Rsine-arc differences (*jyā-cāpāntara-yoga*). This is (how it is done): Multiply this correction by the arc, divide by the radius and subtract the result from the earlier correction arrived at for the Rversine. Then the Rversine-correction (*śara-saṃskāra*) will be more accurate. Multiply this correction to the Rversine by the required arc, and divide by the radius; the result is the correction to the Rsine-arc difference.

It is to be noted that the aim is to arrive at a correction based on the *saṅkalita*-s. Hence, here, in all instances, when the result is multiplied by the arc, what has been divided by the numbers, 1, 2, 3 etc., up to some number, has to be divided by the radius. Here is the method to obtain the difference between the result based on one *saṅkalita* and that derived on the basis of the succeeding *saṅkalita*. Here, note the number of times the arc has been multiplied by itself. For this the divisor is the product of the radius multiplied by itself so many times, and the product of that many numbers starting from one, (i.e., 1, 2, 3 etc). When it is required to get the result succeeding another result, multiply the result once by the required arc and divide once by the radius. Then too the derived result would be the same.

7.5.5 Accurate computation of Rsines and Rversines, without using tables

Thus, since the results were (actually) obtained from the sum of the arcs, while it was required to obtain them from the sum of Rsines, the correction terms are all in excess of the actual results. Therefore, the successive correction terms (*saṃskāra-phala*) at every higher stage should be subtracted from the earlier correction-results. This being the case, the following is the procedure (*kriyā-krama*) to be adopted. The required arc is the first result. When this is squared, halved and divided by the radius, the second result is got. Keep this second result separately. Now multiply the second result also by the arc and divide by 3 and also by radius. Place the result got below the first result. Then multiply this also by the arc and divide by four and the radius. Keep the result below the second result. In this manner, derive successive results by multiplying the previous result by the arc, and dividing by corresponding successive numbers 1, 2, 3 etc. and by radius. Now, place below the first result the odd results, viz., the third, the fifth etc., and place below the second result the even results, viz., the fourth, sixth etc. Then subtract successively the bottom result from the one above it, the remainder from the one still above it. Ultimately, in the first column (*paṅkti*), the resultant first result will be left and in the second column the resultant second result will be left. These will be the required Rsine and Rversine.

The odd results might also be separately calculated and the desired Rsine can be derived, and the even results calculated separately and the desired Rversine calculated separately. This latter is another procedure (for arriving at the results). Now, the procedure to derive the required Rsine: Multiply the required arc by the square of the required arc and divide by the square of radius. Then, divide also by 6, which is the product 2 and 3. The result is Rsine-arc difference. The further results are derived similarly, the multiplier is the square of the arc, and the divisor is the square of the radius. Still another divisor is the product of the (corresponding) even number and the next odd number. This would be the square of the corresponding even

number plus its root, because the succeeding odd number is one more than the even number. Thus, this is another method to derive the required Rsine.

Then applying the same procedure on the second result, by multiplying and dividing the successive results, the required the Rversine would be obtained. The only difference is that here the divisor is the square of the odd number plus that number.

Then, in this manner, Rsines and Rversines are calculated for the entire quadrant and tabulated. Using them (i.e., these tables) derive (the Rsine and Rversine) for any desired arc (in-between them) by the rule of three. (Here is the method therefor): Set out the odd results and even results separately in two columns. In both cases, multiply the last result by the square of the required arc and divide by the square of the radius. Subtract the results from the penultimate in the two columns. Continue this process of multiplication and division and subtract from the results just previous to each. Then subtract from the required arc the last item of the sequence beginning with *vidvān* etc. The remainder would be the required Rsine. Through the same procedure used with the sequence beginning with *stena*, the last result would be the required Rversine. This is the method of deriving the required Rsine and Rversine without using the regular, tabular values (of Rsines and Rversines).

7.6 Obtaining accurate circumference from approximate value

Now, in pursuance of the principles enunciated above, here is a method to evaluate an accurate value of the circumference starting from an approximate value for a given diameter. First construct a circle for any assumed diameter and compute its circumference roughly by the application of the rule of three using approximations such as for 7 (of radius) 22 (of circumference) and so on. Then, taking the required diameter as radius, one-fourth of the presumed

rough circumference will be one-eighth (of the circumference for the new radius). Calculate the Rsine for this in the manner explained above. Then (that Rsine) will be more or less equal to the Rsine of an arc, which is one-eighth of the accurate circumference in a circle of which the desired diameter is (taken to be) the radius. Now, when the Rsine is calculated according to the procedure enunciated in the verses beginning with *nihatya cāpavargeṇa* ('having multiplied with the square of the arc'), instead of taking the square of the radius as the first divisor, as had been instructed previously, take the square of the desired diameter as the divisor. The reason for this is that here it is the square of the radius of a circle whose diameter is twice the (first) diameter. This is the only speciality in the matter of the derivation of the Rsine relating to the desired diameter.

Then, subtract the square of this Rsine from the square of the radius. The remainder will be the square of the Rcosine. The square of the Rsine of one-eighth of the accurate circumference is half of the square of the radius. The square of Rcosine is also the same. Since this eighth is half of the circumference, the *bhujā* and *koṭi* will be equal.

It is possible to derive the Rsine of the difference of one-eighth of the rough circumference and that of the accurate circumference, applying the rule of *jīve paraspara* which will be stated later. Now, multiply the squares of the rough Rsine and rough Rcosine by the squares of the accurate Rsine and Rcosine and divide by the square of the radius. The results got will be half the squares of the approximate Rsine and Rcosine, since the multipliers are half and double. Find their respective square roots and find the difference thereof. The difference is the Rsine of the difference between one-eighth of the rough circumference and that of the accurate circumference. Find the arc of (this Rsine). For that, find the cube of this and divide by six times the square of the radius. Add the result to this Rsine of the difference (of one-eighth circumferences found above). This will be the arc of the difference. Add this to the eighth of the approximate circumference, if the square of the approximate Rsine is smaller than half the square of the diameter, and subtract if greater. The result will be one-eighth of the accurate cir-

cumference of a circle of double the diameter. This will be one-fourth of the circumference for the desired diameter. When this is multiplied by four, the accurate circumference is got. This (as has been explained) is the procedure for rendering the approximate circumference accurate.

7.5 Computations of *Jyā* and *Śara* by *Saṅkalita*-s

7.5.1 First and second order differences of Rsines

So far we had considered the tabular Rsines (*paṭhita-jyā*) obtained by dividing the quadrant of a circle into 24 equal arc-bits. We shall now consider the

[10]If we do not approximate the Rsine and Rcosine, M_2B_2 and OB_2, of the arc EM_2 by the nearest tabulated Rsine and Rcosine, but derive them in terms of the Rsine and Rcosine at the midpoint M_2 of the arc C_iM_2 and then make a similar approximation, we will get the third order approximation,

$$R\sin(EC_i+\Delta) \approx R\sin EC_i+\left(\frac{\Delta}{r}\right)R\cos EC_i-\frac{1}{2}\left(\frac{\Delta}{r}\right)^2 R\sin EC_i+\frac{1}{2.4}\left(\frac{\Delta}{r}\right)^3 R\cos EC_i,$$

$$R\cos(EC_i+\Delta) \approx R\cos EC_i-\left(\frac{\Delta}{r}\right)R\sin EC_i-\frac{1}{2}\left(\frac{\Delta}{r}\right)^2 R\cos EC_i+\frac{1}{2.4}\left(\frac{\Delta}{r}\right)^3 R\sin EC_i.$$

In this context, it should be noted that while the first order approximation, given in the previous footnote, and the second order approximation, given in equations (7.37) and (7.38), appear like the first and second order terms in a Taylor's series expansion, there is no reason that the further terms should correspond to the terms in a Taylor's series expansion.

more general case of a given arc (*iṣṭa-cāpa*) of arc-length s, which is divided into n equal arc-bits. At this stage, the general results derived are equally applicable to the particular case of the quadrant divided into 24 equal parts. Later, we shall obtain approximate results for the Rsine and Rversine of an arbitrary arc, which become more and more accurate as n becomes larger and larger.

We first define the constituent Rsines (*piṇḍa-jyā*, or more accurately *piṇḍa-bhujā-jyā*) and the corresponding Rcosines and Rversines. If $s = r\theta$, then the j-th *piṇḍa-jyā*, B_j is given by

$$B_j = R \sin\left(\frac{js}{n}\right) = r \sin\left(\frac{j\theta}{n}\right) = r \sin\left(\frac{js}{rn}\right). \tag{7.39}$$

The corresponding *koṭi-jyā* K_j, and the *śara* S_j, are given by

$$K_j = R \cos\left(\frac{js}{n}\right) = r \cos\left(\frac{j\theta}{n}\right) = r \cos\left(\frac{js}{rn}\right), \tag{7.40}$$

$$S_j = R \operatorname{vers}\left(\frac{js}{n}\right) = r\left[1 - \cos\left(\frac{j\theta}{n}\right)\right] = r\left[1 - \cos\left(\frac{js}{rn}\right)\right]. \tag{7.41}$$

In Figure 7.7, EC is the desired arc of length s, and C_jC_{j+1} is the $(j+1)$-th arc bit. Then the arc $EC_j = \frac{js}{n}$, and its *piṇḍa-jyā* $B_j = C_jP_j$, and the corresponding *koṭi-jyā* and *śara* are $K_j = C_jT_j$, $S_j = EP_j$. Similarly we have $B_{j+1} = C_{j+1}P_{j+1}$, $K_{j+1} = C_{j+1}T_{j+1}$, and $S_{j+1} = EP_{j+1}$.

Let M_{j+1} be the mid-point of the arc-bit C_jC_{j+1} and similarly M_j the mid-point of the previous (j-th) arc-bit. We shall denote the *piṇḍa-jyā* of the arc EM_{j+1} as $B_{j+\frac{1}{2}}$ and clearly

$$B_{j+\frac{1}{2}} = M_{j+1}Q_{j+1}\ .$$

The corresponding *koṭi-jyā* and *śara* are

$$K_{j+\frac{1}{2}} = M_{j+1}U_{j+1} \quad \text{and} \quad S_{j+\frac{1}{2}} = EQ_{j+1}\ .$$

Similarly,

$$B_{j-\frac{1}{2}} = M_jQ_j,\ K_{j-\frac{1}{2}} = M_jU_j \ \text{ and } \ S_{j-\frac{1}{2}} = EQ_j\ .$$

The full-chord of the equal arc-bits $\frac{s}{n}$ may be denoted α as before. Thus,

$$C_jC_{j+1} = M_jM_{j+1} = \alpha.$$

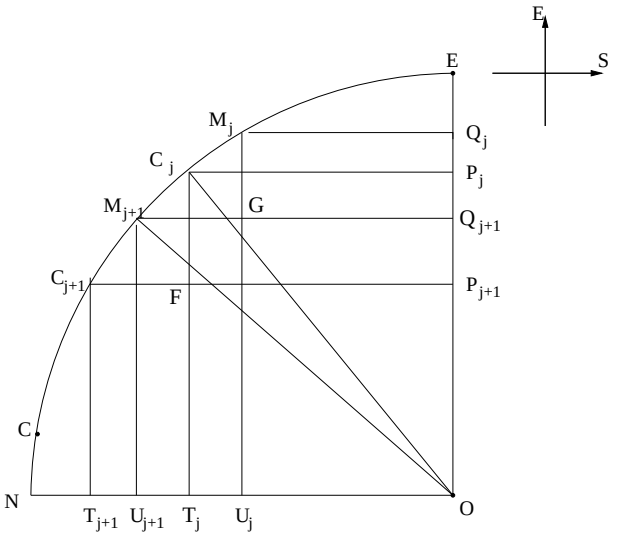

Figure 7.7: Computation of *Jyā* and *Śara* by *Saṅkalita*-s.

Let F be the intersection of C_jT_j and $C_{j+1}P_{j+1}$, and G of M_jU_j and $M_{j+1}Q_{j+1}$. As discussed before, the triangles $C_{j+1}FC_j$ and $OQ_{j+1}M_{j+1}$are similar, as their sides are mutually perpendicular. Thus we have

$$\frac{C_{j+1}C_j}{OM_{j+1}} = \frac{C_{j+1}F}{OQ_{j+1}} = \frac{FC_j}{Q_{j+1}M_{j+1}} \ . \tag{7.42}$$

Hence we obtain

$$B_{j+1} - B_j \ = \ \left(\frac{\alpha}{r}\right) K_{j+\frac{1}{2}}, \tag{7.43}$$

$$K_j - K_{j+1} \ = \ S_{j+1} - S_j = \left(\frac{\alpha}{r}\right) B_{j+\frac{1}{2}}. \tag{7.44}$$

Similarly, the triangles $M_{j+1}GM_j$ and OP_jC_j are similar and we get

$$\frac{M_{j+1}M_j}{OC_j} = \frac{M_{j+1}G}{OP_j} = \frac{GM_j}{P_jC_j}. \tag{7.45}$$

Thus we obtain

$$B_{j+\frac{1}{2}} \ - \ B_{j-\frac{1}{2}} = \left(\frac{\alpha}{r}\right) K_j, \tag{7.46}$$

$$K_{j-\frac{1}{2}} \ - \ K_{j+\frac{1}{2}} = S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}} = \left(\frac{\alpha}{r}\right) B_j \ . \tag{7.47}$$

We define the Rsine-differences (*khaṇḍa-jyā*) Δ_j by

$$\Delta_j = B_j - B_{j-1} \,, \tag{7.48}$$

with the convention that $\Delta_1 = B_1$. From (7.43), we have

$$\Delta_j = \left(\frac{\alpha}{r}\right) K_{j-\frac{1}{2}} \,. \tag{7.49}$$

From (7.47) and (7.49), we also get the second order Rsine-difference (the differences of the Rsine-differences called *khaṇḍa-jyāntara*).[11]

$$\begin{aligned} \Delta_j - \Delta_{j+1} &= (B_j - B_{j-1}) - (B_{j+1} - B_j) \\ &= \left(\frac{\alpha}{r}\right)\left(K_{j-\frac{1}{2}} - K_{j+\frac{1}{2}}\right) \\ &= \left(\frac{\alpha}{r}\right)\left(S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}}\right) \\ &= \left(\frac{\alpha}{r}\right)^2 B_j \,. \end{aligned} \tag{7.50}$$

Now, if the sum of the second-order Rsine-differences, are subtracted from the first Rsine-difference, then we get any desired Rsine-difference. That is

$$\Delta_1 - [(\Delta_1 - \Delta_2) + (\Delta_2 - \Delta_3) + \ldots + (\Delta_{j-1} - \Delta_j)] = \Delta_j \,. \tag{7.51}$$

From (7.50) and (7.51) we conclude that

$$\Delta_1 - \left(\frac{\alpha}{r}\right)^2 (B_1 + B_2 + \ldots + B_{j-1}) = \Delta_j \,. \tag{7.52}$$

7.5.2 Desired Rsines and Rversines from *Jyā-saṅkalita*

We can sum up the Rversine-differences (7.47), to obtain the *śara*, Rversine, at the midpoint of the last arc-bit as follows:

$$\begin{aligned} S_{n-\frac{1}{2}} - S_{\frac{1}{2}} &= \left(S_{n-\frac{1}{2}} - S_{n-\frac{3}{2}}\right) + \ldots\ldots \left(S_{\frac{3}{2}} - S_{\frac{1}{2}}\right) \\ &= \left(\frac{\alpha}{r}\right)(B_{n-1} + B_{n-2} + \ldots + B_1) \,. \end{aligned} \tag{7.53}$$

[11] This formula for the second order Rsine-difference, when applied to the tabulated Rsines obtained by dividing the quadrant of a circle into 24 parts, is nothing but the well-known relation given in the *Āryabhaṭīya* for the second order Rsine-differences. *Āryabhaṭīya* gives the approximate value $\left(\frac{1}{225'}\right)$ for the quantity $\left(\frac{\alpha}{r}\right)^2$ in (7.50); Nīlakaṇṭha (*Tantrasaṅgraha* 2.4) gives the more accurate value $233\frac{1}{2}'$ for the divisor, and Śaṅkara, in his commentary *Laghuvivṛtti*, refines it further to $233'32''$.

Using (7.50), the right hand side of (7.53) can also be expressed as a summation of the second order differences. From (7.52) and (7.53) it follows that the Rversine at the midpoint of the last arc-bit is also given by

$$\left(\frac{\alpha}{r}\right)\left(S_{n-\frac{1}{2}} - S_{\frac{1}{2}}\right) = (\Delta_1 - \Delta_n). \tag{7.54}$$

Now, since the first Rsine-difference $\Delta_1 = B_1$, any desired Rsine can be obtained by adding the Rsine-differences; these Rsine-differences have been obtained in (7.52). Before discussing the general result which is going to be obtained this way, a particular case, the eighth *piṇḍa-jyā*, is evaluated to elucidate the general result. From the definition (7.48) of the Rsine-differences, We have

$$B_8 = \Delta_8 + \Delta_7 + \ldots + \Delta_1 \,. \tag{7.55}$$

By using (7.52), equation (7.55) becomes

$$B_8 = 8\Delta_1 - \left(\frac{\alpha}{r}\right)^2 [(B_1 + B_2 + \ldots + B_7) + (B_1 + B_2 + \ldots + B_6) + \ldots + B_1]$$

$$= 8B_1 - \left(\frac{\alpha}{r}\right)^2 (B_7 + 2B_6 + \ldots + 7B_1). \tag{7.56}$$

In the same way, by making use of (7.52), the last *piṇḍa-jyā* can be expressed as follows:

$$B_n = \Delta_n + \Delta_{n-1} + \ldots + \Delta_1$$

$$= n\Delta_1 - \left(\frac{\alpha}{r}\right)^2 [(B_1 + B_2 \ldots + B_{n-1}) + (B_1 + B_2 \ldots + B_{n-2}) + \ldots + B_1]$$

$$= nB_1 - \left(\frac{\alpha}{r}\right)^2 [B_{n-1} + 2B_{n-2} + \ldots + (n-1)B_1] \,. \tag{7.57}$$

Equations (7.53) and (7.57) express the Rsines and Rversines around the tip of the desired arc in terms of the summations and repeated summations (*saṅkalita*) of the *piṇḍa-jyā*-s, Rsines at the tips of the arc-bits. By employing (7.53), we can also re-express the expression (7.57) for the last (*piṇḍa-jyā*) in the form

$$B_n = nB_1 - \left(\frac{\alpha}{r}\right)\left[S_{n-\frac{1}{2}} + S_{n-\frac{3}{2}} + \ldots + S_{\frac{1}{2}} - nS_{\frac{1}{2}}\right] \tag{7.58}$$

The results (7.46)–(7.58), obtained so far, involve no approximations. It is now shown how better and better approximations to the Rsine and Rversine

can be obtained by taking n to be very large or, equivalently, the arc-bit $\frac{s}{n}$ to be very small. Then, we can approximate the full-chord and the Rsine of the arc-bit by the length of the arc-bit $\frac{s}{n}$ itself. Also, as a first approximation, we can approximate the *piṇḍa-jyā*-s B_j in the equations (7.53), (7.57) or (7.58) by the corresponding arcs themselves. That is

$$B_j \approx \frac{js}{n} \,. \tag{7.59}$$

The result for the Rsine obtained this way is again used to obtain a better approximation for the *piṇḍa-jyā*-s B_j which is again substituted back into the equations (7.53) and (7.57) or (7.58) and thus by a process of iteration successive better approximations are obtained for the Rsine and Rversine. Now, once we take $B_j \approx \frac{js}{n}$, we will be led to estimate the sums and repeated sums of natural numbers (*ekādyekottara-saṅkalita*), when the number of terms is very large. This topic has been dealt with earlier in Chapter 6 and is briefly recalled here.

7.5.3 First, second, etc. repeated summations (*Ādya-dvitīyādi-saṅkalita*)

The first summation (*ādya-saṅkalita*) is that of the natural numbers

$$V_n^{(1)} = n + (n-1) + \ldots + 1. \tag{7.60}$$

This can be easily obtained from the series figure (*saṅkalita-kṣetra*, see Figure 7.8) to be

$$V_n^{(1)} = \frac{n(n+1)}{2} \,. \tag{7.61}$$

The second summation (*dvitīya-saṅkalita*) is given by

$$\begin{aligned} V_n^{(2)} &= V_n^{(1)} + V_{n-1}^{(1)} + \ldots + V_1^{(1)} \\ &= \frac{n(n+1)}{2} + \frac{(n-1)n}{2} + \ldots + \frac{(1.2)}{2} \,. \end{aligned} \tag{7.62}$$

This summation is stated to be

$$V_n^{(2)} = \frac{n(n+1)(n+2)}{1.2.3} \,. \tag{7.63}$$

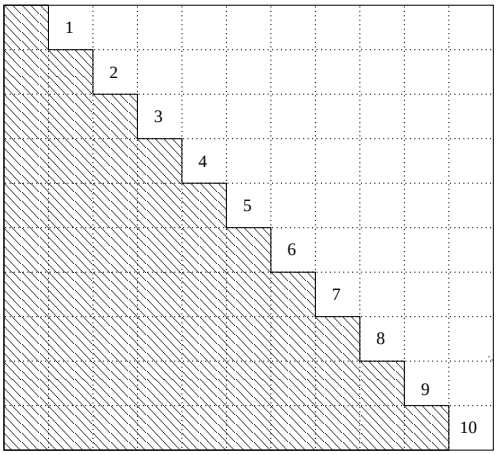

Figure 7.8: *Saṅkalita-kṣetra* for first summation.

Similarly the general k-th (repeated) summation is stated to be [12]

$$\begin{aligned} V_n^{(k)} &= V_n^{(k-1)} + V_{n-1}^{(k-1)} + \ldots + V_1^{(k-1)} \\ &= \frac{n(n+1)(n+2)\ldots(n+k)}{1.2.3.\ldots(k+1)} \,. \end{aligned} \tag{7.64}$$

What is needed for the evaluation of the Rsines and Rversines is the value of these summations for large n. They were derived earlier in Section 6.4.5 and are briefly recalled at this point. Clearly, for large n, the first summation is essentially half the square of the number terms n:

$$V_n^{(1)} \approx \frac{n^2}{1.2} \,. \tag{7.65}$$

Using the estimate (7.65) in (7.62) it follows that

$$V_n^{(2)} \approx \frac{n^2}{1.2} + \frac{(n-1)^2}{1.2} + \ldots \quad . \tag{7.66}$$

Thus, what we have is essentially half the summation of squares (*varga-saṅkalita*). When n is large, the sum of squares has been shown to be

[12] As we noted earlier, the result for the second (repeated) summation is given in *Āryabhaṭīya*, *Gaṇita* 21. The general result (7.64) for the k-th (repeated) summation is also given by Nārāyaṇa Paṇḍita (c.1350) in his *Gaṇitakaumudī*, 3.19.

one-third the cube of the number of terms and thus we get

$$V_n^{(2)} \approx \frac{n^3}{1.2.3} \,. \tag{7.67}$$

This way we can show that for large n, the k-th order (repeated) summation is given by

$$V_n^{(k)} \approx \frac{n^{k+1}}{1.2.3.\ldots.(k+1)} \,. \tag{7.68}$$

7.5.4 Successive corrections to *Jyā* and *Śara*

We first recall the equations (7.53) and (7.57) which give the Rsine and Rversine near the tip of the desired arc in terms of *saṅkalita*-s of the *piṇḍa-jyā*-s:

$$S_{n-\frac{1}{2}} - S_{\frac{1}{2}} = \left(\frac{\alpha}{r}\right)(B_{n-1} + B_{n-2} + \ldots + B_1), \tag{7.69}$$

$$\begin{aligned} B_n &= nB_1 - \left(\frac{\alpha}{r}\right)^2 [(B_1 + B_2 + \ldots + B_{n-1}) \\ &\qquad + (B_1 + B_2 + \ldots + B_{n-2}) + \ldots + B_1]. \end{aligned} \tag{7.70}$$

As we noted earlier, these relations are exact. But now we shall show how better and better approximations to the Rsine and Rversine of any desired arc can be obtained by taking n to be very large or, equivalently, taking the arc-bit $\frac{s}{n}$ to be very small. Then both the full-chord α, and the first Rsine B_1 (the Rsine of the arc-bit), can be approximated by the arc-bit $\frac{s}{n}$ itself, and the Rversine $S_{n-\frac{1}{2}}$ can be taken as S_n and the Rversine $S_{\frac{1}{2}}$ may be treated as negligible. Thus the above relations become

$$S = S_n \approx \left(\frac{s}{nr}\right)(B_{n-1} + B_{n-2} + \ldots + B_1), \tag{7.71}$$

$$\begin{aligned} B = B_n &\approx s - \left(\frac{s}{nr}\right)^2 [(B_1 + B_2 + \ldots + B_{n-1}) \\ &\qquad + (B_1 + B_2 \ldots + B_{n-2}) + \ldots + B_1], \end{aligned} \tag{7.72}$$

where B and S are the Rsine and Rversine of the desired arc of length s and the results will be more accurate, larger the value of n.

It is possible to obtain another approximation for the Rsine by considering equation (7.58). When n is very large, it can be assumed that the Rversines at the mid-points are practically the same as the Rversines at the junctions of the arc-bits, and that the term involving $nS_{\frac{1}{2}}$ can be neglected.[13] We thus obtain

$$B = B_n \approx s - \left(\frac{s}{nr}\right)(S_n + S_{n-1} + \ldots + S_1). \tag{7.73}$$

Either of the equations (7.72) or (7.73) can be used in tandem with equation (7.71) to obtain successive corrections to Rsines and Rversines.

Now, as a first approximation, we take each *piṇḍa-jyā* B_j in (7.71) and (7.72) to be equal to the corresponding arc itself, that is

$$B_j \approx \frac{js}{n}\,. \tag{7.74}$$

Then we obtain for the Rversine

$$\begin{aligned} S &\approx \left(\frac{s}{nr}\right)\left[(n-1)\left(\frac{s}{n}\right) + (n-2)\left(\frac{s}{n}\right) + \ldots\right] \\ &= \left(\frac{1}{r}\right)\left(\frac{s}{n}\right)^2 [(n-1) + (n-2) + \ldots]. \end{aligned} \tag{7.75}$$

Now, from the estimate (7.65) for the *saṅkalita* on the right hand side of (7.75), we obtain

$$S \approx \left(\frac{1}{r}\right)\frac{s^2}{2}\,. \tag{7.76}$$

Equation (7.76) is the first *śara-saṃskāra*, correction to the Rversine. We now substitute our first approximation (7.74) to the *piṇḍa-jyā*-s B_j in (7.72), which gives the Rsine of the desired are as a second order repeated sum of the *piṇḍa-jyā*-s B_j. We then obtain

$$B \approx s - \left(\frac{1}{r}\right)^2\left(\frac{s}{n}\right)^3 [(1+2+\ldots+(n-1))+(1+2+\ldots(n-2))+\ldots]. \tag{7.77}$$

The second term in (7.77) is a *dvitīya-saṅkalita*, the second order repeated sum, and using the estimate (7.67), we obtain

$$B \approx s - \left(\frac{1}{r}\right)^2 \frac{s^3}{1.2.3}\,. \tag{7.78}$$

[13] Unlike the Rsines which are proportional to the arc-bit when the arc-bit is small, the Rversines are proportional to the square of the arc-bit and thus $nS_{\frac{1}{2}}$ can be neglected, unlike nB_1, when the arc-bit $\left(\frac{s}{n}\right)$ is very small.

Thus we see that the first correction obtained in (7.78) to the Rsine-arc-difference (*jyā-cāpāntara-saṃskāra*) is equal to the earlier correction to the Rversine (*śara-saṃskāra*) given in (7.76) multiplied by the arc and divided by the radius and 3.

Another way of deriving (7.78) is by employing the result (7.76) for the Rversines, for the *piṇḍa-śara-jyā*-s S_n, S_{n-1} etc. which appear in (7.73); then we obtain

$$B \approx s - \left(\frac{1}{r}\right)^2 \left(\frac{s}{n}\right)^3 \left(\frac{1}{2}\right) [n^2 + (n-1)^2 + \ldots]. \tag{7.79}$$

Now, the second term in (7.79) can either be estimated using (6.54) as a summation of squares (*varga-saṅkalita*), or by using (7.67) as a *dvitīya-saṅkalita*, second order (repeated) summation, because each individual term there has been obtained by doing a summation as in (7.75). The equation (7.79) then reduces to (7.78). This procedure can be adopted also for deriving the higher order corrections to the Rsine-arc-difference.

It is noted that the results (7.76) and (7.78) are only approximate (*prāyika*), since, instead of the *saṅkalita* of the *piṇḍa-jyā*-s in (7.71) and (7.72), we have only carried out *saṅkalita* of the arc-bits. Now that (7.78) gives a correction to the difference btween the Rsine and the arc (*jyā-cāpāntara-saṃskāra*), we can use that to correct the values of the *piṇḍa-jyā*-s and thus obtain the next corrections to the Rversine and Rsine.

Following (7.78), the *piṇḍa-jyā*-s may now be taken as

$$B_j \approx \frac{js}{n} - \left(\frac{1}{r}\right)^2 \left[\frac{\left(\frac{js}{n}\right)^3}{1.2.3}\right]. \tag{7.80}$$

If we introduce (7.80), in (7.71), we obtain

$$\begin{aligned} S &\approx \left(\frac{1}{r}\right)\left(\frac{s}{n}\right)^2 [(n-1) + (n-2) + \ldots] \\ &- \left(\frac{s}{nr}\right)\left(\frac{1}{r}\right)^2 \left(\frac{s}{n}\right)^3 \left(\frac{1}{1.2.3}\right) [(n-1)^3 + (n-2)^3 + \ldots]. \end{aligned} \tag{7.81}$$

The first term in (7.81) was already evaluated while deriving (7.76). The second term in (7.81) can either be estimated as a summation of cubes (*ghana-saṅkalita*), or as a *tritīya-saṅkalita*, third order (repeated) summation, because each individual term there has been obtained by doing a second-order

(repeated) summation. Hence, recollecting our earlier estimates for these *saṅkalita*-s, we get

$$S \approx \left(\frac{1}{r}\right)\frac{s^2}{1.2} - \left(\frac{1}{r}\right)^3 \frac{s^4}{1.2.3.4} \,. \tag{7.82}$$

Equation (7.82) gives a correction (*śara-saṃskāra*) to the earlier value (7.76) of the Rversine, which is nothing but the earlier correction to the Rsine-arc difference (*jyā-cāpāntara-saṃskāra*) given in (7.78) multiplied by the arc and divided by the radius and 4.

Again, if we use the corrected *piṇḍa-jyā*-s (7.80) in the expression (7.71) for the Rsine, we obtain

$$\begin{aligned} B &\approx s - \left(\frac{1}{r}\right)^2 \left(\frac{s}{n}\right)^3 [(1+2+..+(n-1)) + (1+2+..+(n-2)) + ..] \\ &\quad + \left(\frac{1}{r}\right)^4 \left(\frac{s}{n}\right)^5 \\ &\quad \times \left(\frac{1}{1.2.3}\right) \left[(1^3+2^3+...+(n-1)^3) + (1^3+2^3+...+(n-2)^3) + ..\right] \\ &\approx s - \left(\frac{1}{r}\right)^2 \frac{s^3}{1.2.3} + \left(\frac{1}{r}\right)^4 \frac{s^5}{1.2.3.4.5} \,, \end{aligned} \tag{7.83}$$

where, while evaluating the second order repeated summation in the third term, we have used the estimate (7.68) for the case of a fourth order repeated summation, as each individual term there has been obtained by a second order repeated summation as given in (7.77). Again, we may note that the second correction obtained in (7.83) to the Rsine-arc-difference (*jyā-cāpāntara-saṃskāra*) is equal to the earlier correction to the Rversine (*śara-saṃskāra*) given in (7.82) multitplied by the arc and divided by the radius and 5.

The above process can be repeated to obtain successive higher order corrections for the Rversine and Rsine: By first finding a correction (*jyā-cāpāntara-saṃskāra*) for the difference between the Rsine and the arc, using this correction to correct the *piṇḍa-jyā*-s B_j, and using them in equations (7.71) and (7.72) get the next correction (*śara-saṃskāra*) for the Rversines, and the next correction (*jyā-cāpāntara-saṃskāra*) for the Rsine-arc-difference itself, which is then employed to get further corrections iteratively.

7.5.5 Accurate computation of Rsines and Rversines, without using tables

The successive corrections to Rsine and Rversine, obtained as stated above, are going to be subtractive from the previous correction as they are based upon improving the estimate of the difference between the Rsine and the arc in our expression for the *piṇḍa-jyā*-s. We therefore obtain the following series of correction terms for the Rsine and Rversine:[14]

$$\begin{aligned} R\sin(s) &\approx s - \left(\frac{1}{r}\right)^2 \frac{s^3}{(1.2.3)} + \left(\frac{1}{r}\right)^4 \frac{s^5}{(1.2.3.4.5)} \\ &\qquad - \left(\frac{1}{r}\right)^6 \frac{s^7}{(1.2.3.4.5.7)} + \ldots, \qquad (7.84) \\ R\,\mathrm{vers}(s) &\approx \left(\frac{1}{r}\right)\frac{s^2}{2} - \left(\frac{1}{r}\right)^3 \frac{s^4}{(1.2.3.4)} + \left(\frac{1}{r}\right)^5 \frac{s^6}{(1.2.3.4.6)} - \ldots (7.85) \end{aligned}$$

These can also be re-expressed in the following form given by the rule *nihatya cāpavargeṇa*...(cited also in *Yuktidīpikā*, II. 440–443)

$$\begin{aligned} R\sin(s) &\approx s - s\,\frac{\left(\frac{s}{r}\right)^2}{(2^2+2)} + s\,\frac{\left(\frac{s}{r}\right)^4}{(2^2+2)(4^2+4)} - \ldots, \qquad (7.86) \\ R\,\mathrm{vers}(s) &\approx \frac{r\left(\frac{s}{r}\right)^2}{(1^2+1)} - \frac{r\left(\frac{s}{r}\right)^4}{(1^2+1)(3^2+3)} + \ldots \\ &\approx \frac{r\left(\frac{s}{r}\right)^2}{(2^2-2)} - \frac{r\left(\frac{s}{r}\right)^4}{(2^2-2)(4^2-4)} + \ldots\ . \qquad (7.87) \end{aligned}$$

The above expressions for *jyā* and *śara* can be employed to calculate them without using the tabular values, by using the sequence of numerical values

[14] Using the standard relations (7.4), (7.6) between *jyā* and sine and *śara* and versine, (7.84) and (7.85) reduce to the series

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{(1.2.3)} + \frac{\theta^5}{(1.2.3.4.5)} - \frac{\theta^7}{(1.2.3.4.5.6.7)} + \ldots\ , \\ \mathrm{vers}\,\theta &= \frac{\theta^2}{(1.2)} - \frac{\theta^4}{(1.2.3.4)} + \frac{\theta^6}{(1.2..4.5.6)} - \ldots\ . \end{aligned}$$

given by the formulae, *vidvān* etc and *stena* etc.[15] This is indeed the method of deriving the required Rsine and Rversine without using the regular tabular values.

7.6 Accurate circumference from an approximate value

If d be the diameter of a circle and if we only know the approximate circumference c^*, a way to find a more accurate value of the circumference c, is the following. First evaluate the auxiliary Rsine value B^*, of the arc $\frac{c^*}{4}$ in a circle of radius d, as accurately as is desired. That is,

$$B^* = \frac{c^*}{4} - \left(\frac{1}{d}\right)^2 \frac{1}{1.2.3} \left(\frac{c^*}{4}\right)^3 + \left(\frac{1}{d}\right)^2 \frac{1}{1.2.3.4.5} \left(\frac{c^*}{4}\right)^5 \ldots \quad . \tag{7.88}$$

Let K^* be the corresponding Rcosine, namely

$$K^* = (d^2 - B^{*2})^{\frac{1}{2}}. \tag{7.89}$$

Let Δ be given by

$$\Delta = \left(\frac{B^{*2}}{2}\right)^{\frac{1}{2}} - \left(\frac{K^{*2}}{2}\right)^{\frac{1}{2}}. \tag{7.90}$$

Let δ be the arc corresponding to the Rsine value of Δ, which can be calculated by the relation

$$\Delta + \left(\frac{1}{d}\right)^2 \frac{1}{1.2.3} \Delta^3 \approx \delta. \tag{7.91}$$

[15]The formulae *vidvān* etc. and *stena* etc., are attributed to Mādhava by Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya, Gaṇitapāda*, 17 (see also, *Yuktidīpikā*, II. 437–438). If the quadrant of a circle is assigned the measure $q = 5400'$, then for a given arc s, the corresponding Rsine given by (7.86) can be expressed in the form:

$$R\sin(s) \approx s - \left(\frac{s}{q}\right)^3 \left(u_3 - \left(\frac{s}{q}\right)^2 \left(u_5 - \left(\frac{s}{q}\right)^2 \left(u_7 - \left(\frac{s}{q}\right)^2 \left(u_9 - \left(\frac{s}{q}\right)^2 u_{11}\right)\right)\right)\right),$$

where, the values of u_{11}, u_9, u_7, u_5, and u_3 are given by the formulae *vidvān* etc., as follows: $u_{11} = 44'''$, $u_9 = 33''06'''$, $u_7 = 16'05''41'''$, $u_5 = 273'57''47'''$ and $u_3 = 2220'39''40'''$. The above expression gives the values of Rsine of any arc accurately up to the thirds. Similarly, the formulae *stena* etc., are used to calculate the Rversine of any arc accurately up to the thirds. Mādhava has also given the tabulated sine values (for arcs in multiples of $225'$) accurately to the thirds in the rule *śreṣṭhaṃ nāma variṣṭhānāṃ*...(cited by Nīlakaṇṭha in his *Āryabhaṭīyabhāṣya*, *Gaṇitapāda*, 12).

Then, an accurate circumference is given by

$$c = 4\left[\left(\frac{c^*}{4}\right) + \delta\right]. \tag{7.92}$$

The rationale for this procedure is also explained. If c^* is the accurate circumference itself, then B^{*2} and K^{*2} will be the squares of the Rsine and Rcosine of one-eighth the circumference of a circle with radius (and not diameter) d, and hence will be equal to $\frac{d^2}{2}$. Hence (by the *jīve-paraspara-nyāya* to be stated shortly), Δ is nothing but the Rsine of the difference of one-eighth the true circumference and rough circumference of that circle. Therefore four times the corresponding arc will give the correction to the approximate value of the circumference c^*.

Appendix C

Missing Papers

Papers that should have been in this collection, but were not

Because of their historical and other importance, the following papers should have been included in this collection but could not be included as they were not available in the online world at the time of the compilation of this collection (July 2023).

1. **(1945)**

   K. Mukunda Marar and C.T. Rajagopal, "Gregory's series in the mathematical literature of Kerala", Mathematics Student, 13, 1945, pp. 92–98.

2. **(1948)**

   A. Venkataraman, "Some interesting proofs from Yuktibhāṣā", Mathematics Student, 16, 1948, pp. 1–7.

3. **(1951)**

   C.T. Rajagopal and T. V. Vedamurthi Aiyar, "On the Hindu proof of Gregory's series", Scripta Mathematica, 17, 1951, pp. 65–74.

4. **(1952)**

   C. T. Rajagopal and T. V. Vedamurthi Aiyar, "A Hindu approximation to π", Scripta Mathematica, 18, 1952, pp. 25–30.

5. **(1958)**

   K.V. Sarma, "Jyeṣṭhadeva and his identification as the author of Yuktibhāṣā", Adyar Library Bulletin, 22, 1958, pp. 35–40.

6. **(2010)**

   U. K. V. Sarma, Vanishri Bhat, Venkateswara Pai and K. Ramasubramanian, "The discovery of Mādhava series by Whish: An episode in historiography of science", Gaṇita Bhāratī, 32, 2010, pp. 115–126

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